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A note for the crack problem of functionally graded materials

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Abstract. Employing the power-type function of material properties, a crack lying between the functionally graded materials (FGMs) and homogeneous substrate is studied by an asymptotic analysis from that of bimaterials, J-integral and the numerical calculations. The present results show that when the curve of the material property is concave, i.e. the power (m) of function of material property is great than 1, the stress distribution near the crack-tip is the same as that of homogeneous materials, which is in agreement with previous findings. However, if the curve of the material property is convex corresponding to $0< m <1$, our results show that the stress distribution is strongly affected by m and it can be obtained asymptotically from that of bimaterials.

1. Introduction

The functionally graded materials (FGMs) have attracted much attention for their potential applications in many fields of industries, from aerospace to automobile, to microelectronics etc. Some experiments have showed that there are no distinct layers at a scale of millimeter in FGMs, yet layers exist below this scale if examining with a powerfully distinguishable tool (Watanabe R. and Kawasaki A., 1992; Jung, Churl, et al, 1997). Thus, FGMs can be characterized by spatially varying microstructures that produce continuously changing mechanical properties at the macroscopic level. Based on a model of exponential type function of material properties, Erdogan and coworkers (1983, 1993, 1994, 1996) studied some basic crack problems of FGMs. Among the contributions in this regard are those by Gu, Asaro (1990) and Jin et al. (1994, 1996). By adopting the power-law type function of material properties, G. C. Sih (1981) studied the crack problem as well. It seems that the results concerning the effect of material gradient to the singularity as well as to the angular distribution of stress field near the crack-tip is different from adopting the different function form of material properties. For instance, Jin, et al. obtained that the angular distributions of stresses were the same as that of homogeneous materials, yet G. C. Sih thought they were affected by the material gradient. Consequently, a further research is still needed to clarify this issue. In the present study, based on the multilayered model of FGMs, we correlate the crack problem of bimaterials with that of FGMs and present an asymptotic stress field near the crack-tip in FGMs. Moreover, the analyses from J-integral and numerical calculations are also carried out for illustrating the stress distributions of FGMs in question.

2. The analysis from the conservation of J-integral

The problem considered in the present study is an interfacial crack between FGM coatings and homogeneous substrate. The gradient of FGM coatings is assumed to be in one dimension normal to the crack direction as shown in figure 1. In previous studies, it was shown that the influence of the variation of Poisson's ratio on stress intensity factors (SIFs) is rather insignificant (G.C. Sih, 1981) and therefore ν may be assumed to be constant throughout the medium. As a result, inhomogeneity of FGMs may prevail by assuming the shear modulus μ to be a function of coordinate:

$$\mu(y) = \mu_0 + (\mu_H - \mu_0)(\frac{y}{H})^m \qquad m > 0 \qquad (1)$$

where μ_0 and μ_H represent the material properties at locations $y = 0$ and $y = H$, respectively, and m is a parameter concerning the material gradient. The value of m should be greater than zero and otherwise, if $m < 0$, there exists a discontinuous plane at y=0, which is not compatible with the characteristics of FGMs.

Assume that the strain component being $\varepsilon \sim r^{\lambda}$, based on the elastic constitution relationship and equation (1), the stress component is derived to be $\sigma \sim r^{m+\lambda}$ and r^{λ}, and thereby the density of strain energy $\omega \sim r^{m+2\lambda}$ and $r^{2\lambda}$. According to the works of Jin, Noda (1994) and Gu, Asaro (1990), J-integral of the present problem is independent of path. It is therefore deduced that $\omega \sim r^{-1}$ and thus the order of stress component is $(m-1)/2$ as well as the traditional singular order of $-1/2$. Hence, if $0<m<1$, i.e. the curve of material properties is convex, the stress distributions will be different from those of homogeneous materials and have multi-singularities.

