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DYNAMIC STRESS FIELD AROUND THE MODE Ⅲ CRACK TIP IN AN ORTHOTROPIC FUNCTIONALLY GRADED MATERIAL *

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Abstract: The problem of a Griffith crack in an unbounded orthotropic functionally graded material subjected to antipole shear impact was studied. The shear moduli in two directions of the functionally graded material were assumed to vary proportionately as definite gradient. By using integral transforms and dual integral equations, the local dynamic stress field was obtained. The results of dynamic stress intensity factor show that increasing shear moduli's gradient of FGM or increasing the shear modulus in direction perpendicular to crack surface can restrain the magnitude of dynamic stress intensity factor.

Key words: anisotropic media; functionally graded materials; dynamic stress intensity factor; crack; impact

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Introduction

In recent years, great attentions have been paid to the research of Functionally Graded Materials (FGM). From the viewpoints of applied mechanics, FGM are non-homogeneous solids. The non-homogeneity of FGM has a great influence on their mechanical behavior, especially when the components made of FGM involve some flaws. There has been a considerable bulk of studies concentrated on this influence[1~3]. However, most of the studies are mainly concentrated on static problems. Reports on dynamic fracture mechanics of FGM are very few[4]. In fact, the components made of FGM would be inevitably subjected to time dependent loadings. Therefore, the knowledge of the dynamic fractural behavior of this kind of components is essential

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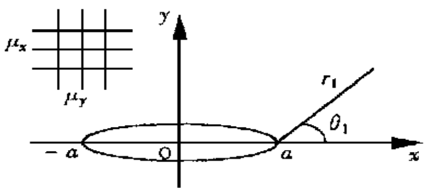
to achieving an in-depth understanding of the failure mechanisms of FGM.

On the other hand, up to now, most of the existing solutions to crack problems related to FGM usually assume that the material is isotropic elastic. However, the nature of the techniques used in processing the FGM are seldom isotropic. For example, processing by a plasma spray technique usually leads to a lamellar structure and processing by electron beam physical vapor deposition generally lead to a highly columnar structure[5]. Thus, it is necessary to consider the anisotropic character of the FGM. Recently, Ozturk and Erdogan studied the mode I static crack problem in an inhomogeneous orthotropic medium[5]. The model in their paper was an exponential form. The singular integral equation technique was used in their study.

In this paper, we studied the problem of a finite crack in an orthotropic FGM subjected to antipole shear impact by applying the method of integral transforms and dual integral equations. The main objective is to obtain the local dynamic stress field and to investigate the effects of material non-homogeneity and orthotropy on the dynamic stress intensity factor.

1 Material Property Model

Due to the mathematical complexity, some simplifications are necessary for the tractable analysis of non-homogeneous materials under impact loadings. However, because of the difficulty in solving ordinary differential equations, the models have been proposed and extensively used to describe the variation of the shear modulus, such as $\mu(y) = \mu_0 \exp(\gamma y)$ [1~3] and $\mu(y) = \mu_0(1 + c|y|)$ [6], can not be applied to the problems of dynamic response. After deep-going consideration, we find that the application of following model can solve the problem.



$$\mu_x(y) = (\mu_x)_0(1 + \alpha|y|)^2$$
$$\mu_y(y) = (\mu_y)_0(1 + \alpha|y|)^2$$

Fig. 1 A crack in an orthotropic functionallygraded material subjected to antipole shear impact

Consider an unbounded functionally graded material as shown in Fig. 1. The coordinates x and y are assumed as the principal axes of orthotropy. The shear moduli μ_x and μ_y are assumed to be functions of y only, and vary proportionately as

$$\mu_x(y) = (\mu_x)_0(1 + \alpha|y|)^2, \quad (1)$$

$$\mu_y(y) = (\mu_y)_0(1 + \alpha|y|)^2, \quad (2)$$

where α are constant ($\alpha > 0$). $(\mu_x)_0$ and $(\mu_y)_0$ are the shear moduli at $y = 0$.

For present consideration, it is assumed that the mass density of FGM is constant.

