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Optimal Bounded Control of First-Passage Failure of Quasi-Integrable Hamiltonian Systems with Wide-Band Random Excitation

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Abstract. The optimal bounded control of quasi-integrable Hamiltonian systems with wide-band random excitation for minimizing their first-passage failure is investigated. First, a stochastic averaging method for multi-degrees-of-freedom (MDOF) strongly nonlinear quasi-integrable Hamiltonian systems with wide-band stationary random excitations using generalized harmonic functions is proposed. Then, the dynamical programming equations and their associated boundary and final time conditions for the control problems of maximizinig reliability and maximizing mean first-passage time are formulated based on the averaged Itô equations by applying the dynamical programming principle. The optimal control law is derived from the dynamical programming equations and control constraints. The relationship between the dynamical programming equations and the backward Kolmogorov equation for the conditional reliability function and the Pontryagin equation for the conditional mean first-passage time of optimally controlled system is discussed. Finally, the conditional reliability function, the conditional probability density and mean of first-passage time of an optimally controlled system are obtained by solving the backward Kolmogorov equation and Pontryagin equation. The application of the proposed procedure and effectiveness of control strategy are illustrated with an example.

Keywords: Nonlinear system, random excitation, first-passage time, stochastic optimal control.

1. Introduction

First-passage time is related to the problem of state transition of randomly excited physical systems and first-passage (first-excursion) failure is a major failure model of mechanical/structural systems under random excitation. Thus, the first-passage problem is of great significance although it is one of the most difficult problems in stochastic dynamics. At present, a mathematical exact solution is possible only if the random phenomenon in question can be treated as a diffusion process. Still, known solutions are limited to the one-dimensional case [1, 2].

The state space of a physical or engineering dynamical system is generally two-dimensional or higher and the random excitation of the system is usually not Gaussian white noise. So, it is difficult to directly apply the diffusion process theory of the first-passage problem. A feasible way in this case is first to apply the stochastic averaging method to reduce a system into averaged Itô equations of lower dimension and then to apply the diffusion process theory of the first-passage problem to an averaged system. In the last three decades, the combination approach of the classical stochastic averaging method and diffusion process theory of the first-passage problem has been applied by many researchers to single degree-of-freedom (SDOF) oscillators with linear or nonlinear restoring force [3–13]. Recently, the combination approach of the stochastic averaging method for quasi-Hamiltonian systems [14–16] and diffusion process theory of first-passage time was proposed and applied to quasi-non-integrable, quasi-integrable and quasi-partially integrable Hamiltonian systems [17–19].

The mathematical theory of stochastic optimal control is quite well developed [20–23]. However, in physical and engineering fields, only the linear quadratic Gaussian (LQG) control strategy has been widely applied until quite recently. In the last few years, a nonlinear stochastic optimal control strategy was proposed by Zhu and co-workers [24–26] based on the stochastic averaging method for quasi-Hamiltonian systems [14–16] and the stochastic dynamical programming principle [20–23]. The LQG control strategy is usually used to reduce the response or to stabilize physical and engineering systems, while the nonlinear stochastic optimal control strategy [24–26] can also be used to minimize the first-passage failure [19, 27] except response reduction [24–26] and stabilization [28]. However, in all these studies, the random excitation is assumed to be Gaussian white noise.

In this paper, a procedure for designing optimal bounded control of quasi-integrable Hamiltonian systems with wide-band random excitations for minimizing first-passage failure is proposed. First, the motion equations of such a system is reduced to averaged Itô equations of lower-dimension by using the stochastic averaging method for multi-degrees-of-freedom, strongly nonlinear quasi-integrable Hamiltonian systems with wide-band random excitations. Then, the dynamical programming equations and their boundary and final-time conditions for the control problems of reliability maximization and mean first-passage time maximization are established. The optimal control law is determined from the dynamical programming equations and control constraints. The relationship between these dynamical programming equations and the backward Kolmogorov equation for the conditional reliability function and Pontryagin equation for mean first-passage time of optimally controlled systems is discussed. Finally, the conditional reliability function, the conditional probability density and mean of first-passage time of optimally controlled systems are obtained by solving the backward Kolmogorov equation and Pontryagin equation. The proposed procedure is illustrated with an example of two coupled Duffing oscillators under external and parametric excitations of stationary wide-band random processes.

2. Stochastic Averaging

Consider a controlled quasi-Hamiltonian system of n degrees of freedom governed by the following motion equations:

$$\begin{aligned}
\dot{Q}_i &= \frac{\partial H}{\partial P_i}, \\
\dot{P}_i &= -\frac{\partial H}{\partial Q_i} - \varepsilon c_{ij}(\mathbf{Q},\mathbf{P})\frac{\partial H}{\partial P_j} + \varepsilon u_i(\mathbf{Q},\mathbf{P}) + \varepsilon^{1/2} f_{ik}(\mathbf{Q},\mathbf{P})\xi_k(t), \\
Q_i(0) &= Q_{i0}, \quad P_i(0) = P_{i0}, \quad i, j = 1, 2, \ldots, n,\ k = 1, 2, \ldots, m,
\end{aligned} \tag{1}$$

where Q_i and P_i are generalized displacements and momenta, respectively; $H = H(\mathbf{Q},\mathbf{P})$ is a twice differentiable Hamiltonian; ε is a small parameter; $\varepsilon c_{ij}(\mathbf{Q},\mathbf{P})$ denotes the coefficients of light quasi-linear damping; $\varepsilon^{1/2} f_{ik}(\mathbf{Q},\mathbf{P})$ denote amplitudes of weak random excitations; $\xi_k(t)$ are stationary wide-band random processes with correlation functions $R_{kl}(\tau) = E[\xi_k(t)\xi_l(t+\tau)]$ or spectral densities $S_{kl}(\omega)$; $\varepsilon u_i(\mathbf{Q},\mathbf{P})$ denotes weak feedback control forces.

Assume that the Hamiltonian system with Hamiltonian H is separable, i.e.,

$$H(\mathbf{q}, \mathbf{p}) = \sum_{i=1}^{n} H_i(q_i, p_i) \tag{2}$$

and for most non-gyroscopic Hamiltonian systems,

$$H_i(q_i, p_i) = p_i^2/2 + U_i(q_i). \tag{3}$$

Besides, each sub-Hamiltonian system with a Hamiltonian H_i has a family of periodic solutions in a domain Ω_i around equilibrium position $(b_i, 0)$, i.e.,

$$q_i(t) = a_i \cos\varphi_i(t) + b_i, \quad p_i(t) = -a_i v_i(a_i, \varphi_i)\sin\varphi_i(t), \quad \varphi_i(t) = \psi_i(t) + \theta_i(t), \tag{4}$$

where

$$v_i(a_i, \varphi_i) = \frac{\mathrm{d}\psi_i}{\mathrm{d}t} = \sqrt{\frac{2[U_i(a_i + b_i) - U_i(a_i\cos\varphi_i + b_i)]}{a_i^2 \sin^2\varphi_i}}. \tag{5}$$

Here a_i and b_i are constants related to H_i as follows:

$$U_i(a_i + b_i) = U_i(-a_i + b_i) = H_i. \tag{6}$$

$\cos\varphi_i$ and $\sin\varphi_i$ are called generalized harmonic functions [29]; a_i are the amplitudes of displacements, and $v_i(a_i, \varphi_i)$ are the instantaneous frequencies of the sub-Hamiltonian oscillators.

Expand $v_i^{-1}(a_i, \varphi_i)$ into Fourier series

$$v_i^{-1}(a_i, \varphi_i) = C_{0i}(a_i) + \sum_{n=1}^{\infty} C_{ni}(a_i)\cos n\varphi_i. \tag{7}$$

Integrating Equation (5) with respect to ψ_i yields

$$t = C_{0i}(a_i)\psi_i + \sum_{n=1}^{\infty} \frac{1}{n} C_{ni}(a_i)\sin n\varphi_i. \tag{8}$$

Letting the integration (8) be from 0 to 2π leads to the average periods

$$T_i(a_i) = 2\pi C_{0i}(a_i) \tag{9}$$

and average frequencies

$$\omega_i(a_i) = 1/C_{0i}(a_i) \tag{10}$$

of the oscillators. Thus, in averaging, the following approximate relations can be used:

$$\varphi_i(t) \approx \omega_i(a_i)t + \theta_i(t). \tag{11}$$

Since the dampings are light, random excitations and feedback control forces are weak, the sample response of system (1) is nearly periodic and of the form

$$\begin{aligned} &Q_i(t) = A_i\cos\Phi_i(t) + B_i, \quad P_i(t) = -A_i v_i(A_i, \Phi_i)\sin\Phi_i(t), \\ &\Phi_i(t) = \Psi_i(t) + \Theta_i(t), \end{aligned} \tag{12}$$

where

$$v_i(A_i, \Phi_i) = \frac{d\Psi_i}{dt} = \frac{\sqrt{2[U_i(A_i + B_i) - U_i(A_i \cos\Phi_i + B_i)]}}{A_i \sin\Phi_i}. \tag{13}$$