3. Asymptotic analysis from the results of bimaterials

In the following, the asymptotic analysis is performed by correlating the problem with that of bimaterials. According to the results of bimaterial interfacial crack originally obtained by Williams (1959), the oscillatory factor ε, which was defined to be a parameter uniquely related to the material properties, was given by:

$$\varepsilon = \frac{1}{2\pi} \ln \frac{\mu_2 \kappa_1 + \mu_1}{\mu_1 \kappa_2 + \mu_2} \qquad (2)$$

where $\kappa_i = 3 - 4\nu_i$ for plane strain, $\kappa_i = \frac{3-\nu_i}{1+\nu_i}$ for plane stress, the subscribe i=1,2 refers to materials 1 and 2, respectively. Since FGMs is generally approximated by the multihomogeneous-layered materials as shown in figure 1, the results of crack of bimaterial can be regarded as an asymptotic solution of

FGMs. An important feature of FGMs different from bimaterials is that the oscillatory factor ε is correlated to the thickness of the divided homogeneous layer h, which is decided by material gradient or the distinguishable size of observation. By replacing the shear modulus μ_1 with the average value of a layer near the crack-tip and μ_2 with μ_0 in formula (2), i.e. $\mu_1 = \frac{\int_0^h \mu dy}{h} = \mu_0 + \frac{\mu_H - \mu_0}{m+1}(\frac{h}{H})^m$, $\mu_2 = \mu_0$, ε in asymptotic results of FGMs is approached as:

$$\varepsilon = \frac{1}{2\pi}\ln\frac{\mu_0(1+\kappa)+\kappa\frac{\mu_H-\mu_0}{m+1}(\frac{h}{H})^m}{\mu_0(1+\kappa)+\frac{\mu_H-\mu_0}{m+1}(\frac{h}{H})^m} = \frac{1}{2\pi}\ln G \tag{3}$$

Concerning the above ε, two cases are discussed as follows. i) If m is big enough, the value of ε is tending to zero (G =1) owing to h/H < 1, and the asymptotic solution will be the same as that of homogeneous materials. This is in agreement with those by adopting the exponential type of material property (Chen and Erdogan, 1996; Jin and Noda, 1994; Gu and Asaro, 1990). ii) If m is small enough so that $(h/H)^m$ tends to a limited value, ε will be nonzero ($G\neq 1$) and the stress distributions will be similar to those of bimaterials, which are influenced by the factor ε. As a result, by substituting ε of formula (3) into the results of bimaterials (Williams, 1959), the asymptotic stress distributions near the crack-tip of FGM are obtained as follows:

$$\begin{aligned}\sigma_r = \frac{K_1}{2\sqrt{2r}}\{G^{\frac{\theta\pm\pi}{2\pi}}[3\cos(\tfrac{1}{2}\theta+\tfrac{\ln G}{2\pi}\ln r)+\frac{\ln G}{\pi}\sin\theta\cos(\tfrac{1}{2}\theta-\tfrac{\ln G}{2\pi}\ln r)\\ -\sin\theta\sin(\tfrac{1}{2}\theta-\tfrac{\ln G}{2\pi}\ln r)]-G^{-\frac{\theta\pm\pi}{2\pi}}\cos(\tfrac{3}{2}\theta+\tfrac{\ln G}{2\pi}\ln r)\}\\ -\frac{K_2}{2\sqrt{2r}}\{G^{\frac{\theta\pm\pi}{2\pi}}[3\sin(\tfrac{1}{2}\theta+\tfrac{\ln G}{2\pi}\ln r)-\sin\theta\cos(\tfrac{1}{2}\theta-\tfrac{\ln G}{2\pi}\ln r)\\ -\frac{\ln G}{\pi}\sin\theta\sin(\tfrac{1}{2}\theta-\tfrac{\ln G}{2\pi}\ln r)]-G^{-\frac{\theta\pm\pi}{2\pi}}\sin(\tfrac{3}{2}\theta+\tfrac{\ln G}{2\pi}\ln r)\}+o(r^0)\end{aligned} \tag{4}$$