2 Formulation of the Problem

As shown in Fig. 1, assume a finite crack of length $2a$ is situated in $y = 0$ plane and subjected to antipole shear impact. Let the components of the displacement in the x, y and z directions be labeled by u_x, u_y and u_z, respectively. For antipole shear motion, u_x and u_y vanish everywhere and u_z is a function of x, y and t, i.e.,

$$u_x = u_y = 0, \quad u_z = w(x, y, t), \quad (3)$$

where t is time. The two nonvanishing stress components τ_{xz} and τ_{yz} are

$$\tau_{xz} = \mu_x \frac{\partial w}{\partial x}, \quad \tau_{yz} = \mu_y \frac{\partial w}{\partial y}, \tag{4}$$

where the shear moduli μ_x, μ_y are assumed to be expressed by Eqs. (1) and (2).

The equation of motion can be written as

$$\frac{\partial^2 w}{\partial x^2} + \frac{\mu_y(y)}{\mu_x(y)} \frac{\partial^2 w}{\partial y^2} + \frac{\mu_y'(y)}{\mu_x(y)} \frac{\partial w}{\partial y} = \frac{\rho}{\mu_x(y)} \frac{\partial^2 w}{\partial t^2}, \tag{5}$$

where $\mu_y'(y)$ is the derivative of $\mu_y(y)$ and ρ is the mass density of the FGM.

Suppose that the material is initially at rest. At time $t = 0$, an antipole shear stress of magnitude τ_0 is suddenly applied to crack surfaces and maintained at the same constant value thereafter. Hence, the boundary conditions are given as follows:

$$\tau_{yz}(x, 0, t) = -\tau_0 \mathrm{H}(t), \qquad 0 \leqslant |x| < a;\ t > 0, \tag{6a}$$

$$w(x, 0, t) = 0, \qquad |x| \geq a;\ t > 0, \tag{6b}$$

where $\mathrm{H}(t)$ is the Heaviside unit step function. The initial conditions are zero.

3 Derivation of Integral Equation

The standard Laplace transform on $f(t)$ is

$$f^*(p) = \int_0^\infty f(t) \mathrm{e}^{-pt} \mathrm{d}t, \tag{7a}$$

whose inversion is

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Br}} f^*(p) \mathrm{e}^{pt} \mathrm{d}p, \tag{7b}$$

where Br denotes the Bromwich path of integration, which is a line on the right-hand side of the P-plane and parallel to the imaginary axis. Applying Eq. (7a) to Eq. (5) yields the transformed equation

$$\frac{\partial^2 w^*}{\partial x^2} + \frac{\mu_y(y)}{\mu_x(y)} \frac{\partial^2 w^*}{\partial y^2} + \frac{\mu_y'(y)}{\mu_x(y)} \frac{\partial w^*}{\partial y} = \frac{\rho p^2}{\mu_x(y)} w^*. \tag{8}$$

Considering the symmetry, it suffices to consider only the first quadrant of the x-y plane Forward, the Fourier cosine transform defined by

$$f^c(s) = \int_0^\infty f(x) \cos(sx) \mathrm{d}x, \tag{9a}$$

$$f(x) = \frac{2}{\pi} \int_0^\infty f^c(s) \cos(sx) \mathrm{d}s, \tag{9b}$$

is applied to the space variable x. Let

$$w^*(x, y, p) = \frac{2}{\pi} \int_0^\infty U(s, y, p) \cos(sx) \mathrm{d}s, \tag{10}$$

then the Eq. (8) can be transformed into

$$\frac{\mu_y(y)}{\mu_x(y)} \frac{\partial^2 U(s, y, p)}{\partial y^2} + \frac{\mu_y'(y)}{\mu_x(y)} \frac{\partial U(s, y, p)}{\partial y} - \left[s^2 + \frac{\rho p^2}{\mu_x(y)} \right] U(s, y, p) = 0. \tag{11}$$

