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NUMERICAL SIMULATION OF STANDING SOLITONS AND THEIR INTERACTION*

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(Paper from ZHOU Xian-chu, Member of Editorial Committee, AMM)

Abstract: Standing soliton was studied by numerical simulation of its governing equation, a cubic Schrödiger equation with a complex conjugate term, which was derived by Miles and was accepted. The value of linear damping in Miles equation was studied. Calculations showed that linear damping effects strongly on the formation of a standing soliton and Laedke and Spatschek stable condition is only a necessary condition, but not a sufficient one. The interaction of two standing solitons was simulated. Simulations showed that the interaction pattern depends on system parameters. Calculations for the different initial condition and its development indicated that a stable standing soliton can be formed only for proper initial disturbance, otherwise the disturbance will disappear or develop into several solitons.

Key words: soliton; standing soliton; cubic Schrödinger equation; numerical simulation

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Introduction

The standing soliton was discovered in 1984 by Dr. WU[1], a Chinese visiting scholar in U.S.A. from Nanjing University. He poured the water into a narrow and long rectangular channel, then put the channel on a loudspeaker and vibrated it vertically or horizontally. Controlling the vibration amplitude, a kind of non-propagating solitary water wave occurred in the channel when the frequency is twice the character frequency of the fluid.

Since the first report of standing soliton by Wu, many theoretical and experimental studies have been done to explain and explore the special soliton. Larraza and Putterman[2] derived a cubic Schrödinger equation from the governing equation of water wave by the multiple scales and got a standing soliton solution in 1984. At the same time Miles[3] made a more perfect theoretical analysis on the standing soliton by variation method. He got a cubic Schrödinger equation with a complex conjugate term, which was satisfied by a complex function r proportion to the amplitude of standing soliton

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Biography: ZHOU Xian-chu (1939～), Professor

$$i(r_\tau + \alpha r) + Br_{XX} + (\beta + A|r|^2)r + \gamma r^* = 0, \quad (1)$$

where r^* is the complex conjugate of r, α linear damping, A, B the parameters related to the geometry character of the system, β and γ are related to the amplitude and frequency of the vibration. Miles Eq. (1) is the governing equation accepted by most people up to now and describes the physical model of a standing soliton better. He considered the forcing vibration of the system and introduced a conjugate term γr^* to represent vibrating excitation. This means that the energy in the system increases by external drive continuously. He also introduced a dissipation term $i\alpha r$ to balance vibration exciting, then a stable solution can be obtained. Actually dissipation will exist in a real physical system.

Laedke et al.[4] considered the linear stability of a small disturbance solution for Miles' Eq. (1), analyzed the stability of a soliton solution in each parameter region, and made a check numerically for special case. They found that a stable soliton can be obtained in some parameters, but in many cases, a soliton solution of the equation will develop into other waves but not the soliton.

There are many research works on the standing soliton in China. Prof. WEI in the Acoustic Institute of Nanjing University mainly studied the theory and experiment of the standing soliton. They researched the chaos of the soliton height[5] and the standing soliton in two-layer fluid[6] and granular material[7]. Prof. CUI[8,9] made serious experiments on the standing soliton. Prof. ZHOU[10,11] considered the effect of the surface tension on the standing soliton carefully. Prof. YAN studied the standing soliton in two-layer fluid by multiple scales[12] and analyzed the interaction of two standing solitons[13].

To sum up, the researches in the standing soliton mainly focused on the qualitative explanation of the experimental study and theory. As I know, there is no reference for numerical simulation. In the theory, there is no satisfactory explanation for the interaction between two standing solitons. In the numerical simulation, the numerical solution of a nonlinear cubic Schrödinger equation with a complex conjugate term is very difficult. ZHOU X. et al.[14] have made a numerical calculation for it. Using the physical parameters that a stable standing soliton could occur in experiments, letting linear damping $\alpha = 0$ which can not be determined in the theory, and letting a soliton solution be an initial condition, they found that the solution of the equation is unstable. So, to make numerical simulation of a standing solution is necessary.

This article started from Miles' theory and solved Miles' nonlinear cubic Schrödinger Eq. (1) numerically to simulate a standing soliton. The value of dissipation coefficient α is studied to calculate a stable standing soliton. Three patterns for the interaction between two standing solitons are simulated. The relationship between the initial disturbance and the standing soliton is calculated and analyzed.