$$\begin{aligned}\sigma_\theta = \frac{K_1}{2\sqrt{2r}}\{G^{\frac{\theta\pm\pi}{2\pi}}[\cos(\tfrac{1}{2}\theta+\tfrac{\ln G}{2\pi}\ln r)-\frac{\ln G}{\pi}\sin\theta\cos(\tfrac{1}{2}\theta-\tfrac{\ln G}{2\pi}\ln r)\\ +\sin\theta\sin(\tfrac{1}{2}\theta-\tfrac{\ln G}{2\pi}\ln r)]+G^{-\frac{\theta\pm\pi}{2\pi}}\cos(\tfrac{3}{2}\theta+\tfrac{\ln G}{2\pi}\ln r)\}\\ -\frac{K_2}{2\sqrt{2r}}\{G^{\frac{\theta\pm\pi}{2\pi}}[\sin(\tfrac{1}{2}\theta+\tfrac{\ln G}{2\pi}\ln r)+\sin\theta\cos(\tfrac{1}{2}\theta-\tfrac{\ln G}{2\pi}\ln r)\end{aligned}$$

$$+\frac{\ln G}{\pi}\sin\theta\sin(\tfrac{1}{2}\theta-\tfrac{\ln G}{2\pi}\ln r)]+G^{-\frac{\theta\pm\pi}{2\pi}}\sin(\tfrac{3}{2}\theta+\tfrac{\ln G}{2\pi}\ln r)\}+o(r^0) \quad (5)$$

$$\tau_{r\theta}=\frac{\mathrm{K}_1}{2\sqrt{2r}}\{G^{\frac{\theta\pm\pi}{2\pi}}[\sin(\tfrac{1}{2}\theta+\tfrac{\ln G}{2\pi}\ln r)-\sin\theta\cos(\tfrac{1}{2}\theta-\tfrac{\ln G}{2\pi}\ln r)$$

$$-\frac{\ln G}{\pi}\sin\theta\sin(\tfrac{1}{2}\theta-\tfrac{\ln G}{2\pi}\ln r)]+G^{-\frac{\theta\pm\pi}{2\pi}}\sin(\tfrac{3}{2}\theta+\tfrac{\ln G}{2\pi}\ln r)\}$$

$$-\frac{\mathrm{K}_2}{2\sqrt{2r}}\{G^{\frac{\theta\pm\pi}{2\pi}}[-\cos(\tfrac{1}{2}\theta+\tfrac{\ln G}{2\pi}\ln r)-\frac{\ln G}{\pi}\sin\theta\cos(\tfrac{1}{2}\theta-\tfrac{\ln G}{2\pi}\ln r)$$

$$+\sin\theta\sin(\tfrac{1}{2}\theta-\tfrac{\ln G}{2\pi}\ln r)]-G^{-\frac{\theta\pm\pi}{2\pi}}\cos(\tfrac{3}{2}\theta+\tfrac{\ln G}{2\pi}\ln r)\}+o(r^0) \quad (6)$$

where $K_1=4\sqrt{\tfrac{2}{\pi}}G^{\pm\frac{1}{2}}(\tfrac{1}{2}K_I+\frac{\ln G}{2\pi}K_{II})$, $K_2=4\sqrt{\tfrac{2}{\pi}}G^{\pm\frac{1}{2}}(\frac{\ln G}{2\pi}K_I+\tfrac{1}{2}K_{II})$, K_I and K_{II} are the stress intensify factor of mode *I* and *II*, respectively, in conventional meaning. And "+" is used for $-\pi<\theta<0$ and "-" for $0<\theta<\pi$, respectively. It is noted that if $m\to 0$, ε will tend to $\frac{1}{2\pi}\ln\frac{\mu_H\kappa+\mu_0}{\mu_0\kappa+\mu_H}$ and from formula (1), the FGMs/substrate in question degenerates into the bimaterial, so are the results of the stress distributions in formulae (4-6). In this sense, the stress distributions in FGMs/substrate are related to the material gradient and they are between those of homogeneous materials and bimaterials when the value of *m* decrease from a large value (>1) to zero. Concerning the influence of the oscillatory factor ε on the stress distributions in bimaterials, it is suggested to consult in some relevant papers (Rice, 1988; Yang and Shih, 1994).