Substituting Eqs. (1) and (2) into Eq. (11), we obtain

$$\frac{\partial^2 U(s, y, p)}{\partial y^2} + \frac{2\alpha}{1 + \alpha y} \frac{\partial U(s, y, p)}{\partial y} - \left[S^2 + \frac{\rho p^2}{(\mu_y)_0 (1 + \alpha y)^2} \right] U(s, y, p) = 0, \tag{12}$$

where $S = s\sqrt{(\mu_x)_0/(\mu_y)_0}$. By defining

$$X = S(1+\alpha y), \quad Y = (1+\alpha y)^{1/2}U. \tag{13}$$

Eq. (12) can be rewritten as

$$\frac{d^2 Y}{d X^2} + \frac{1}{X}\frac{d Y}{d X} - \left[\frac{1}{\alpha^2} + \frac{\beta^2}{X^2}\right] Y = 0, \tag{14}$$

where

$$\beta = \sqrt{\frac{1}{4} + \frac{\rho p^2}{(\mu_y)_0 \alpha^2}}. \tag{15}$$

Eq. (14) is a modified Bessel differential equation. From the solution of Eq. (14) and considering the regularity condition at $y \to \infty$, the solution of Eq. (12) can be expressed as

$$U(s, y, p) = A(s, p)(1+\alpha y)^{-1/2} \mathrm{K}_\beta[(1+\alpha y) S/\alpha], \tag{16}$$

where $\mathrm{K}_\beta(\ )$ is the modified Bessel function of the second kind.

Substituting Eq. (16) into Eq. (10), we obtain

$$w^*(x, y, p) = \frac{2}{\pi}\int_0^\infty A(s, p)(1+\alpha y)^{-1/2} \mathrm{K}_\beta\left[(1+\alpha y)\frac{S}{\alpha}\right]\cos(sx)\, ds. \tag{17}$$

Substituting Eq. (17) into the Laplace transform of the stresses τ_{yz} and τ_{xz} in Eqs. (4), we obtain

$$\tau_{yz}^*(x, y, p) = \mu_y(y)\frac{2}{\pi}\int_0^\infty A(s, p)\left\{-\frac{\alpha}{2}(1+\alpha y)^{-3/2}\mathrm{K}_\beta\left[(1+\alpha y)\frac{S}{\alpha}\right] + S(1+\alpha y)^{-1/2}\mathrm{K}'_\beta\left[(1+\alpha y)\frac{S}{\alpha}\right]\right\}\cos(sx)\, ds, \tag{18a}$$

$$\tau_{xz}^*(x, y, p) = -\mu_x(y)\frac{2}{\pi}\int_0^\infty A(s, p)(1+\alpha y)^{-1/2} \times \mathrm{K}_\beta[(1+\alpha y) S/\alpha]\sin(sx)\, s\, ds. \tag{18b}$$

In the Laplace transform domain, the conditions on the plane $y = 0$ becomes

$$\tau_{yz}^*(x, 0, p) = -\tau_0/p, \qquad 0 \le x < a, \tag{19a}$$

$$w^*(x, 0, p) = 0, \qquad x \ge a. \tag{19b}$$

From Eqs. (17), (18a) and the conditions Eq. (19), a pair of dual integral equations are obtained as

$$\int_0^\infty B(s, p)\cos(sx)\, ds = 0, \qquad x \ge a, \tag{20a}$$

$$\int_0^\infty sB(s, p)\, G(s, p)\cos(sx)\, ds = \frac{\pi\tau_0}{2(\mu_y)_0 p}, \qquad 0 \le x < a, \tag{20b}$$

where

$$B(s, p) = A(s, p)\mathrm{K}_\beta(S/\alpha), \tag{21}$$

$$G(s, p) = \frac{\frac{\alpha}{2}\mathrm{K}_\beta\left(\frac{S}{\alpha}\right) - S\mathrm{K}'_\beta\left(\frac{S}{\alpha}\right)}{s\mathrm{K}_\beta\left(\frac{S}{\alpha}\right)}. \tag{22}$$

The dual integral Eqs. (20) can be solved by applying the method of Copson[7], the solution of Eqs. (20) is found as follows

$$B(s,p) = \frac{\pi\tau_0 a^2}{2(\mu_y)_0 p}\int_0^1 \sqrt{\xi}\Phi^*(\xi,p)J_0(sa\xi)\,d\xi, \tag{23}$$

where J_0 is the zero-order Bessel function of the first kind. The function $\Phi^*(\xi, p)$ is governed by a Fredholm integral equation of the second kind,

$$\Phi^*(\xi,p) + \int_0^1 \Phi^*(\xi,p)M(\xi,\eta,p)\,d\eta = \sqrt{\xi}. \tag{24}$$

The kernel function $M(\xi, \eta, p)$ in Eq. (24) is

$$M(\xi,\eta,p) = \sqrt{\xi\eta}\int_0^\infty s\left[G\left(s/a,p\right) - 1\right]J_0(s\xi)J_0(s\eta)\,ds. \tag{25}$$

The Fredholm integral Eq. (24) can be solved numerically.