1 Establishment of Numerical Model

As mentioned above, we used Miles' theory as a physical model, solved Eq. (1) and simulated the standing soliton numerically. The initial condition and the boundary conditions are as follows

$$r(X,0) = r_0(X), \quad (2)$$

$$r_X(-l,\tau) = 0, \quad (3)$$

$$r_X(l,\tau) = 0. \quad (4)$$

We adopted the following idea to solve Eq. (1) : finite difference was used for the differentiation with respect to space variable and the integral was used for the time variable. This means that we reduced original partial differential equation into an ordinary differential equation of time variable, the differentiation with respect to space variable was replaced by scattered difference. A complex equation is reduced into two real equations. The final equations to be solved are

$$\frac{\mathrm{d}p_j}{\mathrm{d}\tau} = (\gamma - \beta)q_j - \alpha p_j - B\frac{q_{j+1} + q_{j-1} - 2q_j}{\Delta x^2} - Aq_j(p_j^2 + q_j^2), \tag{5}$$

$$\frac{\mathrm{d}q_j}{\mathrm{d}\tau} = (\gamma + \beta)p_j - \alpha q_j + B\frac{p_{j+1} + p_{j-1} - 2p_j}{\Delta x^2} + Ap_j(p_j^2 + q_j^2),$$

$$j = 1, 2, 3, \cdots, N. \tag{6}$$

Where the spatial derivation p_{XX} and q_{XX} are replaced by difference of order 2, Δx is spatial step, $N - 1$ is the total number of the net. Integrating over the time variable τ, Eqs. (5) and (6) can be solved. Gill method with variable step is adopted to integrate equations since it can cancel the accumulative error and is more accurate.

2 Numerical Results and Analysis

2.1 The value of α

Dissipation term is a key term in Miles' equation and has a very important effect on the stability of the solution. Unfortunately, dissipation coefficient α can not be determined in physics in advance, but can only be given by people, whose value could be given to get a stable solution and how is the stable solution related to α? How could α be determined in physics. These are the problems to be solved. So, we are interested in the relationship between α and the solution.

Let $A = 1.0$, $B = 1.0$, $\gamma = 1.1$, $\beta = -1.0$ and α be variable in Eq. (1), and the initial condition be a standing soliton. The calculation results show that α can not be given arbitrarily and has an upper bound and a lower limit if you would like to get a stable standing soliton at last. For parameters mentioned above, the system can maintain a stable standing soliton, only when $0.46 \le \alpha \le 1.09$.

Laedke and Spatschek analyzed Miles' Eq. (1) for the stability in the case $A = 1$, $B = 1$ and got the stability condition

$$\beta < 0, \quad \alpha^2 < \gamma^2 < \alpha^2 + \beta^2. \tag{7}$$

It can be written in the form

$$\gamma^2 - \beta^2 < \alpha^2 < \gamma^2. \tag{8}$$

In the model $\beta = -1.0$ and $\gamma = 1.1$, from inequality (8), $0.458 < \alpha < 1.10$ satisfied Laedke and Spatschek stability condition. Obviously, our result is consistent with the analysis of small disturbance theory in [4]. A real physical model was calculated. The channel is $l \times b \times h = 20 \times 2.5 \times 2\ \mathrm{cm}^3$. The drive is $z = A_e \cos 2\omega t$, where A_e is the driven amplitude, $f = \omega / 2\pi$ is the driven frequency. Experiments showed that stable standing soliton can occur in the channel. Calculations showed that when $0.64 < \alpha < 1.612$, stable standing can be produced. The stability condition in [4] is $0.632 < \alpha < 1.614$. So, they are fairly identical.

