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SIMPLE LATTICE BOLTZMANN MODEL FOR TRAFFIC FLOWS

Yan Guangwu (闫广武)[+,++] Hu Shouxin (胡守信)[++]
[+](*State Key Laboratory of Nonlinear Mechanics, Institute of Mechanics, Chinese Academy of Sciences, Beijing* 100080, *China*)
[++](*Section of Mechanics, Department of Mathematics, Jilin University, Changchun* 130023, *China*)

ABSTRACT: A lattice Boltzmann model with 5-bit lattice for traffic flows is proposed. Using the Chapman-Enskog expansion and multi-scale technique, we obtain the higher-order moments of equilibrium distribution function. A simple traffic light problem is simulated by using the present lattice Boltzmann model, and the result agrees well with analytical solution.

KEY WORDS: lattice Boltzmann method, traffic flows, higher-order moment method

1 INTRODUCTION

In recent years, the lattice Boltzmann method (LBM) has attracted attention as an alternative numerical scheme for simulation of complex fluid flows for the reason of simpler scheme and robust stability[1~3]. In this paper, we propose a lattice Boltzmann model for traffic flows, which can be used to analyze and comprehend the traffic flows from the viewpoint of mesoscopic physics, and give a new numerical technique. The main idea of the lattice Boltzmann method is to construct a simplified kinetic model that incorporates the essential physics of microscopic or mesoscopic processes and the macroscopice variables, and obeys the desired macroscopic equations[2]. The next step is to use the Chapman-Enskog expansion and multi-scale method to get the equilibrium distribution functions. In this letter, we solve two problems: (1) the structure of traffic flux-density, (2) the second- order accuracy of the truncation errors. As the same as in Ref.[2], the conservation law is used in time scale t_0 to get the equilibrium distribution functions.

It is known that the lattice gas cellular automaton (LGCA) can simulate some complex flows. Now, there are some studies on traffic flows[9~12], which presented good results, but there are two pitfalls in practice: (1) we need an exact solution of equilibrium distribution functions to determine the truncation errors and stability; (2) the problem about the ensemble of particles is related to a large amount of the "situations" with same macroscopic states, which may increase the memory or CPU time. If the lattice Boltzmann model is used, these two shortcomings can be overcome. In the present letter, we simulate a simple example. The result agrees well with analytical solution.

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2 LATTICE BOLTZMANN MODEL

Consider a one-dimensional long string of traffic flows. We assume the flow as a continuous flow, and divide it into some nodes. There are five types of cars with velocity $e_\alpha = \{0, c, 2c, 3c, 4c\}$, thus $f_\alpha(x,t)$ is the distribution function of traffic density at time t, node x with type e_α. The definition of the traffic density ρ (e.g. the number of cars) is

$$\rho = \sum_\alpha f_\alpha$$

The distribution function satisfies the lattice Boltzmann equation[3~5]

$$f_\alpha(x+e_\alpha, t+1) - f_\alpha(x,t) = -\frac{1}{\tau}[f_\alpha(x,t) - f_\alpha^{eq}(x,t)] \tag{1}$$

where τ is the single relaxation time factor and $f_\alpha^{eq}(x,t)$ is the equilibrium distribution function at time t, node x with velocity e_α. The conservation condition is

$$\sum_\alpha f_\alpha^{eq} = \sum_\alpha f_\alpha \tag{2}$$

Using a small parameter ε as the time unit in physical unit, which can play the role of the Knudsen number[2], the lattice Boltzmann equation (1) in physical unit is

$$f_\alpha(x+\varepsilon e_\alpha, t+\varepsilon) - f_\alpha(x,t) = -\frac{1}{\tau}[f_\alpha - f_\alpha^{eq}] \tag{3}$$

We apply the Taylor expansion and Chapman-Enskog expansion[6] to Eq.(3), retaining terms up to ε^5, which is applied to f_α under the assumption that the mean free path is of the same order of ε. To discuss changes in different time scales introduced as t_0, t_1, $\cdots$, t_4

$$t_k = \varepsilon^k t$$

thus, we get a series of lattice Boltzmann equations in different time scales

$$\frac{\partial f_\alpha^{(eq)}}{\partial t_0} + e_\alpha \frac{\partial f_\alpha^{(eq)}}{\partial x} = -\frac{1}{\tau} f_\alpha^{(1)} \tag{4}$$

$$\frac{\partial f_\alpha^{(eq)}}{\partial t_1} - \tau\Big(1-\frac{1}{2\tau}\Big)\Big(\frac{\partial}{\partial t_0} + e_\alpha\frac{\partial}{\partial x}\Big)^2 f_\alpha^{(eq)} = -\frac{1}{\tau} f_\alpha^{(2)} \tag{5}$$