Treating Equation (12) as generalized van der Pol transformations from Q_i, P_i to A_i, Θ_i, one obtains the following standard equations for A_i, Θ_i:

$$\begin{aligned}
\frac{dA_i}{dt} &= \varepsilon F_i^{(1)}(\mathbf{A}, \mathbf{\Theta}, \mathbf{u}) + \varepsilon^{1/2} G_{ik}^{(1)}(\mathbf{A}, \mathbf{\Theta})\xi_k(t), \\
\frac{d\Theta_i}{dt} &= \varepsilon F_i^{(2)}(\mathbf{A}, \mathbf{\Theta}, \mathbf{u}) + \varepsilon^{1/2} G_{ik}^{(2)}(\mathbf{A}, \mathbf{\Theta})\xi_k(t), \\
A_i(0) &= a_{i0}, \quad \Theta_i(0) = \theta_{i0}, \quad i = 1, 2, \ldots, n; \quad k = 1, 2, \ldots, m,
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
F_i^{(1)}(\mathbf{A}, \mathbf{\Theta}, \mathbf{u}) &= F_{i1}^{(1)}(\mathbf{A}, \mathbf{\Theta}) + F_{i2}^{(1)}(\mathbf{A}, \mathbf{\Theta}, \mathbf{u}) \\
&= \frac{-A_i}{g_i(A_i + B_i)(1 + h_i)} \\
&\quad \times [-c_{ij}(\mathbf{A}, \mathbf{\Theta}) A_j v_j(A_j, \Phi_j) \sin\Phi_j + u_i(\mathbf{A}, \mathbf{\Theta})] v_i(A_i, \Phi_i) \sin\Phi_i, \\
F_i^{(2)}(\mathbf{A}, \mathbf{\Theta}, u) &= F_{i1}^{(2)}(\mathbf{A}, \mathbf{\Theta}) + F_{i2}^{(2)}(\mathbf{A}, \mathbf{\Theta}, \mathbf{u}) \\
&= \frac{-1}{g_i(A_i + B_i)(1 + h_i)} \\
&\quad \times [-c_{ij}(\mathbf{A}, \mathbf{\Theta}) A_j v_j(A_j, \Phi_j) \sin\Phi_j + u_i(\mathbf{A}, \mathbf{\Theta})] v_i(A_i, \Phi_i)(\cos\Phi_i + h_i), \\
G_{ik}^{(1)}(\mathbf{A}, \mathbf{\Theta}) &= \frac{-A_i}{g_i(A_i + B_i)(1 + h_i)} f_{ik}(\mathbf{A}, \mathbf{\Theta}) v_i(A_i, \Phi_i) \sin\Phi_i, \\
G_{ik}^{(2)}(\mathbf{A}, \mathbf{\Theta}) &= \frac{-1}{g_i(A_i + B_i)(1 + h_i)} f_{ik}(\mathbf{A}, \mathbf{\Theta}) v_i(A_i, \Phi_i)(\cos\Phi_i + h_i),
\end{aligned} \tag{15}$$

$c_{ij}(\mathbf{A}, \mathbf{\Theta})$, $f_{ik}(\mathbf{A}, \mathbf{\Theta})$ and $u_i(\mathbf{A}, \mathbf{\Theta})$ are $c_{ij}(\mathbf{Q}, \mathbf{P})$, $f_{ik}(\mathbf{Q}, \mathbf{P})$ and $u_i(\mathbf{Q}, \mathbf{P})$, respectively, with $\mathbf{Q}$, $\mathbf{P}$ replaced by $\mathbf{A}$, $\mathbf{\Theta}$ according to transformations (12), and

$$g_i = \frac{\partial U_i}{\partial Q_i}, \quad h_i = \frac{dB_i}{dA_i} = \frac{g_i(-A_i + B_i) + g_i(A_i + B_i)}{g_i(-A_i + B_i) - g_i(A_i + B_i)}. \tag{16}$$

Suppose that system (1) has no internal resonance, i.e.,

$$k_i \omega_i(a_i) \neq 0(\varepsilon), \tag{17}$$

where k_i are small integers. Based on the Stratonovich–Khasminskii limit theorem [30–32], A in Equation (14) converges weakly to an n-dimensional vector diffusion process as $\varepsilon \to 0$ in a time internal $[0, T]$, where $T \sim O(\varepsilon^{-1})$. This limiting diffusion process is governed by the following averaged Itô equations:

$$\begin{aligned}
dA_i &= [m_{i1}(\mathbf{A}) + \varepsilon \langle F_{i2}^{(1)}(\mathbf{A}, \mathbf{\Theta}, \mathbf{u}) \rangle_t]\, dt + \sigma_{ik}(\mathbf{A})\, dB_k(t), \\
A_i(0) &= a_{i0}, \quad i = 1, 2, \ldots, n; k = 1, 2, \ldots, m,
\end{aligned} \tag{18}$$

where

$$m_{i1}(\mathbf{A}) = \varepsilon \left\langle F_{i1}^{(1)} + \int_{-\infty}^{0} \left(\frac{\partial G_{ik}^{(1)}}{\partial A_j}\bigg|_t G_{jl}^{(1)}\big|_{t+\tau} + \frac{\partial G_{ik}^{(1)}}{\partial \Theta_j}\bigg|_t G_{jl}^{(2)}\big|_{t+\tau} \right) R_{kl}(\tau)\, \mathrm{d}\tau \right\rangle_t ,$$

$$b_{ij}(\mathbf{A}) = \sigma_{ik}(\mathbf{A})\sigma_{jk}(\mathbf{A}) = \varepsilon \left\langle \int_{-\infty}^{\infty} \left(G_{ik}^{(1)}\big|_t G_{jl}^{(1)}\big|_{t+\tau} \right) R_{kl}(\tau)\, \mathrm{d}\tau \right\rangle_t . \tag{19}$$

$\langle\cdot\rangle_t$ denotes the time averaging operation, i.e.,

$$\langle\cdot\rangle_t = \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle\cdot\rangle\, \mathrm{d}t. \tag{20}$$

To obtain the explicit expressions for m_{i1} and b_{ij}, first expand F_i, G_{ik} into n-fold Fourier series with respect to Φ_i, integrate with respect to τ and then average with respect to Φ_i using Equation (11). Note that the averaging of $m_{i2}(\mathbf{A}) = \langle F_{i2}^{(1)}(\mathbf{A}, \mathbf{\Theta}, \mathbf{u})\rangle_t$ is not completed at this stage, since $\mathbf{u}$ is unknown.

Equation (18) is the averaged Itô equation for the amplitudes of generalized displacements Q_i. In some cases, the averaged Itô equations for the first integrals (energies of each degree of freedom) H_i, $i = 1, 2, \ldots, n$ are needed. They can be obtained from Equation (18) by using Itô differential rule, noting that

$$H_i = U_i(A_i + B_i). \tag{21}$$

The result is

$$\begin{aligned} \mathrm{d}H_i &= [\bar{m}_{i1}(\mathbf{H}) + \varepsilon \bar{m}_{i2}(\mathbf{H}, \mathbf{u})]\, \mathrm{d}t + \bar{\sigma}_{ik}(\mathbf{H})\, \mathrm{d}B_k(t), \\ H_i(0) &= H_{i0}, \quad i = 1, 2, \ldots, n; k = 1, 2, \ldots, m, \end{aligned} \tag{22}$$

where $\mathbf{H} = [H_1, H_2, \ldots, H_n]^T$,

$$\begin{aligned} \bar{m}_{i1}(\mathbf{H}) &= \Bigg[g_i(A_i + B_i)(1 + h_i)m_{i1}(\mathbf{A}) \\ &\quad + \frac{1}{2}\frac{\mathrm{d}}{\partial A_i}[g_i(A_i + B_1)(1 + h_i)]\sigma_{ik}(\mathbf{A})\sigma_{ik}(\mathbf{A}) \Bigg]\Bigg|_{A_i = U_i^{-1}(H_i) - B_i}, \\ \bar{m}_{i2}(\mathbf{H}, \mathbf{u}) &= [g_i(A_i + B_i)(1 + h_i)\langle F_{i2}^{(1)}(\mathbf{A}, \mathbf{\Theta}, \mathbf{u})_t\rangle]\big|_{A_i = U_i^{-1}(H_i) - B_i} = \langle u_i P_i\rangle_t, \\ \bar{b}_{ij}(\mathbf{H}) &= [\bar{\sigma}_{ik}(\mathbf{H})\bar{\sigma}_{jk}(\mathbf{H})] \\ &= [g_i(A_i + B_i)g_j(A_j + B_j)(1 + h_i)(1 + h_j)\sigma_{ik}(\mathbf{A})\sigma_{jk}(\mathbf{A})]\big|_{A_i = U_i^{-1}(H_i) - B_i} . \end{aligned} \tag{23}$$

Note that $\mathbf{A}(t)$ in Equation (18) and $\mathbf{H}(t)$ in Equation (22) are homogeneous diffusion processes. So, the theory of first passage time of diffusion processes can be applied to them. Furthermore, the dimension of averaged equations in Equations (18) or (22) is only a half of that of the original equation (1) and in the averaged equations only the slowly varying process A_i or H_i are retained. Note that time averaging in Equation (19) smoothens out only

