Stress Wave Propagation in a Gradient Elastic Medium *

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The gradient elastic constitutive equation incorporating the second gradient of the strains is used to determine the monochromatic elastic plane wave propagation in a gradient infinite medium and thin rod. The equation of motion, together with the internal material length, has been derived. Various dispersion relations have been determined. We present explicit expressions for the relationship between various wave speeds, wavenumber and internal material length.

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There is abundant and increasing evidence that microstructures of materials have a major influence on wave propagation if the wavelength is of the same order as the characteristic size of the microstructure. This is of importance in applications of microelectromechanical systems (MEMS) for the detection of material internal defects or other related purposes. Suhubi and Eringen[1] have studied the Rayleigh waves in micropolar theory, in which the microrotations were taken as free variables in their analysis. Ottosen *et al*[2] investigated the propagation of Rayleigh waves in an elastic medium described by the indeterminate couple-stress theory.

To predict the scale effect, the constitutive equations must include some material intrinsic length scales. There are several means to introduce a length scale into constitutive equations, i.e., non-local theory, couple-stress theory, plastic strain gradient theory, and addition of higher-order gradients. Mindlin[3] proposed a linear theory for the description of solid deformation, in which the density of strain energy was the function of strain as well as of its first and second gradients. Considering the second gradient of strain, Mindlin claimed the incorporation of both cohesive forces and surface tension into the linear elasticity. The corresponding modification of Hooke's law reads

$$\underline{\underline{\sigma}} = \lambda\left(\underline{\underline{I}} - c_1\underline{\underline{I}}\nabla^2 - c_2\nabla\nabla\right)\left(\mathrm{tr}\underline{\underline{\varepsilon}}\right) + 2\mu\left(1 - c_3\nabla^2\right)\underline{\underline{\varepsilon}}. \tag{1}$$

Here c_1, c_2 and c_3 are the three independent gradient coefficients having the dimension of length squared; λ and μ are the Lamé constants; $\underline{\underline{\sigma}}$ and $\underline{\underline{\varepsilon}}$ are the elastic stress and strain tensors, respectively; $\underline{\underline{I}}$ is the unit tensor; and ∇^2 is the Laplacian operator. The tensor order is here denoted by the number of underbars. By taking $c_1 = c_2 = c_3 = 0$, the standard Hooke law is retrieved. In the special case of $c_2 = 0$, $c_1 = c_3 = \bar{l}^2$, using Poisson's ratio ν and $\lambda = \mu\frac{2\nu}{1-2\nu}$, Eq. (1) reduces to

$$\begin{aligned}\sigma_{ij} = 2\mu&\left(\varepsilon_{ij} + \tfrac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right)\\ &-\bar{l}^2\nabla^2\left[2\mu\left(\varepsilon_{ij} + \tfrac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right)\right],\end{aligned} \tag{2}$$

where δ_{ij} is the Kronecker delta, and $\bar{l}$ is the internal material length that may be deduced from the resulting wave dispersion equation as compared with a corresponding dispersion relation of lattice dynamics.[4] As a matter of fact, Eq. (2) was proposed by Aifantis *et al.*[5,6] to eliminate the strain singularity at the crack tip. They have also shown that, for a crystalline lattice, the intrinsic material length is taken to be of the order of the interatomic distance a, specifically $\bar{l} \cong a/4$. Other estimates for $\bar{l}$ are possibly dependent on the lattice or atomic chain model used and the interatomic potentials assumed. This model has been successfully used to predict the scale-dependent phenomena of dislocation, fracture, interfacial mechanics and failure of solids.

Substituting the strain tensor $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ into Eq. (2) and then into the equation of motion, $\sigma_{ij,j} = \rho\ddot{u}_i$ yields

$$\begin{aligned}&\mu\left(u_{i,jj} + \frac{1}{1-2\nu}u_{j,ji}\right) - \mu\bar{l}^2\left(u_{i,jjll} + \frac{1}{1-2\nu}u_{j,jill}\right)\\ &= \rho\ddot{u}_i,\end{aligned} \tag{3}$$

where u_i is the displacement component, ρ is the mass density, and the superimposed dot denotes the time derivative. For a longitudinal wave, Eq. (3) is reduced to

$$c_L^2\left(u_{i,ii} - \bar{l}^2u_{i,iiii}\right) = \ddot{u}_i, \tag{4}$$

where

$$c_L = \sqrt{\frac{2\mu}{\rho}\frac{1-\nu}{1-2\nu}}$$

is the longitudinal plane wave speed in gradient-free isotropic infinite elastic medium. For the transverse wave, however, Eq. (3) reduces to