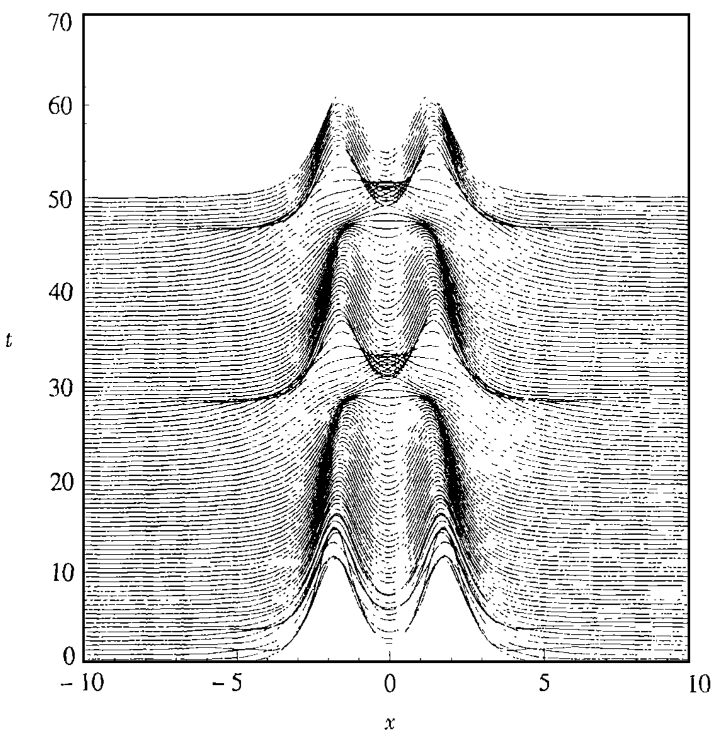
Though stability condition by Laedke and Spatschek came from small disturbance theory, there is almost no difference with our calculation. So, the stability condition in [4] can be used to the nonlinear analysis. It can be recognized as a condition to restrict the dissipation term and excitation.

Notice that $|\beta| < |\gamma|$ in the model mentioned above. If $|\beta| > |\gamma|$, the stable condition (8) becomes $\alpha < \gamma$, i.e. (8) can be satisfied even if $\alpha = 0$. It means that from stable condition in [4], the system can get and maintain the standing soliton only if $|\beta| > |\gamma|$ even if there is no dissipation term. It can not be believed in physics. If the exciting term in Miles' Eq. (1) is not balanced by a dissipation term, the wave amplitude will increase and the system will lose the balance at last. We regulate the linear resistance α to calculate a model with parameters $A = 1.0$, $B = 1.0$, $\beta = -1.0$, $\gamma = 1.1$ and $|\beta| > |\gamma|$. The calculation showed that only when $0.6 < \alpha < 1.1$, the system can maintain a stable standing soliton. It is consistent with the physical analysis qualitatively and is different from the stable condition in [4]. So, the stable condition in [4] is a necessary condition, not a sufficient condition. It is also the reason of defeat to simulate a standing soliton in [14]. From above, how to determine the value of α remains to be further discussed.

2.2 The interaction between two standing solitons

1) For real physical model

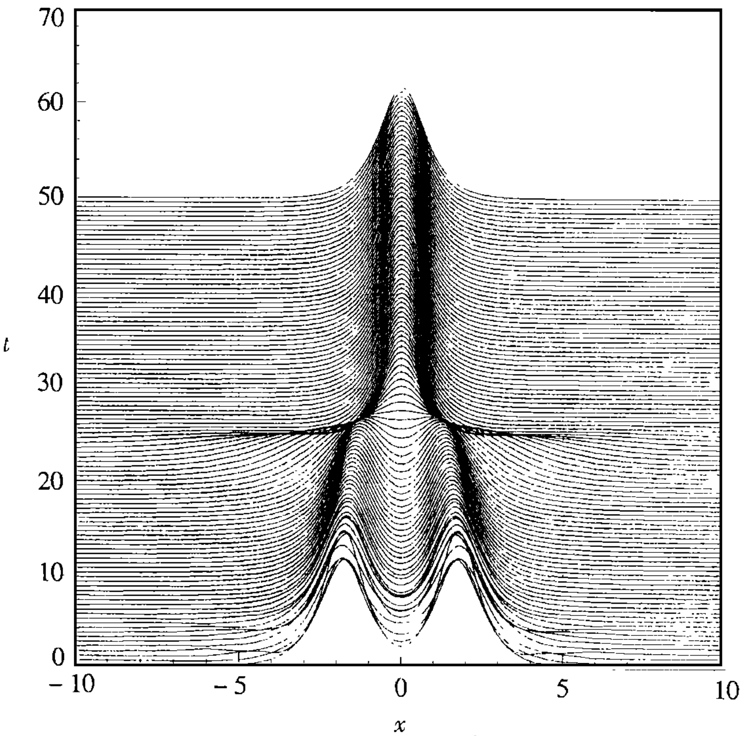
The real physical model mentioned in 2.1 is used again. The geometry of the channel is $l \times b \times h = 20 \times 2.5 \times 2\ \mathrm{cm}^3$, the drive is $z = A_e \cos 2\omega t$, where A_e is driven amplitude, $\omega/2\pi = f$ is the driven frequency. Experiments showed that a stable standing soliton can be produced in the system and interaction between two standing solitons can be observed.



