A Method to Generate New Exact Solutions from a Known Stationary Solution *

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(Received 19 November 1998)

By combining the Backlund transformations and the AKNS system[Study in Appl. Math. 53 (1974) 249.] which is a linear eigenvalue problem of the corresponding evolution equation, a method to find new exact solutions from known stationary solutions for nonlinear integrable equations is proposed. As an example, Korteweg de Vries (KdV) equation is used to illustrate this method, and a class of new exact solutions of KdV equation is obtained.

PACS: 02.30.Jr, 02.30.Hq

Recently, the dynamical behavior of complicated systems is considered as one which stands at the frontier of nonlinear dynamics. This problem is concerned with not only the behavior of time evolution, but also the structure of space, therefore, the research is very difficult, Cross and Hohenberg[1] made a good review on this problem and pointed out that there are four ways to research it, i.e., (1) numerical simulation, (2) qualitative analysis, (3) perturbation method and (4) to find new exact solutions.

During the last three decades, construction of exact solutions for a wide class of nonlinear integrable systems has been developed rapidly. At present, there are many methods to construct new exact solutions, such as the inverse scattering method,[2] the Backlund transformations,[3,4] the Painleve analysis,[5] the Lie group methods,[6] the direct algebraic method[7] and tangent hyperbolic method,[8] the perturbation theory.[9–11]

In this letter, a method to find new exact solutions from known stationary solutions for nonlinear integrable equations is proposed by associating the AKNS system[2] and the Backlund transformations.

Firstly, we describe the frame of method. It is known that many nonlinear integrable systems can be derived from the AKNS system, which is a linear eigenvalue problem and is defined by

$$\Phi_x = M\Phi, \quad \Phi_t = N\Phi, \tag{1}$$

where $\Phi = (\Phi_1, \Phi_2)^{\mathrm{T}}$, $M = \begin{pmatrix} \eta & q \\ r & -\eta \end{pmatrix}$, $N = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}$, η is any eigenvalue associated with problem to consider, q and r are functions of x and t. The equivalent form of Eq. (1) is

$$\Phi_{1x} = \eta\Phi_1 + q\Phi_2, \tag{2a}$$
$$\Phi_{2x} = r\Phi_1 - \eta\Phi_2, \tag{2b}$$
$$\Phi_{1t} = A\Phi_1 + B\Phi_2, \tag{2c}$$
$$\Phi_{2t} = C\Phi_1 - A\Phi_2. \tag{2d}$$

From $\Phi_{xt} = \Phi_{tx}$, the following relations hold:

$$-A_x + qC - rB = 0, \tag{3a}$$
$$q_t - B_x - 2qA + 2\eta B = 0, \tag{3b}$$
$$r_t - C_x - 2\eta C + 2rA = 0, \tag{3c}$$

where A, B, and C are functions of q, r, and η.

Now, the stationary solutions of nonlinear integrable equations are considered, namely, q and r are only functions of x. This implies that A, B, and C are also only functions of x, so Eq. (3) can be rewritten as

$$-A_x + qC - rB = 0, \tag{3a$'$}$$
$$-B_x - 2qA + 2\eta B = 0, \tag{3b$'$}$$
$$-C_x - 2\eta C + 2rA = 0. \tag{3c$'$}$$

From Eqs. (2b), (2d), and (3c$'$), we get

$$C\Phi_{2x} - r\Phi_{2t} = \frac{1}{2}C_x\Phi_2. \tag{4}$$

By the method of characteristics, one may find the solutions of Eq. (4) as follows:

$$\Phi_2 = k_1 C^{1/2}, \tag{5}$$
$$k_2 = \sigma(x) - t, \tag{6}$$

where k_1 and k_2 are integration constants, $\sigma(x) = -\int(r/C)\mathrm{d}x$.

From Eqs. (5) and (6), the general solution of Eq. (4) is

$$\Phi_2 = F(\xi)C^{1/2}, \tag{7}$$

where $\xi = \sigma(x) - t$. Substituting Eq. (7) into Eq. (2b), one may obtain the general solution of Φ_1:

$$\Phi_1 = -C^{1/2}[F'(\xi) - AF(\xi)]. \tag{8}$$

In order to determine the function $F(\xi)$, substituting Eqs. (7) and (8) into Eq. (2a), we find it satisfies the following ordinary differential equation:

$$F''(\xi) - (A^2 + BC)F(\xi) = 0. \tag{9}$$

*Supported by the National Natural Science Foundation of China under Grant No. 19872044.