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$$c_T^2\left(u_{i,jj}-\bar{l}^2u_{i,jjjj}\right)=\ddot{u}_i, \tag{5}$$

where $c_T=\sqrt{\mu/\rho}$ is the transverse wave speed in the gradient-free isotropic infinite elastic medium.

Here we consider a plane monochromatic longitudinal elastic wave propagating in an infinite medium, and we assume the following displacement vector[7]

$$u_i=\mathrm{Re}\left[a_i\mathrm{e}^{\,\mathrm{i}q(n_sx_s\pm\bar{c}_Lt)}\right], \tag{6}$$

where Re denotes the real part, a_i is the constant vector representing the direction of u_i, n_s is the unit vector normal to the wave front, x_s is the position vector, q is the wavenumber, and $\bar{c}_L$ is the phase speed of the longitudinal wave in the gradient elastic medium. The relation between the position vector x_s, the wavenumber q and the wave speed $\bar{c}_L$ makes this function actually satisfy Eq. (4). Substituting Eq. (6) into Eq. (4) and noticing $n_in_i=1$, we obtain

$$\bar{c}_L^2=c_L^2\left(1+\bar{l}^2q^2\right). \tag{7}$$

For the plane monochromatic transverse wave propagation in the elastic infinite medium, we seek solutions in the form

$$u_i=\mathrm{Re}\left[a_i\mathrm{e}^{\,\mathrm{i}q(n_sx_s\pm\bar{c}_Tt)}\right], \tag{8}$$

where $\bar{c}_T$ is the phase speed of the transverse wave in the gradient elastic medium. The relation between the position vector x_s, the wavenumber q and the wave speed $\bar{c}_T$ makes this function actually satisfy Eq. (5). Substituting Eq. (8) into Eq. (5), we have

$$\bar{c}_T^2=c_T^2\left(1+\bar{l}^2q^2\right). \tag{9}$$

Longitudinal waves in a thin rod (uniform over any cross section) are simply extensively or compressively propagated along its length, thus it is a one-dimensional problem. The gradient constitutive relation of Eq. (2) reduces to

$$\sigma_{xx}=E\left(\varepsilon_{xx}-\bar{l}^2\varepsilon_{xx,xx}\right), \tag{10}$$

for one-dimensional problem, and $\varepsilon_{xx}=u_{x,x}$, with u_x being the displacement along the x-axis. Substituting Eq. (10) into the equation of motion, $\rho\ddot{u}_x=\sigma_{xx,x}$, we obtain the equation of motion in terms of displacement u_x and the material intrinsic length, i.e.

$$\ddot{u}_{tt}=c_1^2\left(u_{x,xx}-\bar{l}^2u_{x,xxxx}\right), \tag{11}$$

where $c_1=\sqrt{E/\rho}$ is the one-dimensional elastic wave speed.

We consider the propagation of monochromatic plane waves in the thin rod

$$u_x=u_{x0}\mathrm{e}^{\,\mathrm{i}q(x-\bar{c}_1t)}, \tag{12}$$

where u_{x0} is some constant, q is the wavenumber again, and $\bar{c}_1$ is the corresponding phase velocity. Substituting Eq. (12) into Eq. (11) we obtain

$$\left(\frac{\bar{c}_1}{c_1}\right)^2=\left(1+\bar{l}^2q^2\right), \tag{13}$$

which is equivalent to

$$\bar{c}_1^2=c_1^2\left[1+\left(\frac{2\pi\bar{l}}{\lambda}\right)^2\right], \tag{14}$$

where λ is the wavelength of the longitudinal wave.