4 Dynamic Stress Field Around the Crack Tip

Integrating $B(s, p)$ in Eq. (23) by parts, it gives

$$B(s,p) = \frac{\pi\tau_0 a}{2(\mu_y)_0 p}\frac{1}{s}\left\{\Phi^*(1,p)J_1(sa) - \int_0^1 \xi J_1(sa\xi)\frac{d}{d\xi}\left[\xi^{-1/2}\Phi^*(\xi,p)\right]d\xi\right\}. \tag{26}$$

From Eqs. (26), (21) and Eqs. (18), we obtain

$$\tau^*_{yz}(x,y,p) = \gamma\frac{\tau_0 a\mu_y(y)}{(\mu_y)_0}\frac{\Phi^*(1,p)}{p}\int_0^\infty \frac{(1+\alpha y)^{-1/2}K_\beta\left[(1+\alpha y)S/\alpha\right]}{K_\beta\left(S/\alpha\right)}\times J_1(sa)\cos(sx)\,ds + \cdots, \tag{27a}$$

$$\tau^*_{xz}(x,y,p) = -\frac{\tau_0 a\mu_x(y)}{(\mu_y)_0}\frac{\Phi^*(1,p)}{p}\int_0^\infty \frac{(1+\alpha y)^{-1/2}K_\beta\left[(1+\alpha y)S/\alpha\right]}{K_\beta\left(S/\alpha\right)}\times J_1(sa)\sin(sx)\,ds + \cdots, \tag{27b}$$

where $\gamma = \sqrt{(\mu_x)_0/(\mu_y)_0}$.

Noting that the integrands are finite and continuous for any given values of s, the divergence of the integrals at the crack tips must be due to behavior as $s \to \infty$. Carrying out the expansion for large s and considering the following asymptotic behavior of $K_\beta(x)$ and $K'_\beta(x)$ when $x \to \infty$,

$$K_\beta(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\left[1 + O\left(\frac{1}{x}\right)\right], \tag{28a}$$

$$K'_\beta(x) = -\sqrt{\frac{\pi}{2x}}e^{-x}\left[1 + O\left(\frac{1}{x}\right)\right], \tag{28b}$$

we obtain the lower-order terms of the stresses

$$\tau^*_{yz}(x,y,p) = -\gamma\frac{\tau_0 a\mu_y(y)}{(\mu_y)_0}\frac{\Phi^*(1,p)}{p}(1+\alpha y)^{-1}\int_0^\infty J_1(sa)\exp(-Sy)\cos(sx)\,ds + \cdots = -\gamma\frac{\Phi^*(1,p)}{p}\tau_0 a(1+\alpha y)\int_0^\infty J_1(sa)\exp(-Sy)\cos(sx)\,ds + \cdots, \tag{29a}$$

$$\tau^*_{xz}(x,y,p) = -\frac{\tau_0 a\mu_x(y)}{(\mu_y)_0}\frac{\Phi^*(1,p)}{p}(1+\alpha y)^{-1}\int_0^\infty J_1(sa)\exp(-Sy)\sin(sx)\,ds + \cdots =$$

$$- \frac{(\mu_x)_0}{(\mu_y)_0} \frac{\Phi^*(1, p)}{p} \tau_0 a(1 + \alpha y) \int_0^\infty J_1(sa) \exp(-Sy) \sin(sx) \, ds + \cdots . \quad (29b)$$

Define complex variable $z_0 = x + i\gamma y$, we obtain

$$\int_0^\infty J_1(sa) \exp(i z_0 s) \, ds = -\frac{1}{\sqrt{2 r_1 a}} \frac{1}{\sqrt{\cos\theta_1 + i\gamma\sin\theta_1}} + O(r_1^0). \quad (30)$$

The polar coordinates r_1 and θ_1 are defined in Fig. 1.