Trough: $l \times b \times h = 20 \times 2.5 \times 2\ \mathrm{cm}^3$; $\alpha = 1.16$;
Drive: $Z = A_e \cos 2\omega t$; $\omega = 2\pi f$; $A_e = 0.95\mathrm{mm}$; $f = 5.03\mathrm{Hz}$

Fig. 1 A pair of NPSW of the same polarity oscillate about each other

First, let $\alpha = 1.16$, $f = 5.03$ Hz, and A_e be variable. When driven amplitude is in 0.95 mm～0.96 mm, we simulated the interaction between two standing solitons. At the beginning, two standing solitons attracted each other, then overlapped, separated, and attracted again. The process went round and round. We call it 2S-1S-2S simply (see Fig. 1). When driven amplitude is less than 0.95mm, the standing solitons attract each other, then overlapped, became one standing soliton and never separated (2S-1S, see Fig. 2); Two standing solitons had no interaction and maintained their original shapes when driven amplitude is larger than 0.96mm (2S-2S, see Fig. 3).



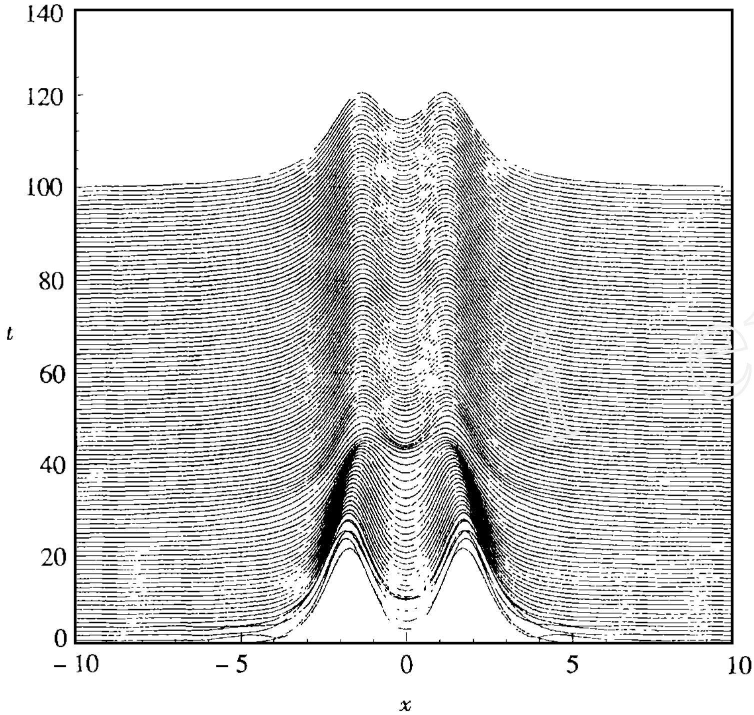
Trough: $l \times b \times h = 20 \times 2.5 \times 2\ cm^3$; $\alpha = 1.16$;
Drive: $Z = A_e \cos 2\omega t$; $\omega = 2\pi f$; $A_e = 0.94$mm; $f = 5.03$Hz

Fig. 2 A pair of NPSW of the same polarity combine into a single and never separated NPSW

Secondly, let $\alpha = 1.16$, $A_e = 0.95$mm and f be variable. When driven frequency f is in 5.02～5.03Hz, 2S-1S-2S picture can be obtained in the system. If $f < 5.02$Hz, the computational result is 2S-1S. If $f > 5.03$Hz, the result is 2S-2S. It is basically consistent with WU's experimental observation. They found that two standing solitons will combine into one soliton and will never separate again if driven frequency is smaller, the interaction between two standing solitons will go round and round only if driven frequency is near a proper value. But they didn't mention that there is 2S-2S phenomenon if driven frequency is larger.