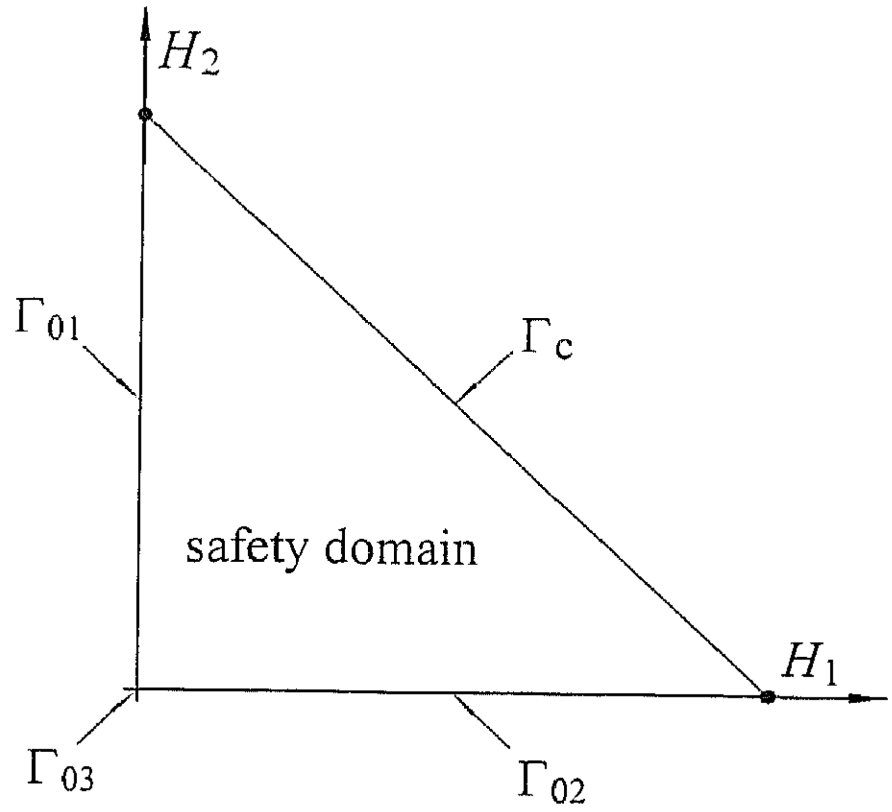

Figure 1. Safety domain Ω and its boundary on plane H_1 and H_2 for system (54).

the rapidly temporal fluctuation in the response but not the transient response. So, the averaged equations (18) and (22) can depict the averaged transient response as well as averaged stationary response. Since the first-passage failure of engineering systems rarely occurs and it is a long-term behavior, the averaged equations are specially suitable for studying the first-passage problem. In the following, the averaged Itô equations in Equation (22) rather than the original equation (1), are used to study the optimal control of first-passage failure.

3. Dynamical Programming Equations

For most mechanical and structural dynamical systems, the Hamiltonian $H(t)$ represents the total energy and H_i the energy of the rth degree-of-freedom of the system. Each H_i may vary in some sub-interval of $[0, \infty)$. The state of averaged system (22) varies randomly in the n-dimensional domain defined by the direct product of n H_i sub-intervals and the safety domain Ω is a bounded region with boundary Γ within the n-dimensional H_r domain. Suppose that the lower boundary of safety domain for each H_i is at zero, then the boundary Γ of safety domain Ω consists of Γ_0 (at least one of H_i vanishes) and critical boundary Γ_c. One example of the safety domain and its boundary is shown in Figure 1. The first-passage failure occurs when $\mathbf{H}(t)$ reaches the critical boundary Γ_c for the first time and it is characterized by a reliability function, the probability, density, or moments of first-passage time.

For the control problem of reliability maximization, introduce the value function

$$V(t, \mathbf{H}) = \sup_{\mathbf{u}\in U} P\{\mathbf{H}(\tau, \mathbf{u}) \in \Omega, \tau \in [t, t_f]\}, \tag{24}$$

where $\mathbf{u} \in U$ denotes the control constraint; 'sup' is the abbreviation of the word 'supremum'. Equation (24) implies that $V(t, \mathbf{H})$ is the reliability function of the optimally controlled system. Suppose that $\mathbf{H}(\tau, \mathbf{u})$ is governed by the averaged Itô equation (22). Based on the stochastic dynamical programming principle [20–22], the following dynamical programming equation can be derived:

$$\sup_{\mathbf{u}\in U}\left\{\frac{\partial}{\partial t}+[\bar{m}_{i1}(\mathbf{H})+\varepsilon\langle u_iP_i\rangle_t]\frac{\partial}{\partial H_i}+\frac{1}{2}\bar{b}_{ij}(\mathbf{H})\frac{\partial^2}{\partial H_i\partial H_j}\right\}V(t,\mathbf{H})=0,$$
$$0\le t\le t_f,\mathbf{H}\in\Omega. \tag{25}$$

The boundary conditions associated with Equation (25) are

$$V(t,\Gamma_c)=0, \tag{26}$$

$$V(t,\Gamma_0)=\text{finite}, \tag{27}$$

and the final time condition is

$$V(t_f,\mathbf{H})=1,\quad \mathbf{H}\in\Omega. \tag{28}$$

Equations (25–28) are the mathematical formulation for the problem of feedback maximization of the reliability of averaged system (22). Both the optimal control law and the reliability function of optimally controlled system (22) can be obtained from solving these equations.

The control problem of maximizing the mean first-passage time of averaged system (22) can be similarly formulated. Let $E[\tau(\mathbf{H},\mathbf{u})]$ denote the mean first-passage time of a controlled system. Define the value function

$$V_1(\mathbf{H})=\sup_{\mathbf{u}\in U}E[\tau(\mathbf{H},\mathbf{u})], \tag{29}$$

which implies $V_1(\mathbf{H})$ is the mean first-passage time of optimally controlled system. Based on the dynamical programming principle, the following dynamical programming equation for value function $V_1(\mathbf{H})$ can be derived from Equation (22):

$$\sup_{\mathbf{u}\in U}\left\{[\bar{m}_{i1}(\mathbf{H})+\varepsilon\langle u_iP_i\rangle_t]\frac{\partial}{\partial H_i}+\frac{1}{2}\bar{b}_{ij}(\mathbf{H})\frac{\partial^2}{\partial H_i\partial H_j}\right\}V_1(\mathbf{H})=-1. \tag{30}$$

The boundary conditions associated with Equation (30) are

$$V_1(\Gamma_c)=0, \tag{31}$$

$$V_1(\Gamma_0)=\text{finite}. \tag{32}$$

Solving Equations (30–32) yields both the optimal control law and the mean first-passage time of optimally controlled system (22).

Note that boundary conditions (27) and (32) imply that $\mathbf{H}(t)$ should not cross boundary Γ_c. They are qualitative and can be made to be quantitative by using Equation (25) or (30) and by examining the behavior of the drift and diffusion coefficients of Equation (22) at boundary Γ_0. This will be illustrated in the following example.

The optimal control law can be determined from maximizing the left-hand side of Equation (25) or (30) with respect to $u\in U$. Suppose that the control constraints are of the form

$$-c_i\le u_i\le c_i,\quad i=1,2,\ldots,n, \tag{33}$$

where c_i are positive constants. The terms $u_i P_i \partial V/\partial H_i$ will be maximum when $|u_i| = c_i$ and each term $u_i P_i \partial V/\partial H_i$ (no summation over i) is positive. Thus, the optimal control law is

$$u_i^* = c_i \text{sign}\left(P_i \frac{\partial V}{\partial H_i}\right), \quad i = 1, 2, \ldots, n. \tag{34}$$

From [18] it is seen that the conditional reliability function is a monotonically decreasing function of H_i. So, $\partial V/\partial H_i < 0$. Thus, Equation (34) is reduced to

$$u_i^* = -c_i \text{sign}\, P_i = -c_i \text{sign}(\dot{Q}), \quad i = 1, 2, \ldots, n. \tag{35}$$

Equation (35) implies that the optimal control law is bang-bang control. u_i^* has a constant magnitude of c_i. It is in the opposite direction of $\dot{Q}$ and changes its direction at $\dot{Q}_i = 0$.

Inserting Equation (35) into Equation (25) to replace u_i and then averaging $u_i^* P_i$, one obtains the final dynamical programming equation for the control problem of reliability maximization

$$\left[\frac{\partial}{\partial t} + \bar{m}_i(\mathbf{H})\frac{\partial}{\partial H_i} + \frac{1}{2}\bar{b}_{ij}(\mathbf{H})\frac{\partial^2}{\partial H_i \partial H_j}\right] V(t, \mathbf{H}) = 0, \quad 0 \le t \le t_f, \mathbf{H} \in \Omega, \tag{36}$$

where

$$\bar{m}_i(\mathbf{H}) = \bar{m}_{i1}(\mathbf{H}) + \varepsilon \langle u_i^* P_i \rangle_t. \tag{37}$$

The boundary and finial time conditions are still those used in Equations (26–28). Similarly, the finial dynamical programming equation for the control problem of mean first-passage time maximization is

$$\left[\bar{m}_i(\mathbf{H})\frac{\partial}{\partial H_i} + \frac{1}{2}\bar{b}_{ij}(\mathbf{H})\frac{\partial^2}{\partial H_i \partial H_j}\right] V_1(\mathbf{H}) = -1, \quad \mathbf{H} \in \Omega. \tag{38}$$

The boundary conditions are still those used in Equations (31) and (32).