From Eqs. (3a′)−(3c′), one may easily find

$$\frac{d}{dx}(A^2+BC)=0,$$

so $\alpha \equiv A^2+BC$ is a constant. Generally, α is a function of η and integration constants. Therefore, the solution of $F(\xi)$ in Eq (9) may be the following two cases:

$$F(\xi)=c_1\xi+c_2, \quad \text{for} \quad \alpha=0 \tag{10a}$$

$$F(\xi)=c_1\cosh[\omega(\xi+c_2)], \quad \text{for} \quad \alpha\neq 0, \tag{10b}$$

where c_1 and c_2 are any constants, $\omega^2=(\omega_1+i\,\omega_2)^2\equiv\alpha$. Thus the solutions of Φ_1 and Φ_2 can be determined by various forms of $F(\xi)$.

It is shown[12] that the Backlund transformations of many nonlinear integrable equations have following relation:

$$\widetilde{q}=q+W(\eta,\Gamma), \tag{11}$$

where $\widetilde{q}$ is a new solution of the nonlinear integrable equations, W is a concrete function from the Backlund transformation, and

$$\Gamma=\frac{\Phi_1}{\Phi_2}. \tag{12}$$

Hence by a known solution q, one may obtain a new solution from Eq. (11).

As an example, a class of new exact solutions of Korteweg de Vries (KdV) equation is derived by this method. We consider KdV equation:

$$q_t+6qq_x+q_{xxx}=0, \tag{13}$$

its stationary solutions satisfy

$$3q^2+q_{xx}=a_1, \tag{14}$$

where a_1 is a constant. An equivalent form of Eq. (14) is

$$q_x=p, \quad p_x=-3q^2+a_1, \tag{15}$$

which is a Hamilton system with Hamiltonian

$$H=\frac{1}{2}p^2+q^3-a_1q=h, \tag{16}$$

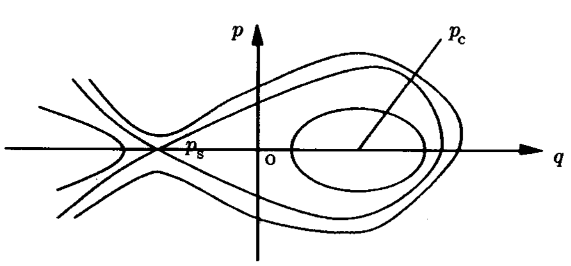

Fig. 1. Phase potrait of the system (15), where P_s is a saddle and P_c is a center.

where h is integration constant. We only discuss the case with $a_1>0$, because the other case is trivial. At the moment, from the planar Hamiltonian system theory, the system (15) admits a center $p_c\left(0,\sqrt{a_1/3}\right)$ and a saddle point $p_s\left(0,-\sqrt{a_1/3}\right)$, and its phase portait is roughly given in Fig. 1.

When $|h|<h_0$, where $h_0=(2a_1/3)\sqrt{a_1/3}$, there are a family of periodic orbits surrounding p_c, whose parametric expressions are

$$q_h(x)=a\,\mathrm{cn}^2\left(\frac{\sqrt{2(a-c)}}{2}x,k\right)+b\,\mathrm{sn}^2\left(\frac{\sqrt{2(a-c)}}{2}x,k\right), \tag{17}$$

where cn and sn are elliptic functions, and $a=2\sqrt{a_1/3}\cos(\theta/3)$, $b=2\sqrt{a_1/3}\cos[(4\pi+\theta)/3]$, $c=2\sqrt{a_1/3}\cos[(2\pi+\theta)/3]$, $k^2=(a-b)/(a-c)=\sin[(2\pi+\theta)/3]/\sin[(\pi+\theta)/3]$, $\theta=\arccos\{h/[(2a_1/3)\sqrt{a_1/3}]\}$. Note that the subscript h in $q_h(x)$ denotes a family, the follows including $\sigma_h(x)$ are the same. When $h=h_0$, there is a homoclinic orbit connecting p_s, whose parametric expression is

$$q_0(x)=\sqrt{\frac{a_1}{3}}\left[2-3\mathrm{th}^2\left(\frac{\sqrt{6}}{2}\left(\frac{a_1}{3}\right)^{1/4}x\right)\right]. \tag{18}$$