$$\frac{\partial f_\alpha^{(eq)}}{\partial t_2} + (1-2\tau)\Big(\frac{\partial}{\partial t_0} + e_\alpha\frac{\partial}{\partial x}\Big)\frac{\partial f_\alpha^{(eq)}}{\partial t_1} + \Big(\tau^2 - \tau + \frac{1}{6}\Big)\Big(\frac{\partial}{\partial t_0} + e_\alpha\frac{\partial}{\partial x}\Big)^3 f_\alpha^{(eq)} = -\frac{1}{\tau} f_\alpha^{(3)} \tag{6}$$

$$\frac{\partial f_\alpha^{(eq)}}{\partial t_3} + \Big(2\tau^2 - \frac{5}{2}\tau + \frac{1}{2}\Big)\frac{\partial}{\partial t_1}\Big(\frac{\partial}{\partial t_0} + e_\alpha\frac{\partial}{\partial x}\Big)^2 f_\alpha^{(eq)} + (1-2\tau)\frac{\partial}{\partial t_2}\Big(\frac{\partial}{\partial t_0} + e_\alpha\frac{\partial}{\partial x}\Big) f_\alpha^{(eq)} +$$
$$\Big(-\tau^3 + \frac{3}{2}\tau^2 - \frac{7}{12}\tau + \frac{1}{24}\Big)\Big(\frac{\partial}{\partial t_0} + e_\alpha\frac{\partial}{\partial x}\Big)^4 f_\alpha^{(eq)} + \frac{\partial f_\alpha^{(2)}}{\partial t_1} + \frac{1}{2}\frac{\partial^2 f_\alpha^{(2)}}{\partial t_1^2} = -\frac{1}{\tau} f_\alpha^{(4)} \tag{7}$$

Taking the summation in Eqs.(4)~(7) about α, we obtain

$$\frac{\partial\rho}{\partial t_0}+\frac{\partial\sum\limits_\alpha f_\alpha^{(eq)}e_\alpha}{\partial x}=0 \tag{8}$$

$$\frac{\partial\rho}{\partial t_1}+\Big(\frac{1}{2}-\tau\Big)\sum_\alpha\Big(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\Big)^2 f_\alpha^{(eq)}=0 \tag{9}$$

$$\frac{\partial\rho}{\partial t_2}+\Big(\tau^2-\tau+\frac{1}{6}\Big)\sum_\alpha\Big(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\Big)^3 f_\alpha^{(eq)}=0 \tag{10}$$

$$\begin{aligned}&\frac{\partial\rho}{\partial t_3}+(1-2\tau)\sum_\alpha\frac{\partial}{\partial t_2}\Big(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\Big)f_\alpha^{(eq)}+\Big(1-\frac{1}{2\tau}\Big)\sum_\alpha\frac{\partial f_\alpha^{(2)}}{\partial t_1}+\\&\qquad\Big(2\tau^2-2\tau+\frac{1}{4}\Big)\sum_\alpha\frac{\partial}{\partial t_1}\Big(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\Big)^2 f_\alpha^{(eq)}+\\&\qquad\Big(-\tau^3+\frac{3}{2}\tau^2-\frac{7}{12}\tau+\frac{1}{24}\Big)\sum_\alpha\Big(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\Big)^4 f_\alpha^{(eq)}=0\end{aligned} \tag{11}$$

The rearrangement of (8)+(9)$\times\varepsilon$ +(10)$\times\varepsilon^2$ +(11)$\times\varepsilon^3$ results in

$$\begin{aligned}&\frac{\partial\rho}{\partial t}+\frac{\partial\sum\limits_\alpha f_\alpha^{(eq)}e_\alpha}{\partial x}+\varepsilon\Big(\frac{1}{2}-\tau\Big)\sum_\alpha\Big(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\Big)^2 f_\alpha^{(eq)}+\\&\varepsilon^2\Big(\tau^2-\tau+\frac{1}{2}\Big)\sum_\alpha\Big(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\Big)^3 f_\alpha^{(eq)}+\\&\varepsilon^3\Big[\frac{\partial\rho}{\partial t_2}+(1-2\tau)\sum_\alpha\frac{\partial}{\partial t_2}\Big(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\Big)f_\alpha^{(eq)}+\\&\Big(1-\frac{1}{2\tau}\Big)\sum_\alpha\frac{\partial f_\alpha^{(2)}}{\partial t_1}+\Big(2\tau^2-2\tau+\frac{1}{4}\Big)\sum_\alpha\frac{\partial}{\partial t_1}\Big(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\Big)^2 f_\alpha^{(eq)}+\\&\Big(-\tau^3+\frac{3}{2}\tau^2-\frac{7}{12}\tau+\frac{1}{24}\Big)\sum_\alpha\Big(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x}\Big)^4 f_\alpha^{(eq)}\Big]=O(\varepsilon^4)\end{aligned} \tag{12}$$

Denote

$$q^0=\sum_\alpha f_\alpha^{(eq)}e_\alpha \qquad \pi^0=\int\Big(\frac{\partial q^0}{\partial\rho}\Big)^2\mathrm{d}\rho \tag{13}$$

$$P^0=\int\frac{\partial\pi^0}{\partial\rho}\frac{\partial q^0}{\partial\rho}\mathrm{d}\rho \qquad L^0=\int\frac{\partial P^0}{\partial\rho}\frac{\partial q^0}{\partial\rho}\mathrm{d}\rho+\lambda\rho \tag{14}$$