2) Let all physical parameters be fixed, but linear damping α regulated

Let $A = 1.0$, $B = 1.0$, $\beta = -1.0$ and $\gamma = 1.1$ be fixed, i.e. the channel geometry, fluid depth, driven frequency and amplitude fixed, we calculated the interaction of standing solitons with regulated linear damping α. It was found that 2S-1S-2S occurred if $0.83 < \alpha < 0.87$, 2S-2S occurred if α<0. 83 and 2S-1S occurred if α>0. 87. This means that though stable standing soliton can simulate if α satisfies linear stability condition, α will be limited further if you calculate the interaction of standing solitons.



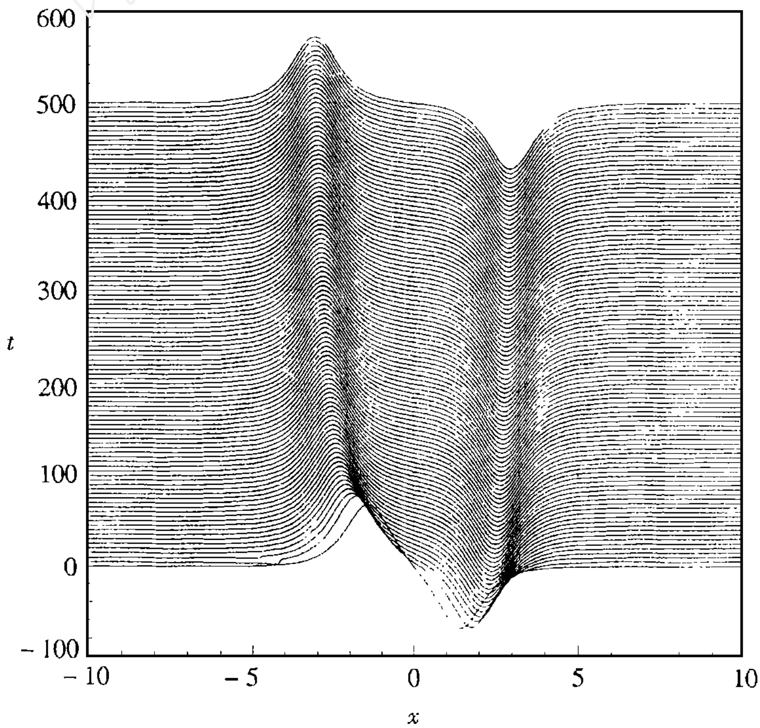
Trough: $l \times b \times h = 20 \times 2.5 \times 2\ cm^3$; $\alpha = 1.16$;
Drive: $Z = A_e \cos 2\omega t$; $\omega = 2\pi f$; $A_e = 0.98$ mm; $f = 5.03$ Hz

Fig. 3 A pair of NPSW of the same polarity maintain independent development without interaction

We obtained typical interaction of standing solitons in our numerical simulation. Two standing solitons in same phase attract each other first, then overlap and separate, then attract again. It will go round and round. But to get such interaction, parameters required by calculation models are very strict and sensitive. Three patterns of the interaction between two standing solitons in our numerical simulation are qualitatively consistent with experiments. Though to obtain the interaction between two standing solitons in experiments there is really a requirement of the drive, the required ranges of parameters are not so strict, variable ranges of the driven amplitude and frequency in experiments are larger than that in calculations. So, to describe the interaction of standing solitons better, there is much work to do in the numerical simulation.

In Miles' equation, α is the dissipation term and represents the linear damping. The reason to introduce α is as follows. There is energy inputting into the system continuously when the channel is vibrated by external drive. If there is no energy dissipation, the wave amplitude will increase continuously. The dissipation exists in the real physical system too. So, to get stable solution, the only thing to do is to introduce a dissipation term to balance the vibration exciting.