For KdV equation, in its AKNS system, $r=-1$, $A=-4\eta^3-2\eta q-q_x$, $C=4\eta^2+2q$, $B=-4\eta^2q-2\eta q_x-q_{xx}-2q^2$ (to see Ref. 12), so from $\sigma(x)=-\int(r/C)dx$, we obtain, respectively,

$$\sigma_h(x)=\int\frac{dx}{4\eta^2+q_h(x)}=\frac{\sqrt{2(a-c)}}{(4\eta^2+2a)(a-c)}\Pi(\varphi,\beta^2,k), \tag{19}$$

and

$$\sigma_0(x)=\int\frac{dx}{4\eta^2+q_0(x)}=\frac{1}{\sqrt{6}(a_1/3)^{1/4}(2\eta^2-\sqrt{a_1/3})}\cdot\left[\ln(\mathrm{tg}\varphi_0+\sec\varphi_0)-\beta\ln\sqrt{\frac{1+\beta\sin\varphi_0}{1-\beta\sin\varphi_0}}\right], \tag{20}$$

where Π is the normal elliptic integral of the third kind,

$\beta=\left[\frac{2(a-b)}{4\eta^2+2a}\right]^{1/2}$, $\varphi=\mathrm{am}\left(\frac{\sqrt{2(a-c)}}{2}x,k\right)$, $\varphi_0=\arcsin\left[\mathrm{th}\left(\frac{\sqrt{6}}{2}\left(\frac{a_1}{3}\right)^{1/4}x\right)\right]$, where am(·,·) is a sine amplitude of elliptic functions. In addition, by calculation the constant α is obtained:

$$\alpha=A^2+BC=16\eta^6-4\eta^2a_1+2h. \tag{21}$$

According to certain boundary conditions of problem, the value of η can be selected, then α is determined. While for KdV equation, the function $W(\eta,\Gamma)$

in Eq. (11) is $-2\Gamma_x$ (Ref. 12). Thus new solutions can be constructed with different α.

By an easy calculation, one may find

$$\Gamma = -\frac{1}{4\eta^2+2q}\left[\frac{1}{\xi+c_0}+4\eta^3+2\eta q+q_x\right], \quad \text{for } \alpha=0,$$

$$\Gamma = -\frac{1}{4\eta^2+2q}[\omega\coth[\omega(\xi+c_0)]+4\eta^3+2\eta q+q_x], \quad \text{for } \alpha\neq 0,$$

where c_0 is a constant, ω is given Eq. (10).

Then, by Eq. (11), we find new solutions from $q_h(x)$ and $q_0(x)$, respectively:
for $\alpha=0$,

$$\begin{aligned}\widetilde{q}_h(x) =& q_h(x) - \frac{q_h'(x)}{[2\eta^2+q_h(x)]^2}\left[\frac{1}{\xi_h+c_0}+4\eta^3 + 2\eta q_h(x)+q_h'(x)\right] + \frac{1}{2\eta^2+q_h(x)} \\ &\cdot\left[\frac{-\sigma_h'(x)}{(\xi_h+c_0)^2}+2\eta q_h'(x)+q_h''(x)\right],\end{aligned} \tag{22}$$

$$\begin{aligned}\widetilde{q}_0(x) =& q_0(x) - \frac{q_0'(x)}{[2\eta^2+q_0(x)]^2}\left[\frac{1}{\xi_0+c_0}+4\eta^3 + 2\eta q_0(x)+q_0'(x)\right] + \frac{1}{2\eta^2+q_0(x)} \\ &\cdot\left[\frac{-\sigma_0'(x)}{(\xi_0+c_0)^2}+2\eta q_0'(x)+q_0''(x)\right],\end{aligned} \tag{23}$$

for $\alpha\neq 0$,

$$\begin{aligned}\widetilde{q}_h(x) =& q_h(x) - \frac{q_h'(x)}{[2\eta^2+q_h(x)]^2}[\omega\coth(\omega\xi_h+\omega c_0) + 4\eta^3+2\eta q_h(x)+q_h'(x)] + \frac{1}{2\eta^2+q_h(x)} \\ &\cdot\left[-\omega^2\sigma_h'(x)\mathrm{csch}^2(\omega\xi_h+\omega c_0) +2\eta q_h'(x)+q_h''(x)\right],\end{aligned} \tag{24}$$