Inserting $\bar{m}_i(\mathbf{H})$ into Equation (22) to replace its drift coefficient yields completely averaged Itô equations

$$\mathrm{d}H_i = \bar{m}_i(\mathbf{H})\,\mathrm{d}t + \bar{\sigma}_{ik}(\mathbf{H})\,\mathrm{d}B_k(t), \quad i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m. \tag{39}$$

Based on theorems due to Kushner [33], the optimal control for averaged system (22) will be quasi-optimal for the original system (1). For simplification, it is called the optimal control for both the original and averaged systems.

4. The Backward Kolmogorov Equation and Pontryagin Equation of Optimally Controlled System

Equation (39) is the completely averaged Itô equations of the optimally controlled system (22). The conditional reliability function of the optimally controlled system is defined as

$$R_{\text{opt}}(t_1 \mid \mathbf{H}_0) = P\{\mathbf{H}(\tau, \mathbf{u}^*) \in \Omega, \tau \in [0, t_1] \mid \mathbf{H}_0 \in \Omega\}. \tag{40}$$

Since $H(t)$ is a homogeneous diffusion process, R_{opt} is governed by the following backward Kolmogorov equation

$$\left[-\frac{\partial}{\partial t_1} + \bar{m}_i(\mathbf{H}_0)\frac{\partial}{\partial H_{i0}} + \frac{1}{2}\bar{b}_{ij}(\mathbf{H}_0)\frac{\partial^2}{\partial H_{i0}\partial H_{j0}}\right] R_{\text{opt}} = 0, \quad \mathbf{H}_0 \in \Omega, \tag{41}$$

with boundary conditions

$$R_{\rm opt}(t_1 \mid \Gamma_c) = 0, \tag{42}$$

$$R_{\rm opt}(t_1 \mid \Gamma_0) = \text{finite}, \tag{43}$$

and initial condition

$$R_{\rm opt}(0 \mid \mathbf{H}_0) = 1, \quad \mathbf{H}_0 \in \Omega, \tag{44}$$

where $\bar{m}_i(\mathbf{H}_0)$ and $\bar{b}_{ij}(\mathbf{H}_0)$ are obtained from $\bar{m}_i(\mathbf{H})$ and $\bar{b}_{ij}(\mathbf{H})$ with $\mathbf{H}$ replaced by $\mathbf{H}_0$. A comparison between Equations (24) and (40) reveals that

$$R_{\rm opt}(t_f \mid \mathbf{H}_0) = V(0, \mathbf{H}_0). \tag{45}$$

Note that in Equation (41), t_1 is a forward time running from 0, while in Equation (36) t is a backward time running from t_f. Introducing the transformation

$$\begin{aligned} &t_1 = t_f - t, \\ &V(t, \mathbf{H}) \to R_{\rm opt}(t_1 \mid \mathbf{H}_0), \end{aligned} \tag{46}$$

Equation (36) will be of the same form of Equation (41) and the final time condition (28) becomes initial condition (44), while boundary conditions (26), (27) become (42), (43).

The conditional probability of first-passage failure of the optimally controlled system is

$$P_{\rm opt}(t_1 \mid \mathbf{H}_0) = 1 - R_{\rm opt}(t_1 \mid \mathbf{H}_0). \tag{47}$$

The conditional probability density of first-passage time is then

$$p_{\rm opt}(T \mid \mathbf{H}_0) = \left.\frac{\partial P_{\rm opt}(t_1 \mid \mathbf{H}_0)}{\partial t_1}\right|_{t_1=T} = -\left.\frac{\partial R_{\rm opt}(t_1 \mid \mathbf{H}_0)}{\partial t_1}\right|_{t_1=T}. \tag{48}$$

The mean of the first-passage time is defined by

$$\mu_{1,\rm opt}(\mathbf{H}_0) = \int_0^\infty T p_{\rm opt}(T \mid \mathbf{H}_0)\,\mathrm{d}T = \int_0^\infty R_{\rm opt}(T \mid \mathbf{H}_0)\,\mathrm{d}T. \tag{49}$$

The following Pontryagin equation for the mean first-passage time of the optimally controlled system can be derived from Equations (41) and (49):

$$\left[\bar{m}_i(\mathbf{H}_0)\frac{\partial}{\partial H_{i0}} + \frac{1}{2}\bar{b}_{ij}(\mathbf{H}_0)\frac{\partial^2}{\partial H_{i0}\partial H_{j0}}\right]\mu_{1,\rm opt} = -1. \tag{50}$$

The boundary conditions associated with Equation (50) are

$$\mu_{1,\rm opt}(\Gamma_c) = 0, \tag{51}$$

$$\mu_{1,\rm opt}(\Gamma_0) = \text{finite}. \tag{52}$$

Obviously, dynamical programming equation (38) and its boundary conditions (31) and (32) are of the same form of Equations (50–52). Thus, we have

$$\mu_{1,\rm opt}(\mathbf{H}_0) = V_1(\mathbf{H})|_{\mathbf{H}=\mathbf{H}_0}. \tag{53}$$

Therefore, we can first solve dynamical programming equation (36) together with boundary conditions (26), (27) and final-time condition (28) to obtain $V(t, H)$ and then obtain the conditional reliability function and conditional probability density of the first-passage time of the optimally controlled system by using Equations (46) and (48), respectively. Or, we can first transform dynamical programming equation (36) into the backward Kolmogorov equation (41) by using transformation (46) and then solve Equation (41) together with boundary conditions (42), (43) and initial condition (44) to obtain the conditional reliability function, and finally obtain the conditional probability density of the first-passage time of the optimally controlled system by using Equation (48). As for the conditional mean first-passage time of the optimally controlled system, we can either first solve dynamical programming equation (38) together with boundary conditions (31), (32) to obtain value function $V_1(\mathbf{H})$ and then obtain it by using Equation (53), or first transform the dynamical programming equation (38) into Pontryagin equation (50) and then solve Equation (50) together with boundary conditions (51), (52) to obtain it. It is also possible to obtain the conditional mean first-passage time of the optimally controlled system from the conditional probability density of first-passage time by using Equation (49).

Note that here we are interested in the probabilistic and statistical measures of the first-passage time, which are consistent with the weak convergence in stochastic averaging. As indicated in the Introduction, the stochastic averaging has been applied by many authors [3–13] to the first passage problem. The accuracy of the stochastic averaging method for the prediction of the first-passage failure probability can be checked only by comparison with the result obtained from the Monte Carlo simulation of the original system, and it depends on the magnitude of damping, the bandwidth, and the intensity of the random excitation as well as the initial condition of the system. As indicated in [13], for light damping and weakly Gaussian white noise excitation, it is quite accurate when the initial system state is far from the boundary of safety domain and the relative error increases as the initial system state approaches the boundary. Note that the stiffness nonlinearity is concealed in **A** or **H** and the nonlinearity characteristics of the original system is carried over to the averaged one. So, the effect of stiffness nonlinearity on the probability offirst-passage failure and statistics is taken into account in this method.

5. Example

Consider the optimal bounded control of the first-passage failure of two Duffing oscillators coupled by both linear dampings and external and parametric excitations of stationary wide-band random processes. The motion equations of the system are

$$\begin{aligned}
&\ddot{X}_1 + \beta_{11}\dot{X}_1 + \beta_{12}\dot{X}_2 + \omega_1^2 X_1 + \alpha_1 X_1^3 = f_{11}X_1\xi_1(t) + f_{12}\xi_2(t) + u_1,\\
&\ddot{X}_2 + \beta_{12}\dot{X}_1 + \beta_{22}\dot{X}_2 + \omega_2^2 X_2 + \alpha_2 X_2^3 = f_{22}X_2\xi_1(t) + f_{21}\xi_2(t) + u_2,\\
&X_1(0) = X_{10}, \quad \dot{X}_1(0) = \dot{X}_{10},\\
&X_2(0) = X_{20}, \quad \dot{X}_2(0) = \dot{X}_{20},
\end{aligned} \tag{54}$$

where β_{ij}, ω_i, α_i, f_{ij} $(i, j = 1, 2)$ are constants; $\xi_k(t)$ are independent stationary random processes with correlation functions $R_k(\tau)$ and spectral density

$$S_k(\omega) = \frac{D_k}{\pi(\omega^2 + \bar{\omega}_k^2)}, \quad k = 1, 2. \tag{55}$$

D_k and $\bar{\omega}_k$ are constants. β_{ij}, D_k and u_i are assumed to be of the same order of ε.