Note that the integrals in Eqs. (29a) and (29b) are

$$\int_0^\infty J_1(sa) \exp(-Sy) \cos(sx) \, ds = \mathrm{Re}\left[\int_0^\infty J_1(sa) \exp(i z_0 s) \, ds\right], \quad (31a)$$

$$\int_0^\infty J_1(sa) \exp(-Sy) \sin(sx) \, ds = \mathrm{Im}\left[\int_0^\infty J_1(sa) \exp(i z_0 s) \, ds\right]. \quad (31b)$$

Then we obtain the local stress field

$$\tau_{yz}^*(r_1, \theta_1, p) = \frac{K_{\mathrm{III}}^*(p)}{\sqrt{2\pi r_1}} \mathrm{Re}\left[\frac{1}{\sqrt{\cos\theta_1 + i\gamma\sin\theta_1}}\right] + O(r_1^0), \quad (32a)$$

$$\tau_{xz}^*(r_1, \theta_1, p) = -\frac{K_{\mathrm{III}}^*(p)}{\sqrt{2\pi r_1}} \mathrm{Re}\left[\frac{i\gamma}{\sqrt{\cos\theta_1 + i\gamma\sin\theta_1}}\right] + O(r_1^0). \quad (32b)$$

The Laplace transform of the dynamic stress intensity factor $K_{\mathrm{III}}^*(p)$ in Eqs. (32) is

$$K_{\mathrm{III}}^*(p) = \sqrt{\frac{(\mu_x)_0}{(\mu_y)_0}} \tau_0 \sqrt{\pi a} \frac{\Phi^*(1, p)}{p}, \quad (33)$$

in which $\Phi^*(1, p)$ is the value of $\Phi^*(\xi, p)$ evaluated at the crack tip corresponding to $\xi = 1$.

The dynamic stress intensity factor in time domain can be obtained by

$$K_{\mathrm{III}}^*(t) = \sqrt{\frac{(\mu_x)_0}{(\mu_y)_0}} \tau_0 \sqrt{\pi a} \frac{1}{2\pi i} \int_{Br} \frac{\Phi^*(1, p)}{p} e^{pt} \, dp. \quad (34)$$

5 Results and Discussion

The functional dependence of the stresses on r_1 and θ_1 as shown in Eqs. (32) reveals that the local dynamic stresses near the crack tip in orthotropic functionally graded materials also possess the inverse square root singularity in terms of r_1 and that the angular distribution in θ_1 is the same as the case in orthotropic homogeneous materials[8]. Eq. (34) shows that the expressional form of the dynamic stress intensity factor for orthotropic functionally graded materials is different from that for homogeneous materials. A coefficient $\sqrt{(\mu_x)_0/(\mu_y)_0}$ is multiplied in this case for orthotropic FGM. Thus, it is clear that the influence of the material orthotropy is significant.

By using the numerical Laplace transform inversion scheme described by Miller and Guy[9], the dynamic stress intensity factor expressed by Eq. (33) can be evaluated. The influences of non-homogeneity parameter αa and orthotropic parameter γ on the normalized dynamic stress intensity factor $K_{\mathrm{III}}(t)/\tau_0\sqrt{\pi a}$ are shown in Fig. 2, where $(c_y)_{20} = \sqrt{(\mu_y)_0/\rho}$. It is observed that all the curves reach a peak and then oscillating about the static values with decreasing magnification. For definite values of γ, the values of the dynamic stress intensity factor are less for larger values of αa. For definite values of αa, the values of the dynamic stress intensity factor are larger for larger values of γ. This means that the increase of the shear moduli's gradients in FGM can always reduce the magnitude of dynamic stress intensity factor no matter how the orthotropy is . The increase of the shear modulus in direction perpendicular to crack surface is beneficial to reducing the dynamic stress intensity factor in orthotropic FGM.