The interaction between two standing solitons is related to the nonlinear and dissipation term in the system. When the vibration frequency is fixed, the driven amplitude represents the energy obtained by channel system from external exciting, α reflects the dissipation and the fluid viscosity of the system in fact. The stronger the dissipation is, the easier to break the formed stable wave is. Therefore, if driven amplitude is too small, i.e. the input energy from outside is not large, two standing solitons will combine into one and will not separate again because of the fluid viscous when α is fixed. With the increase of driven amplitude, the input energy from outside raises and can balance with system dissipation, two standing solitons will attract each other, overlap, separate, and attract again. It will go round and round. If input energy is larger, i.e. driven amplitude is larger, the action of the system dissipation and the fluid viscous is smaller and is not large enough to break the stable standing solitons. So two standing solitons will maintain their own stable waveform because of the support of driven input energy and will not combine into one wave. When input energy is too large, i.e. driven amplitude does not satisfy the stable condition, not only soliton pattern is excited, but also other wave patterns are excited. The competition and interaction among all patterns made the standing soliton disappear. Of course, the results obtained in the case of the fixed driven amplitude and variable frequency can be explained in the same way.



Trough: $l \times b \times h = 20 \times 2.5 \times 2\ cm^3$; $\alpha = 1.16$;

Drive: $Z = A_e \cos 2\omega t$; $\omega = 2\pi f$; $A_e = 0.95mm$; $f = 5.03Hz$

Fig. 4 A pair of NPSW of opposite polarity maintain a without interaction

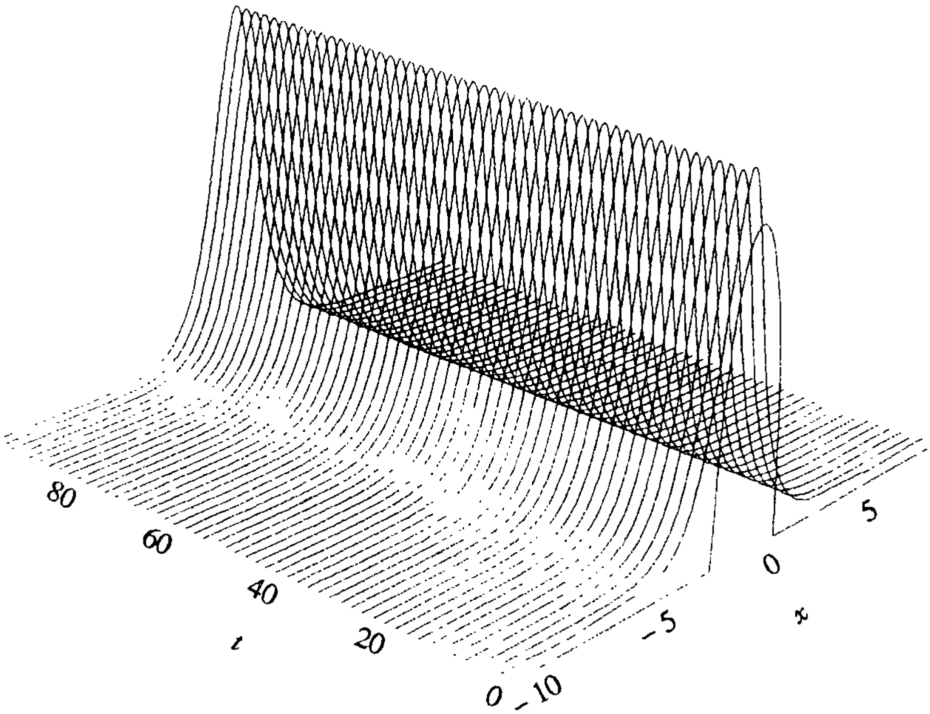
3) Up to now, the interactions mentioned above are for standing solitons in same phase

Now we considered the interaction of standing solitons in opposite phase. The real physical model mentioned in 3.2 was also used. It was found that two were developed by their own way and are inclined to separate very slowly. They didn't disturb each other and has no interaction (see Fig. 4), it is consistent with the real physical phenomenon.

2.3 Initial disturbance and standing soliton

According to experiments, an arbitrary disturbance could develop into one or many standing solitons. Two types of initial disturbance have been calculated.

For half circular initial disturbance, two models were simulated. Parameters of the first one are $A = B = 1.0$, $\beta = -1.0$, $\gamma = 1.1$. The other model is what WU used in [1]. The initial disturbance developed into a standing soliton very soon in either case (see Fig. 5).