Letting $X_1 = Q_1$, $\dot{X}_1 = P_1$, $X_2 = Q_2$, $\dot{X}_2 = P_2$, Equation (54) can be rewritten as equations in the form of Equation (1). The Hamiltonian associated with system (54) is

$$H = H_1 + H_2,$$
$$H_i = \frac{1}{2}p_i^2 + \frac{1}{2}\omega_i^2 q_i^2 + \frac{1}{4}\alpha_i q_i^4, \quad i = 1, 2, \tag{56}$$

which is of the form of Equations (2) and (3) with

$$U_i(q_i) = \frac{1}{2}\omega_i^2 q_i^2 + \frac{1}{4}\alpha_i q_i^4, \quad i = 1, 2. \tag{57}$$

Since $U_i(q_i)$ is symmetrical with respect to 0, one can obtain from Equation (16):

$$g_i = g_i(A_i) = \omega_i^2 A_i + \alpha_i A_i^3, \quad B_i = h_i = 0. \tag{58}$$

The sub-Hamiltonian oscillators associated with system (54) have a periodic solution in the form of Equation (4) throughout the whole phase plane (q_i, p_i) if ω_i^2, $\alpha_i > 0$. The instantaneous frequencies are

$$\begin{aligned} v_i(A_i, \Phi_i) &= [(\omega_i^2 + 3\alpha_i A_i^2/4)(1 + \lambda_i \cos 2\Phi_i)]^{1/2}, \\ \lambda_i &= (\alpha_i A_i^2/4)/(\omega_i^2 + 3\alpha_i A_i^2/4) \le 1/3. \end{aligned} \tag{59}$$

v_i can be expanded into Fourier series. To simplify the calculation, the series is truncated and v_i is approximated by the following finite sum with a relative error less than 0.03%:

$$v_i(A_i, \Phi_i) \approx b_{0i}(A_i) + b_{2i}(A_i)\cos 2\Phi_i + b_{4i}(A_i)\cos 4\Phi_i + b_{6i}(A_i)\cos 6\Phi_i, \tag{60}$$

where

$$\begin{aligned} b_{0i}(A_i) &= (\omega_i^2 + 3\alpha_i A_i^2/4)^{1/2}(1 - \lambda_i^2/16), \\ b_{2i}(A_i) &= (\omega_i^2 + 3\alpha_i A_i^2/4)^{1/2}(\lambda_i/2 + 3\lambda_i^3/64), \\ b_{4i}(A_i) &= (\omega_i^2 + 3\alpha_i A_i^2/4)^{1/2}(-\lambda_i^2/16), \\ b_{6i}(A_i) &= (\omega_i^2 + 3\alpha_i A_i^2/4)^{1/2}(\lambda_i^3/64). \end{aligned} \tag{61}$$

The averaged frequencies are

$$\omega_i(A_i) = b_{0i}(A_i). \tag{62}$$

By using transformations (12), the following equations for A_i and Θ_i are obtained from Equation (54):

$$\begin{aligned} \frac{dA_i}{dt} &= F_{i1}^{(1)}(\mathbf{A}, \mathbf{\Phi}) + F_{i2}^{(1)}(\mathbf{A}, \mathbf{\Phi}, \mathbf{u}) + G_{i1}^{(1)}(\mathbf{A}, \mathbf{\Phi})\xi_{2i-1}(t) + G_{i2}^{(1)}(\mathbf{A}, \mathbf{\Phi})\xi_{2i}(t), \\ \frac{d\Theta_i}{dt} &= F_{i1}^{(2)}(\mathbf{A}, \mathbf{\Phi}) + F_{i2}^{(2)}(\mathbf{A}, \mathbf{\Phi}, \mathbf{u}) + G_{i1}^{(2)}(\mathbf{A}, \mathbf{\Phi})\xi_{2i-1}(t) + G_{i2}^{(2)}(\mathbf{A}, \mathbf{\Phi})\xi_{2i}(t), \\ A_i(0) &= a_{i0}, \quad \Theta_i(0) = \theta_{i0}, \quad i = 1, 2, \end{aligned} \tag{63}$$

where

$$F_{i1}^{(1)} = \frac{-A_i}{g_i}(\beta_{i1}A_1v_1\sin\Phi_1 + \beta_{i2}A_2v_2\sin\Phi_2)v_i\sin\Phi_i,$$

$$F_{i1}^{(2)} = \frac{-1}{g_i}(\beta_{i1}A_1v_1\sin\Phi_1 + \beta_{i2}A_2v_2\sin\Phi_2)v_i\cos\Phi_i,$$

$$F_{i2}^{(1)} = \frac{-A_i}{g_i}u_iv_i\sin\Phi_i; \quad F_{i2}^{(2)} = \frac{-1}{g_i}u_iv_i\cos\Phi_i,$$

$$G_{i1}^{(1)} = \frac{-f_{ii}A_i^2}{g_i}v_i\sin\Phi_i\cos\Phi_i, \quad G_{i1}^{(2)} = \frac{-f_{ii}A_i}{g_i}v_i\cos^2\Phi_i,$$

$$G_{i2}^{(1)} = \frac{-f_{i,3-i}A_i}{g_i}v_i\sin\Phi_i, \quad G_{i2}^{(2)} = \frac{-f_{i,3-i}}{g_i}v_i\cos\Phi_i. \tag{64}$$

By inserting Equation (60) into Equation (64) and applying the stochastic averaging, the following averaged Itô equations are obtained from Equation (63):

$$\mathrm{d}A_i = [m_{i1}(A) + \langle F_{i2}^{(1)}(\mathbf{A}, \boldsymbol{\Phi}, \mathbf{u})\rangle_t]\,\mathrm{d}t + \sigma_{ik}(A)\,\mathrm{d}B_k(t),$$
$$A_i(0) = a_{i0}, \quad i, k = 1, 2. \tag{65}$$

where m_{i1} and σ_{ik} are obtained by using formulas in Equation (19) and are given in Appendix 1.

The relations between A_i and H_i obtained from Equation (56) are

$$A_i = U_i^{-1}(H_i) = \sqrt{\frac{\sqrt{\omega_i^4 + 4\alpha_i H_i} - \omega_i^2}{\alpha_i}}. \tag{66}$$

The averaged Itô equations for H_i are then

$$\mathrm{d}H_i = [\bar{m}_{i1}(\mathbf{H}) + \langle u_i P_i\rangle_t]\,\mathrm{d}t + \bar{\sigma}_{ik}(\mathbf{H})\,\mathrm{d}B_k(t),$$
$$H_i(0) = H_{i0}, \quad i, k = 1, 2, \tag{67}$$

where

$$\bar{m}_{i1}(\mathbf{H}) = \left[(\omega_i^2A_i + \alpha_iA_i^3)m_{i1}(\mathbf{A}) + \frac{1}{2}(\omega_i^2 + 3\alpha_iA_i^2)\sigma_{ii}^2(\mathbf{A})\right]\Bigg|_{A_i=U_i^{-1}(H_i)},$$

$$\bar{b}_{ii}(\mathbf{H}) = \sigma_{ik}(\mathbf{H})\alpha_{ik}(\mathbf{H}) = [(\omega_i^2A_i + \alpha_iA_i^3)^2\sigma_{ii}^2(\mathbf{A})]\big|_{A_i=U_i^{-1}(H_i)}, \quad \bar{b}_{ij} = 0, i \neq j. \tag{68}$$

The dynamical programming equation for the control problem of reliability maximization is of the same form as Equation (25) with $\bar{b}_{11}$, $\bar{b}_{22}$, $\bar{b}_{12} = \bar{b}_{21} = 0$, and $\bar{m}_{11}$, $\bar{m}_{21}$ defined by Equation (68). Suppose that the control constraints are of the same form as Equation (33), then the optimal control forces u_i^* are in the form of Equation (35) and the final dynamical programming equation is in the form of Equation (36) with $\bar{m}_i(\mathbf{H}) = \bar{m}_{i1}(\mathbf{H}) + \langle u_i^* P_i\rangle_t$, where

$$\langle u_i^* P_i\rangle_t = -\frac{2A_ic_i}{\pi}\left(b_{0i} - \frac{b_{2i}}{3} - \frac{b_{4i}}{15} - \frac{b_{6i}}{35}\right)\Bigg|_{A_i=U_i^{-1}(H_i)}. \tag{69}$$

Following the discussion of the last section, the conditional reliability function of the optimally controlled system can be obtained from solving the following backward Kolmogorov equation:

$$\left[-\frac{\partial}{\partial t}+\bar{m}_1(\mathbf{H}_0)\frac{\partial}{\partial H_{10}}+\bar{m}_2(\mathbf{H}_0)\frac{\partial}{\partial H_{20}}+\frac{1}{2}\bar{b}_{11}(\mathbf{H}_0)\frac{\partial^2}{\partial H_{10}^2}+\frac{1}{2}\bar{b}_{22}(\mathbf{H}_0)\frac{\partial^2}{\partial H_{20}^2}\right]R_{\text{opt}}=0. \quad (70)$$