The simplification of $c_2=0$ and $c_1=c_3=\bar{l}^2$ in Eq. (1) results in a stable material model, while $c_1=c_3=-\bar{l}^2$ can lead to a destabilized model.[8]

It is obvious that the wave speeds in the Laplacian gradient elastic medium are dispersive and always larger than the conventional wave speeds. We also have the following relationship

$$\frac{\bar{c}_L}{\bar{c}_T}=\frac{c_L}{c_T}=\sqrt{2\frac{1-\nu}{1-2\nu}}. \tag{15}$$

It is seen from Eq. (15) that the velocity of the plane longitudinal waves is always greater than that of plane transverse waves in the Laplacian gradient infinite medium, as shown in Fig. 1. We always have $\bar{c}_L/\bar{c}_T>\sqrt{2}$. For comparison, Ottosen *et al.*[2] studied the wave propagation in a couple-stress elastic medium and found that, for plane wave in an infinite medium, the longitudinal wave speed is the same as the conventional theory, i.e. $\bar{c}_L=c_L$, and exhibits no dispersion. It is interesting that the transverse wave speed is also given by Eq. (14) for a couple-stress elastic medium. As a result, for couple-stress theory when $\bar{l}^2q^2>\frac{1}{1-2\nu}$, the transverse waves move faster than the longitudinal waves. Moreover, diffuse wave motion may occur when $\bar{l}^2q^2=\frac{1}{1-2\nu}$, as illustrated in Fig. 2.

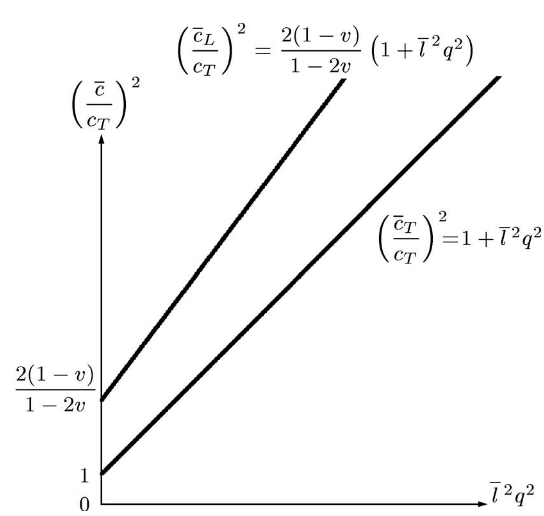

Fig. 1. Plane wave speeds in a Laplacian gradient elastic medium.

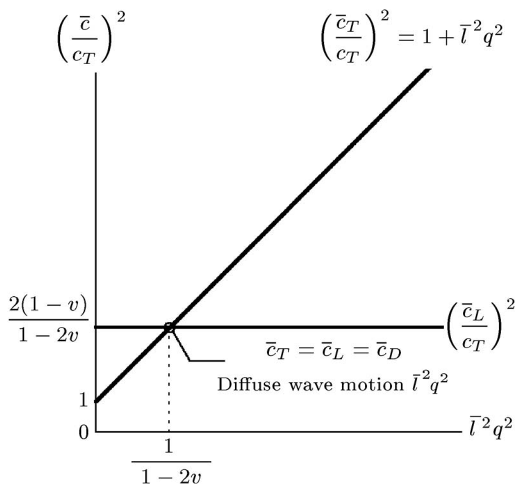

Fig. 2. Plane wave speeds in a couple-stress elastic medium.

From Eqs. (7), (9) and (13) we know that the gradient effect can be neglected if the wavelength satisfies $\lambda \gg 2\pi\bar{l}$. For a crystalline lattice, if $\lambda \gg 1.57a$, then the gradient effect can be negligible.

In conclusion, the gradient elastic constitutive equation incorporating the second gradient of the strains is used to determine the monochromatic elastic plane wave propagation in a gradient infinite medium and in a thin rod. Analytical results show that consideration of the stable gradient coefficient (intrinsic material length squared) enhances the phase speeds of various waves. In the gradient elastic infinite medium, the longitudinal wave speed is always greater than that of the transverse wave speed. This differs from the results predicted by the so-called indeterminate couple-stress theory. For crystalline lattices, the gradient effect should be considered when the wavelength is of the same order of or less than the lattice size, and this effect can be neglected when the wavelength is much greater than the lattice size.

It is noted that the study of wave propagation in the gradient medium is important for MEMS and other applications.[9] In order to have a better understanding of material failure at microscale, further work on elastic–plastic wave propagation in the gradient medium is to be done.

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