4. Numerical analysis using finite element method

We also perform the numerical analysis to study the variation of SIFs with the material gradient. The FGMs/substrate sample studied is the Ti-Al matrix mixed with ZrO_2 grains versus the Ti-Al alloy. The values of elastic constants for the substrate (E_0) and the ZrO_2 ceramic (E_H) are taken to be 105 GPa and 165 GPa respectively, and elastic parameter of FGMs is adopted as formula (1). In addition, the Poisson's ratio ν is assumed to be 0.3 throughout the medium. The crack length is 5 mm, the tensile load is 60 N and h/H = 0.1. The gradient coefficient *m* is selected to be 0, 0.01, 0.05, 0.1, 0.25, 0.5, 1.0, 5.0,10.0, 20.0 respectively for studying its role in influencing SIFs. The numerical analysis is performed using the commercial code ANSYS. Figure 2 shows the calculated variations of SIFs with the parameter *m* together with those of homogeneous materials (substrate) for comparison. It can be seen that for the FGMs/substrate sample, an inflexion point appears at $m = 1$ and that when $m > 1$, the SIFs of both modes *I* and *II* approach to their respective SIFs of the homogenous materials with increasing m, suggesting that the stress field of the FGMs/substrate is similar to that of the homogenous materials. Carefully examining figure 2, an important finding is that when $0 < m < 1$, there are sharp increase of K_{II} with decreasing m, implying that the stress distribution is quite different from that of homogenous materials. The numerical results are in agreement with the above findings from J-integral and asymptotic analysis of bimaterials. Besides, the above numerical results also suggest that when $m = 1$, that is, the material properties change linearly, the SIF has minimum value, which is similar to the finding upon thermal load (Noda, 1997).

5. Conclusion

By adopting the power type function of material properties, J-integral and asymptotic analysis reveal that the stress distributions are determined by the parameter m of material gradient. That is, if the material property curve is convex corresponding to $0 < m < 1$, the gradient cannot be omitted near the crack-tip and the stress distribution can be obtained asymptotically from that of bimaterials. On the contrary, when the material property curve is concave corresponding to $m > 1$, the material gradient is small near the crack-tip and the stress distributions are the same as those of homogeneous materials, which is in good agreement with the results from adopting the exponential function with concave curve of the material properties. Besides, the numerical results also give a support to the above findings.

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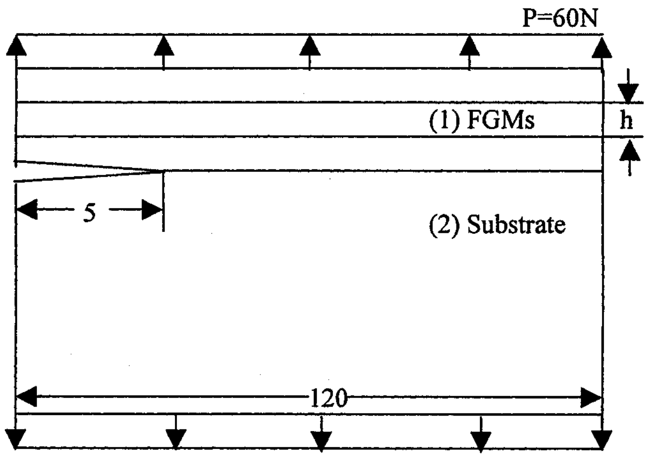

Fig.1 Schematic diagram of FGMs / substrate.

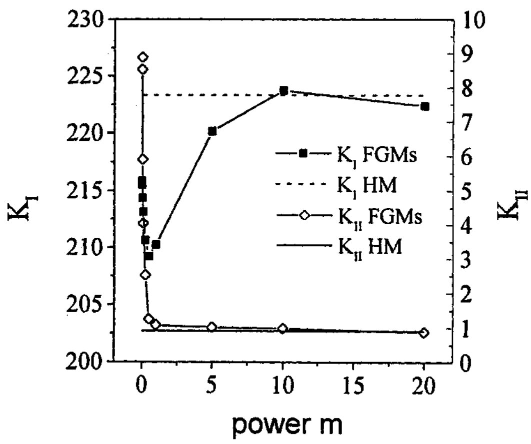

Fig.2 The variations of K_I and K_{II} with the power m of material property for the functionally graded materials (FGMs) and the homogeneous materials (HM).