It is seen from Equation (56) that H_i may vary randomly in $[0, \infty)$ if $\alpha_i \geq 0$. Generally, the boundary of safety domain of system (54) is a function of H_1 and H_2. To be specific, suppose that the boundary is

$$\Gamma_c: \quad H_1 + H_2 = H_c, \quad H_1, H_2 \geq 0. \quad (71)$$

The safety domain is the inside of the right triangle consisting of boundary Γ_c defined by Equation (71) and Γ_0 defined by

$$\begin{aligned}
\Gamma_0 &= \Gamma_{01}+\Gamma_{02}+\Gamma_{03},\\
\Gamma_{01} &: \quad H_1=0, \quad 0<H_2<H_c,\\
\Gamma_{02} &: \quad H_2=0, \quad 0<H_1<H_c,\\
\Gamma_{03} &: \quad H_1=H_2=0
\end{aligned} \quad (72)$$

(see Figure 1). Thus, one boundary condition for Equation (70) is Equation (42) with Γ_c defined by Equation (71). The other qualitative boundary condition (43) can be converted into a quantitative one by using Equation (70) and the limiting the behavior of $\bar{m}_i$ and $\bar{b}_{ij}$ at Γ_0 defined by Equation (72). Since $\bar{b}_{11}(\mathbf{H}_0) \to 0$ as $H_{10} \to 0$ and $\bar{b}_{22}(\mathbf{H}_0) \to 0$ as $H_{20} \to 0$, the quantitative boundary condition at Γ_0 is

$$\left[-\frac{\partial}{\partial t}+\bar{m}_1(\mathbf{H}_0)\frac{\partial}{\partial H_{10}}+\bar{m}_2(\mathbf{H}_0)\frac{\partial}{\partial H_{20}}+\frac{1}{2}\bar{b}_{22}(\mathbf{H}_0)\frac{\partial^2}{\partial H_{20}^2}\right]R_{\text{opt}}=0 \quad \text{at } H_{10}=0 \quad (73)$$

or

$$\left[-\frac{\partial}{\partial t}+\bar{m}_1(\mathbf{H}_0)\frac{\partial}{\partial H_{10}}+\bar{m}_2(\mathbf{H}_0)\frac{\partial}{\partial H_{20}}+\frac{1}{2}\bar{b}_{11}(\mathbf{H}_0)\frac{\partial^2}{\partial H_{10}^2}\right]R_{\text{opt}}=0 \quad \text{at } H_{20}=0 \quad (74)$$

or

$$\left[-\frac{\partial}{\partial t}+\bar{m}_1(\mathbf{H}_0)\frac{\partial}{\partial H_{10}}+\bar{m}_2(\mathbf{H}_0)\frac{\partial}{\partial H_{20}}\right]R_{\text{opt}}=0 \quad \text{at } H_{10}=H_{20}=0. \quad (75)$$

Equation (70) is solved numerically together with boundary conditions (42), (73)–(75) and initial condition (44) to yield the conditional reliability function of the optimally controlled system (54). Then, the probability density of the first-passage time of the optimally controlled system (54) is obtained by using Equation (48).

Similarly, the mean first-passage time of optimally controlled system (54), $\mu_{1,\text{opt}}$ can be obtained from solving the following Pontryagin equation:

$$\left[\bar{m}_1(\mathbf{H}_0)\frac{\partial}{\partial H_{10}}+\bar{m}_2(\mathbf{H}_0)\frac{\partial}{\partial H_{20}}+\frac{1}{2}\bar{b}_{11}(\mathbf{H}_0)\frac{\partial^2}{\partial H_{10}^2}+\frac{1}{2}\bar{b}_{22}(\mathbf{H}_0)\frac{\partial^2}{\partial H_{20}^2}\right]\mu_{1,\text{opt}}=-1. \quad (76)$$

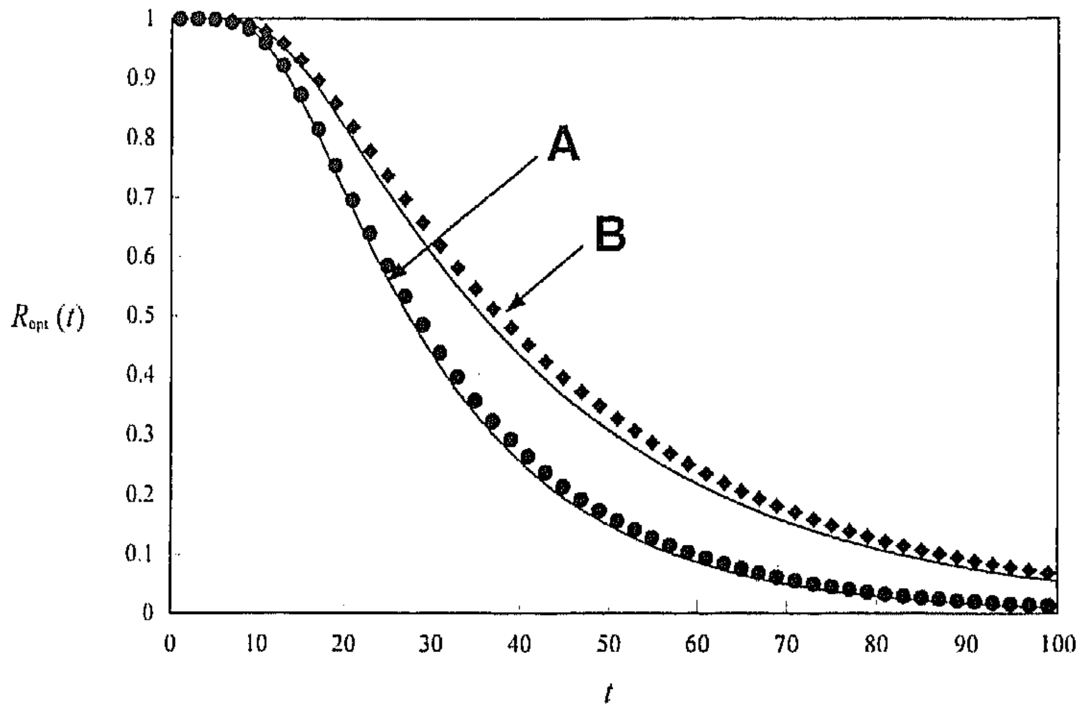

Figure 2. Reliability function of system (54) for given initial condition. The parameters of system are $\beta_{11} = 0.01$, $\beta_{12} = 0.03$, $\omega_1 = 1$, $\alpha_1 = 0.1$, $f_{11} = 1.2$, $f_{12} = 0.7$, $\beta_{21} = 0.02$, $\beta_{22} = 0.02$, $\omega_2 = 1.414$, $\alpha_2 = 0.2$, $f_{22} = 1.5$, $f_{12} = 0.7$, $H_c = 0.8$, $H_{10} = H_{20} = 0$. The parameters of external excitations are $2D_2 = 2$, $\bar{\omega}_2 = 7$. The parameters of parametric excitations are $2D_1 = 2.6$, $\bar{\omega}_1 = 8$. The control parameters are $c_1 = c_2 = 0$ for **A**; $c_1 = 0.01$, $c_2 = 0.015$ for **B**. — analytical result by using the proposed procedure; • ♦ from digital simulation.

One boundary condition of Equation (76) is Equation (51) with Γ_c defined by Equation (71). The other qualitative boundary condition, Equation (52), is converted into quantitative one by using Equation (76) and $\bar{m}_i$, $\bar{b}_{ij}$ at Γ_0. It is

$$\left[\bar{m}_1(\mathbf{H}_0)\frac{\partial}{\partial H_{10}} + \bar{m}_2(\mathbf{H}_0)\frac{\partial}{\partial H_{20}} + \frac{1}{2}\bar{b}_{22}(\mathbf{H}_0)\frac{\partial^2}{\partial H_{20}^2}\right]\mu_{1,\text{opt}} = -1 \quad \text{at } H_{10} = 0 \tag{77}$$

or

$$\left[\bar{m}_1(\mathbf{H}_0)\frac{\partial}{\partial H_{10}} + \bar{m}_2(\mathbf{H}_0)\frac{\partial}{\partial H_{20}} + \frac{1}{2}\bar{b}_{11}(\mathbf{H}_0)\frac{\partial^2}{\partial H_{10}^2}\right]\mu_{1,\text{opt}} = -1 \quad \text{at } H_{20} = 0 \tag{78}$$

or

$$\left[\bar{m}_1(\mathbf{H}_0)\frac{\partial}{\partial H_{10}} + \bar{m}_2(\mathbf{H}_0)\frac{\partial}{\partial H_{20}}\right]\mu_{1,\text{opt}} = -1 \quad \text{at } H_{10} = H_{20} = 0. \tag{79}$$

Equation (76) is solved together with Equations (51), (77–79) to yield the mean first-passage time of optimally controlled system (54).