$$\begin{aligned}\widetilde{q}_0(x) =& q_0(x) - \frac{q_0'(x)}{[2\eta^2+q_0(x)]^2}[\omega\coth(\omega\xi_0+\omega c_0) + 4\eta^3+2\eta q_0(x)+q_0'(x)] + \frac{1}{2\eta^2+q_0(x)} \\ &\cdot\left[-\omega^2\sigma_0'(x)\mathrm{csch}^2(\omega\xi_0+\omega c_0) +2\eta q_0'(x)+q_0''(x)\right],\end{aligned} \tag{25}$$

where

$$\xi_h=\sigma_h(x)-t, \quad \xi_0=\sigma_0(x)-t,$$

$$q_h'(x) = \sqrt{2(a-c)}(b-a)\mathrm{sn}\left(\frac{\sqrt{2(a-c)}}{2}x,k\right)\cdot\mathrm{cn}(\cdot,\cdot)\mathrm{dn}(\cdot,\cdot),$$

$$q_h''(x) = (b-a)(a-c)[\mathrm{cn}^2(\cdot,\cdot)\mathrm{dn}^2(\cdot,\cdot)-\mathrm{sn}^2(\cdot,\cdot)\mathrm{dn}^2(\cdot,\cdot) - k^2\mathrm{sn}^2(\cdot,\cdot)\mathrm{cn}^2(\cdot,\cdot)],$$

$$\sigma_h'(x) = \frac{1}{4\eta^2+2a\,\mathrm{cn}^2(\cdot,\cdot)+2b\,\mathrm{sn}^2(\cdot,\cdot)},$$

$$\sigma_0'(x) = \frac{1}{4\eta^2+2\sqrt{a_1/3}\{2-3\mathrm{th}^2[(\sqrt{6}/2)(a_1/3)^{1/4}x]\}},$$

$$q_0'(x) = -3\sqrt{6}\left(\frac{a_1}{3}\right)^{3/4}\mathrm{th}\left(\frac{\sqrt{6}}{2}\left(\frac{a_1}{3}\right)^{1/4}x\right)\cdot\mathrm{sech}^2\left(\frac{\sqrt{6}}{2}\left(\frac{a_1}{3}\right)^{1/4}x\right),$$

$$q_0''(x) = 3a_1\mathrm{sech}^2\left(\frac{\sqrt{6}}{2}\left(\frac{a_1}{3}\right)^{1/4}x\right)\cdot\left[3\mathrm{th}^2\left(\frac{\sqrt{6}}{2}\left(\frac{a_1}{3}\right)^{1/4}x\right)-1\right].$$

$\mathrm{dn}(\cdot,\cdot)$ is an elliptic function.

REFERENCES

1 M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. 65 (1993) 851.

2 M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, Study in Appl. Phys. 53 (1974) 249, be refered to as AKNS; M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolutions and Inverse Scattering* (Cambridge University Press, Cambridge, 1991).

3 C. Rogers and W. E. Shadwick, *Backlund Transformations and Their Application* (Academic Press, New York, 1982).

4 A. H. Khater and O. H. El-kalauwy, Chaos, Solitons and Fractals, 8 (1997) 1901.

5 J. Weiss, M. Tabor and G. Garnevale, J. Math. Phys. 24 (1983) 522.

6 P. J. Olver, *Applications of Lie Groups to Differential Equations* (Springer-Verlag, Berlin, 1986).

7 W. Herman and M. J. Takaoka, J. Phys. A 23 (1990) 4805.

8 W. Malfliet, J. Phys. A. 26 (1993) L 723.

9 CHEN Xiang-jun and CHEN Zhi-de, Chin. Phys. Lett. 15 (1998) 504.

10 WU Ke-xue and CHEN Shi-rong, Chin. Phys. Lett. 15 (1998) 654.

11 LÜ Xiang-qing, Chin. Phys. Lett. 14 (1997) 561.

12 K. Konno and M. Wodati, Prog. Theor. Phys. 53 (1975) 1652.