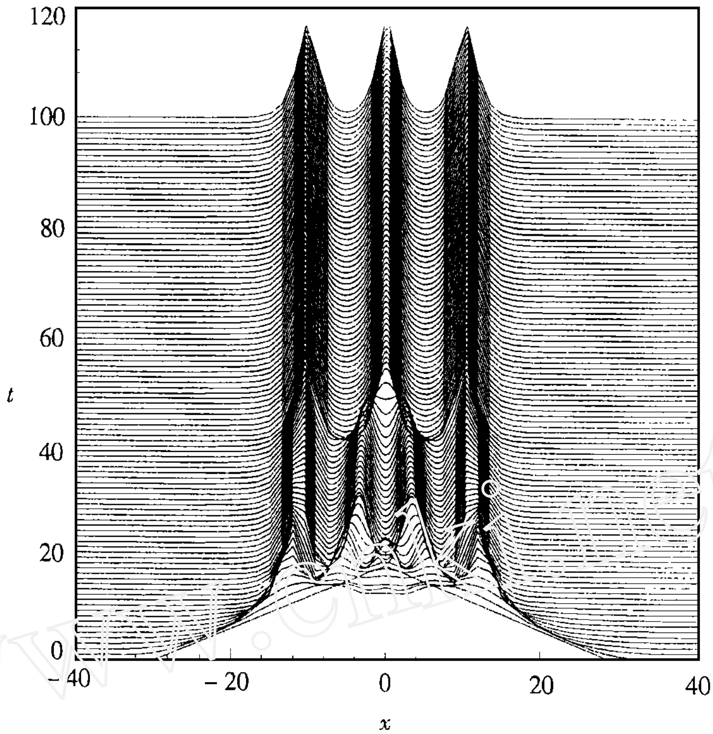


Trough: $l \times b \times h = 38 \times 2.53 \times 2\ cm^3$; $\alpha = 1.16$; $\varepsilon = 0.06$;

Drive: $Z = A_e \cos 2\omega t$; $\omega = 2\pi f$; $A_e = 0.95mm$; $f = 5.03Hz$

Fig. 5 An initial semicircle disturbance develop into a stable NPSW

For triangular initial disturbance, many models were calculated and analyzed. Parameters are the same with that of circular initial condition. First, we fixed the botton length of the triangle as well as the wave width, then let the height of the triangle be variable. From the calculation, we know that when the peak value of the disturbance is 0.95～1.552 of the wave height of a standing soliton, the triangular disturbance will develop into a standing soliton at last. Otherwise the disturbance will disappear. Secondly, we fixed the height of the triangular disturbance, and let the bottom length be variable. To get a singular soliton, the bottom length must be limited in a certain region. Otherwise, the disturbance will disappear or develop into many standing solitons that interact with each other (see Fig. 6).



Parameters : $\alpha = 1.0$, $\beta = - 1.0$, $\gamma = 1.1$, $A = B = 1.0$

Fig.6 A broad initial triangle disturbance develop into three NPSW

3 Conclusions

1) The dissipation coefficient in Miles' Eq. (1) (the linear damping α) strongly affects the formation of a stable standing soliton. In some cases, Laedke and Spatschek stable condition is very well consistent with our calculation, but in other cases, it is not the same. So, Laedke and Spatschek stable condition is only a necessary condition, not a sufficient one. Laedke and Spatschek stable condition for the solution of Miles' equation was obtained from linear small disturbance theory, and was confirmed by our nonlinear numerical simulation in some cases.

2) The numerical simulation of the interaction between two standing solitons in the same phase showed that system parameters are limited strictly. In case of fixed channel geometry, fluid depth and linear damping α, two standing solitons attract each other, overlap, separate, attract again and go round and round only for certain driven amplitude and frequency. In case of fixed channel geometry, fluid depth, driven amplitude and frequency, 2S-1S-2S interaction can be simulated only for certain linear damping α in a very narrow range. Two waves may be combined into one standing soliton and never separate or do not interact with each other and develop separately in their own way if parameters are not suitable. Compared with experiments, the condition required by numerical simulation is harsher, the range of driven amplitude and frequency is narrower. It is examined too that two standing solitons in reversal phase do not interact.

3) Various initial disturbances can develop into one or more standing solitons. If initial

disturbance is too small, standing soliton can not be produced. If initial disturbance is large enough, it will disappear or develop into two or more standing solitons and interact with each other.

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