Some numerical results obtained by using the proposed procedure for the conditional reliability function, the conditional probability density and mean of the first-passage time of uncontrolled and optimally controlled system (54) are shown in Figures 2–4. Similar results from digital simulation of the original system (54) are also shown in the figures for comparison. From these figures it is seen that the two results are in rather good agreement and the control improves the reliability of the system greatly. Some more results for the reliability function, the probability density and mean of first-passage time of optimally controlled system (54) as functions of the initial condition are shown in Figures 5–7. It is seen from these figures that both the reliability function and mean first-passage time of optimally controlled system (54) are monotonously decreasing functions of H_{10} and H_{20}, which justifies the derivation from Equation (34) to (35).

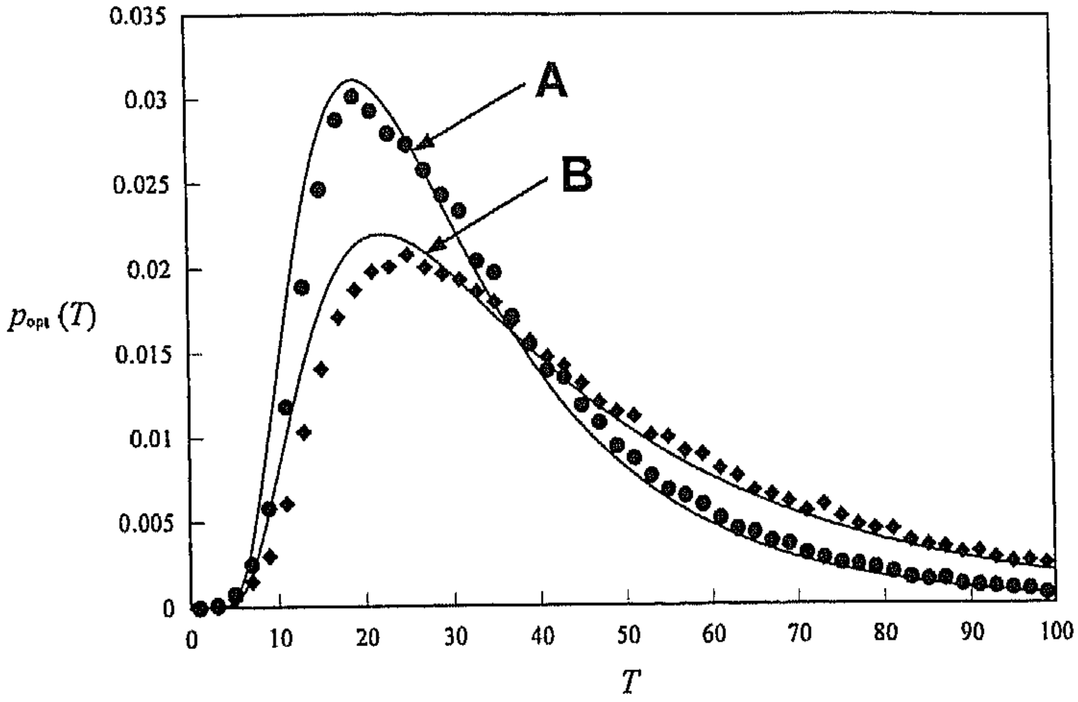

Figure 3. Probability density of first-passage time of system (54) for a given initial condition. The parameters and symbols are the same as those in Figure 2.

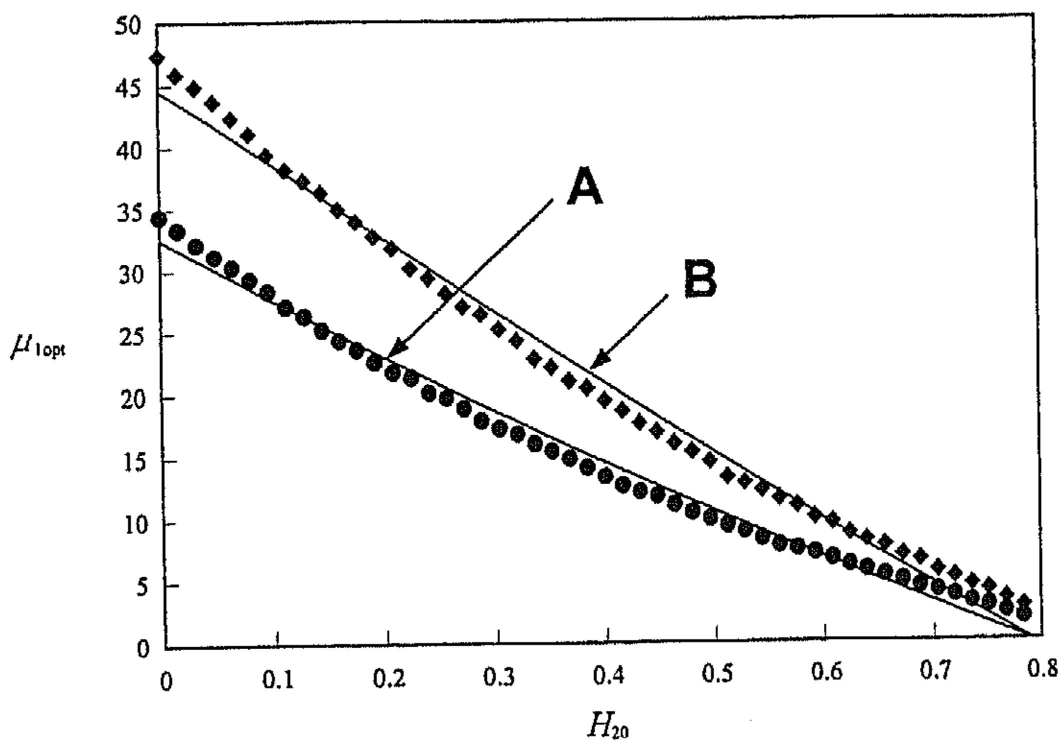

Figure 4. Mean first-passage time of system (54) as a function of H_{20} for given $H_{10} = 0$. The other parameters and symbols are the same as those in Figure 2.

6. Conclusions

In this paper, a procedure for designing the optimally bounded control of quasi-integrable Hamiltonian systems with wide-band random excitations for minimizing the first-passage failure is proposed. The procedure consists of applying the stochastic averaging method for quasi-Hamiltonian systems with wide-band random excitations, establishing the dynamical programming equations for maximizing the reliability function and for maximizing mean first-passage time based on the averaged Itô equations using the dynamical programming principle, determining the optimally control from the dynamical programming equations and control constraints, and obtaining the conditional reliability function, conditional probability density and mean of first-passage time of the optimally controlled system from solving the associated backward Kolmogorov equation and Pontryagin equation. An example has been worked out

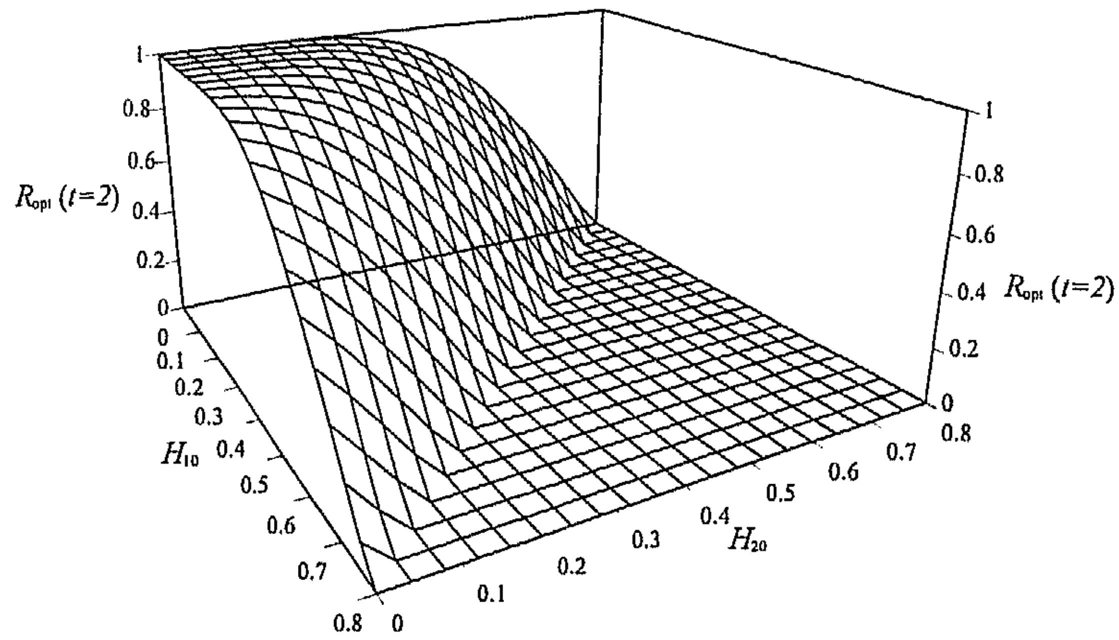

Figure 5. Reliability of optimally controlled system (54) at $t = 2$ as a function of H_{10} and H_{20}. The other parameters are the same as those in Figure 2(**B**).