By simpler algebra, we have

$$\frac{\partial\rho}{\partial t}+\frac{\partial q^0}{\partial x}=R+O(\varepsilon^4) \tag{15}$$

where

$$R=\varepsilon^3\lambda\Big(\tau^3-\frac{3}{2}\tau^2+\frac{7}{12}\tau-\frac{1}{24}\Big)\frac{\partial^4\rho}{\partial x^4}+O(\varepsilon^4) \tag{16}$$

The equilibrium distribution functions are

$$f_1^{(eq)} = \frac{-26c^2\pi^0 - L^0 + 9cP^0 + 24c^3q^0}{6c^4} \qquad f_2^{(eq)} = \frac{19c^2\pi^0 + L^0 - 8cP^0 - 12c^3q^0}{4c^4} \tag{17}$$

$$f_3^{(eq)} = \frac{-14c^2\pi^0 - L + 7cP^0 + 8c^3q^0}{6c^4} \qquad f_4^{(eq)} = \frac{11c^2\pi^0 + L^0 - 6cP^0 - 6c^3q^0}{24c^4} \tag{18}$$

$$f_0^{(eq)} = \rho - (f_1^{(eq)} + f_2^{(eq)} + f_3^{(eq)} + f_4^{(eq)}) \tag{19}$$

3 NUMERICAL TEST

A very long string of traffic is backed up at a traffic light (see Fig.1). Find the resulting motion when the light finally turns green. The density for $x < 0$ is ρ_0 and for $x > 0$ is zero, with an initial discontinuity at the origin[7]. We suppose the flow-density curve is parabolic, and $q(\rho)$ can be represented by the equation

$$q = q_{\max} - \rho_0 c_0 \left(\frac{\rho}{\rho_0} - \frac{1}{2}\right)^2$$

where, $q_{\max} = \rho_0 c_0/4$, ρ_0 is the jammed density. Thus, higher moments of equilibrium distribution function are

$$\pi^0 = -\frac{4}{3}c_0^2\rho_0\left(\frac{\rho}{\rho_0} - \frac{1}{2}\right)^3 \tag{20}$$

$$P^0 = -2c_0^3\rho_0\left(\frac{\rho}{\rho_0} - \frac{1}{2}\right)^4 \tag{21}$$

$$L^0 = -\frac{16}{5}c_0^4\rho_0\left(\frac{\rho}{\rho_0} - \frac{1}{2}\right)^5 + \lambda\rho \tag{22}$$

The motion is a centered rarefaction example, and we find the theoretical result is

$$\frac{\rho}{\rho_0} = \frac{1}{2}\left(1 - \frac{x}{c_0 t}\right) \tag{23}$$

The error in the left hand and right hand is near zero, and the result is satisfactory.

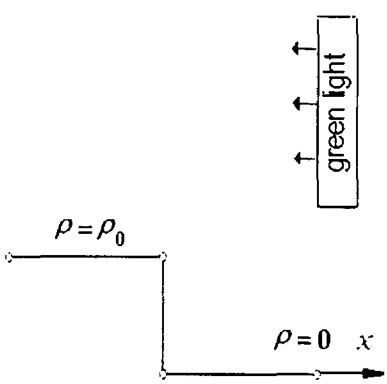

Fig.1 Density distribution in initial time

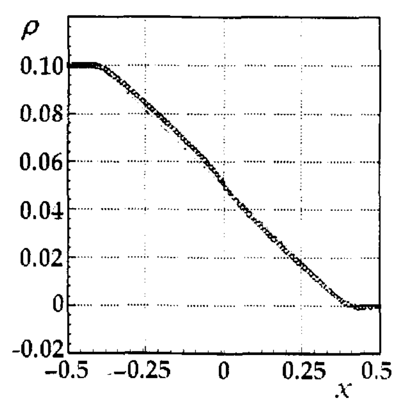

Fig.2 Density distribution

Figure 2 the comparisons between numerical and theoretical results, Exact solution (line), numerical results (circle). The parameters: $\rho_0 = 0.1$, $c_0 = 0.1$, $\lambda = -1.0$, $c = 3.0$, $1/\tau = 1.3$, Lattice size:100, time steps: $1200\Delta t$

4 DISCUSSIONS

In this paper, we propose a simple lattice Boltzmann model for traffic flows. It is easy to construct other flux-density curve by using higher moments method (e.g. Greenberg's curve[8]). We do not consider any complex situation such as higher effect in response time, the flux equations, etc. The result is good enough, but, in view of the Lighthill-Whitham model with simpler flux-density curve, the result may be true compared with observing car motions. It is important that flux-density cuvre may be related to many things, and our results can not be regarded as real traffic flow, but we have only tried to treat traffic problems. It is suitable that the lattice Boltzmann method be used to traffic flows, because there are new physical phenomena which are not yet easily described by macroscopic equations.

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