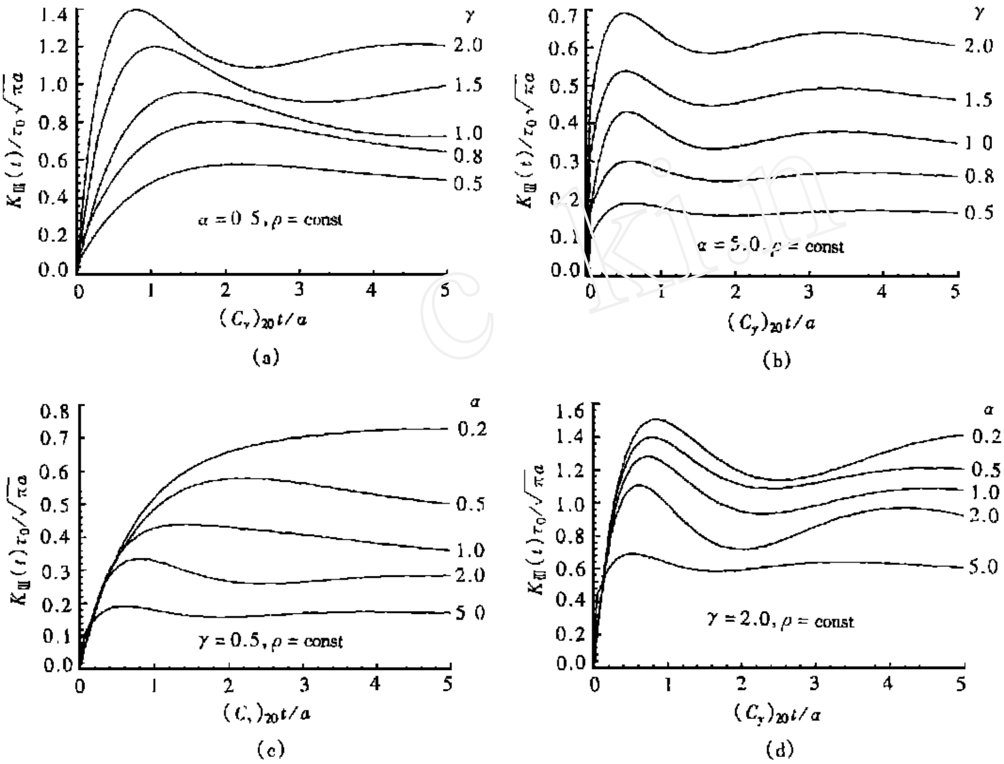


Fig. 2 The variations of $K_{III}(t)$ at different non-homogeneity and orthotropy

6 Conclusion

In this paper, an orthotropic FGM with a finite crack under antipole shear impact is studied. The theoretical analysis show that local stress field around the crack tip in an orthotropic FGM is found to be similar to that in an orthotropic homogeneous material. The dynamic stress intensity factor obtained in time domain show that the non-homogeneity and orthotropy of FGM has a considerable influence on the fracture behavior of anisotropic FGM. The peak values of the dynamic stress intensity factor decrease with the increasing shear moduli's gradient of FGM. Increasing the shear modulus in direction perpendicular to crack surface can also restrain the magnitude of dynamic stress intensity factor.

References :

[1] Erdogan F. The crack problem for bonded non-homogeneous materials under antipole shear loading [J]. ASME Journal of Applied Mechanics, 1985, **52**(4): 823～828.

[2] Konda N, Erdogan F. The mixed mode crack problem in a non-homogeneous elastic medium [J]. Engineering Fracture Mechanics, 1994, **47**(4): 533～545.

[3] Erdogan F. Fracture mechanics of functionally graded materials [J]. Composites Engineering, 1995, **5**(7): 753～770.

[4] Wang Baolin, Han Jiecai, Du Shanyi. Dynamic fracture mechanics analysis for composite material with material non-homogeneity in thickness direction [J]. Acta Mechanica Solida Sinica, 1998, **11**(1): 84～93.

[5] Ozturk M, Erdogan F. Mode Ⅰ crack problem in an inhomogeneous orthotropic medium [J]. Int J Eng Sci, 1997, **35**(9): 869～883.

[6] Gerasoulis A, Srivastav R P. A Griffith crack problem for a non-homogeneous medium [J]. Int J Eng Sci, 1980, **18**(2): 239～247.

[7] Copson E T. On certain dual integral equations [A]. In: Proceedings Glasgow Mathematical Association, 1961, **5**: 19～24.

[8] Sih G C, Chen E P. Mechanics of Fracture 6: Cracks in composite materials [M]. The Hague: Martinus Nijhoff Publishers, 1981.

[9] Miller M K, Guy W T. Numerical inversion of the Laplace transform by use of Jacobi polynomials [J]. SIAM Journal of Numerical Analysis, 1966, **3**(6): 624～635.