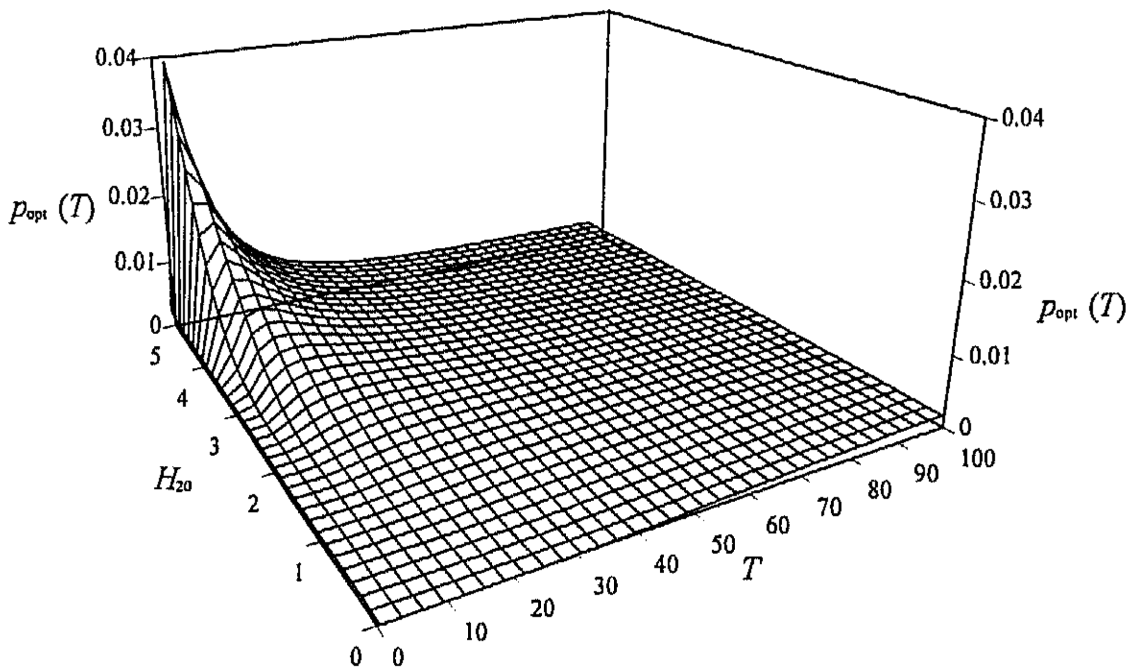

Figure 6. Probability density of first-passage time of optimally controlled system (54) as a function of H_{20} and t for given $H_{10} = 0$. The other parameters are the same as those in Figure 2(**B**).

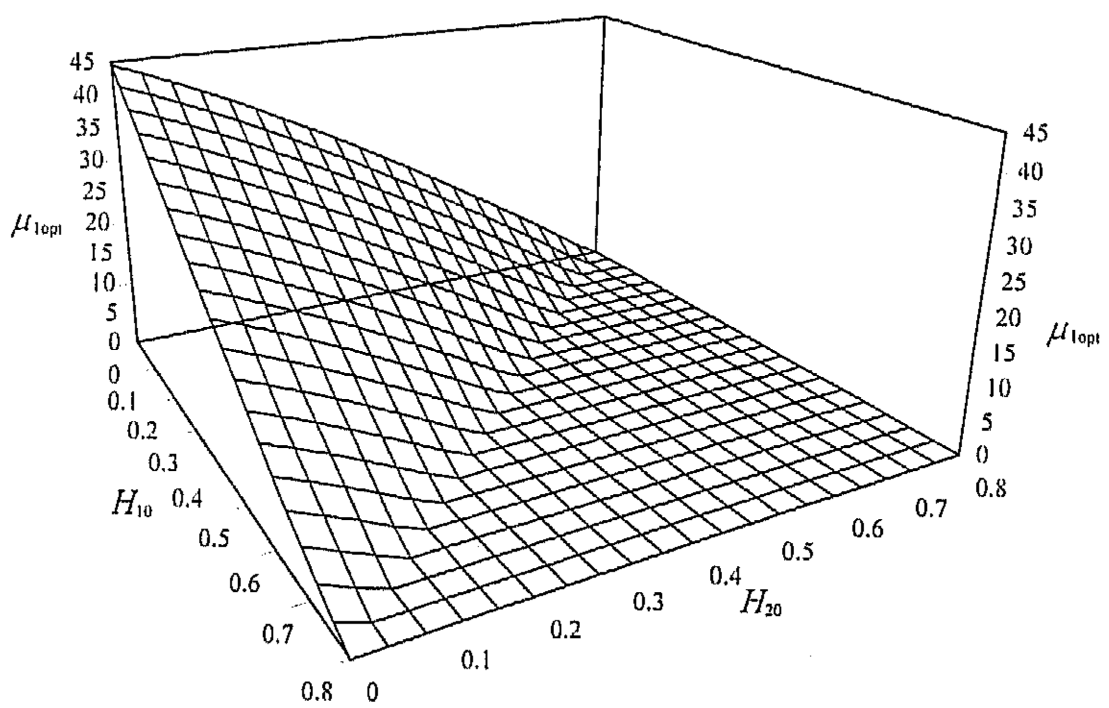

Figure 7. Mean first-passage time of system optimally controlled (54) as a function of H_{10} and H_{20}. The other parameters are the same as those in Figure 2(**B**).

in detail to illustrate the application of the proposed procedure. The comparison between the analytical results and those from digital simulation shows that the proposed procedure works quite well.

Appendix 1

The drift and diffusion coefficients in Equation (65):

$$
\begin{aligned}
m_{i1} &= \frac{-A_i^2}{8g_i}\beta_{i1}(4\omega_i^2+\frac{5}{2}\alpha_i A_i^2)+\frac{\pi f_{ii}^2 A_i^2}{32g_i}\Bigg\{(2b_{0i}-b_{4i}) \\
&\times\left[\frac{\mathrm{d}}{\mathrm{d}A_i}\left[\frac{A_i^2(2b_{0i}-b_{4i})}{g_i}\right]+\frac{2A_i}{g_i}(2b_{0i}+2b_{2i}+b_{4i})\right]S_{11}(2\omega_i) \\
&+(b_{2i}-b_{6i})\left[\frac{\mathrm{d}}{\mathrm{d}A_i}\left[\frac{A_i^2(b_{2i}-b_{6i})}{g_i}\right]+\frac{4A_i}{g_i}(b_{2i}+2b_{4i}+b_{6i})\right]S_{11}(4\omega_i) \\
&+b_{4i}\left[\frac{\mathrm{d}}{\mathrm{d}A_i}\left[\frac{A_i^2 b_{4i}}{g_i}\right]+\frac{6A_i}{g_i}(b_{4i}+2b_{6i})\right]S_{11}(6\omega_i) \\
&+b_{6i}\left[\frac{\mathrm{d}}{\mathrm{d}A_i}\left[\frac{A_i^2 b_{6i}}{g_i}\right]+\frac{8A_i}{g_i}b_{6i}\right]S_{11}(8\omega_i)\Bigg\} \\
&+\frac{\pi f_{i,3-i}^2 A_i}{8g_i}\Bigg\{(2b_{0i}-b_{2i})\left[\frac{d}{dA_i}\left[\frac{A_i(2b_{0i}-b_{2i})}{g_i}\right]+\frac{(2b_{0i}+b_{2i})}{g_i}\right]S_{22}(\omega_i) \\
&+(b_{2i}-b_{4i})\left[\frac{\mathrm{d}}{\mathrm{d}A_i}\left[\frac{A_i(b_{2i}-b_{4i})}{g_i}\right]+\frac{3(b_{2i}+b_{4i})}{g_i}\right]S_{22}(3_i\omega_i) \\
&+(b_{4i}-b_{6i})\left[\frac{\mathrm{d}}{\mathrm{d}A_i}\left[\frac{A_i(b_{4i}-b_{6i})}{g_i}\right]+\frac{5(b_{4i}+b_{6i})}{g_i}\right]S_{22}(5\omega_i) \\
&+b_{6i}\left[\frac{\mathrm{d}}{\mathrm{d}A_i}\left[\frac{A_i b_{6i}}{g_i}\right]+\frac{7b_{6i}}{g_i}\right]S_{22}(7\omega_i)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{ii}^2 &= [\sigma\sigma^T]_{ii}=\frac{\pi f_{ii}^2 A_i^4}{16g_i^2}[(2b_{0i}-b_{4i})^2 S_{11}(2\omega_i)+(b_{2i}-b_{6i})^2 S_{11}(4\omega_i)+b_{4i}^2 S_{11}(6\omega_i) \\
&+b_{6i}^2 S_{11}(8\omega_i)]+\frac{\pi f_{i,3-i}^2 A_i^2}{4g_i^2}[(2b_{0i}-b_{2i})^2 S_{22}(\omega_i)+(b_{2i}-b_{4i})^2 S_{22}(3\omega_i) \\
&+(b_{4i}-b_{6i})^2 S_{22}(5\omega_i)+b_{6i}^2 S_{22}(7\omega_i)] \\
\sigma_{ij}^2 &= [\sigma\sigma^T]_{ij}=0,\quad i\neq j.
\end{aligned}
$$

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