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Torsional impact of transversely isotropic solid with functionally graded shear moduli and a penny-shaped crack

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Abstract

The torsional impact response of a penny-shaped crack in an unbounded transversely isotropic solid is considered. The shear moduli are assumed to be functionally graded such that the mathematics is tractable. Laplace transform and Hankel transform are used to reduce the problem to solving a Fredholm integral equation. The crack tip stress fields are obtained. Investigated are the influence of material nonhomogeneity and orthotropy on the dynamic stress intensity factor. The peak value of the dynamic stress intensity factor can be suppressed by increasing the shear moduli's gradient and/or increasing the shear modulus in a direction perpendicular to the crack surface.

Keywords: Anisotropic media; Functionally graded material; Dynamic stress intensity factor; Crack; Impact

1. Introduction

The so-called Functionally Gradient Material (FGM) has revived the interest in solving boundary value problems with crack-like discontinuity. The nonhomogeneity of the elastic body is assumed to depend on the coordinates while the resulting equations could still be solved analytically. While such an approach has been used to solve nonhomogeneous elasticity problems in the past, they did not receive the attention as in recent years because of the advent of composites such that FGMs could now be made and used in applications. Materials possessing functionally graded nonhomogeneity and containing cracks have been studied extensively [1] for the isotropic case. They include both nonhomogeneity in the axial and radial direction while the crack configuration includes the penny-shaped and flat elliptical crack. Solution to a class of problems for anti-plane shear and in-plane extension can be found in [1]. Depending on the nature of nonhomogeneity, stress singularity other than the inverse square root type was found. Because of the techniques used to process the FGMs, they are seldom isotropic. A plasma spray technique would usually lead to a lamellar structure while electron beam vapor deposition can be used to yield a highly columnar structure. It is therefore necessary to consider the anisotropic character of the FGMs. The Mode I static crack problem in an nonhomogeneous orthotropic medium has been analyzed [2]. An exponential form was used. In what follows, the torsional impact of a penny-shaped crack in a transversely isotropic FGM is considered. The objective is to obtain the local dynamic stress field and to examine the effects of material nonhomogeneity and orthotropy.

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2. Material property model

Assumed in the FGM model are different variations of the shear modulus. In [1], both types $\mu(z) = \mu_0|z|^m (m > 0)$ and $\mu(z) = \mu_0(c + |z|)^m (c \neq 0)$ have been assumed, where m can be positive and negative in the latter case. In the former case, if $m \neq 0$ the order of the stress singularity was found to depend on $(1 - m)/2$ [1], while the singularity for radial nonhomogeneity of the same functional form is the same as the homogeneous material. The cases for $\mu(y) = \mu_0(1 + c|y|)$ and $\mu(y) = \mu_0 \exp(\gamma y)$ were considered in [3–5], respectively.

Consider an unbounded orthotropic functionally graded material as shwon in Fig. 1. The coordinates r and z are assumed to be the principal axes of orthotropy. The shear moduli μ_r and μ_z are assumed to be functions of z only, and vary proportionately as

$$\mu_r(z) = (\mu_r)_0(1 + \alpha|z|)^2, \tag{1}$$

$$\mu_z(z) = (\mu_z)_0(1 + \alpha|z|)^2, \tag{2}$$

where α is a constant $(\alpha > 0)$, $(\mu_r)_0$ and $(\mu_z)_0$ are the shear moduli at $z = 0$.

3. Problem formulation

Fig. 1 considers a penny-shaped crack of diameter $2a$. It is embedded in a functionally graded material and lies in the $z = 0$ plane. The solid extends to infinity in all directions. In cylindrical polar coordinates (r, θ, z), the displacements are denoted u_r, u_θ and u_z. For the present problem, we have

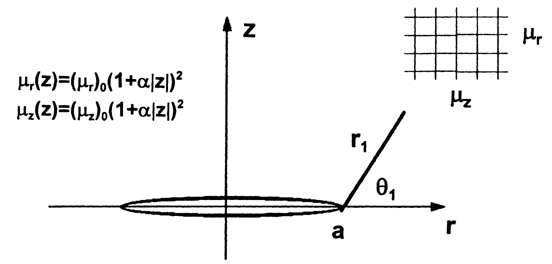

Fig. 1. Penny-shaped crack in transversely isotropic FGM.

$$u_r = u_z = 0, \quad u_\theta = u_\theta(r, z, t), \tag{3}$$

where t is time. The nonvanishing stress components $\tau_{\theta z}$ and $\tau_{r\theta}$ are

$$\tau_{\theta z} = \mu_z \frac{\partial u_\theta}{\partial z}, \quad \tau_{r\theta} = \mu_r \left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right), \tag{4}$$

where the shear moduli μ_r and μ_z are assumed to vary according to Eqs. (1) and (2).

Two of the equations of motion are identically satisfied and the remaining one gives

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\mu_z(z)}{\mu_r(z)}\frac{\partial^2 u_\theta}{\partial z^2} + \frac{\mu_z'(z)}{\mu_r(z)}\frac{\partial u_\theta}{\partial z} = \frac{\rho}{\mu_r(z)}\frac{\partial^2 u_\theta}{\partial t^2}. \tag{5}$$

Prime denotes derivative. The mass density is ρ being a constant.

Suppose that the material is initially at rest. At time $t = 0$, a shear linearly proportional to r, is applied suddenly to crack surfaces and maintained constant thereafter. This is equivalent to a torque. Hence, the boundary conditions are

$$\tau_{\theta z}(r, 0, t) = -\tau_0 H(t) r/a, \quad 0 \leqslant r < a; \; t > 0, \tag{6}$$

$$u_\theta(r, 0, t) = 0, \quad r \geqslant a; \; t > 0, \tag{7}$$

where $H(t)$ is the Heaviside unit step function.

4. Integral equation

The Laplace transform of $f(t)$ is

$$f^*(p) = \int_0^\infty f(t) \mathrm{e}^{-pt} \, \mathrm{d}t \tag{8}$$

whose inversion is

$$f(t) = \frac{1}{2\pi i} \int_{\mathrm{Br}} f^*(p) \, \mathrm{e}^{pt} \, \mathrm{d}p. \tag{9}$$

The Bromwich path of integration is denoted by Br which is a line on the right-hand side of the p-plane and parallel to the imaginary axis. Applying the transform Eq. (8) to Eq. (5) results in the transformed equation

$$\frac{\partial^2 u_\theta^*}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta^*}{\partial r}-\frac{u_\theta^*}{r^2}+\frac{\mu_z(z)}{\mu_r(z)}\frac{\partial^2 u_\theta^*}{\partial z^2}+\frac{\mu_z'(z)}{\mu_r(z)}\frac{\partial u_\theta^*}{\partial z} = \frac{\rho p^2}{\mu_r(z)}u_\theta^*. \quad (10)$$

Because of the symmetry, it suffices to consider only the upper half-space $z > 0$. Moreover, consider the pair of Hankel transforms of the first order,

$$V(s,z,p) = \int_0^\infty u_\theta^*(r,z,p)J_1(sr)r\,\mathrm{d}r, \quad (11)$$

$$u_\theta^*(r,z,p) = \int_0^\infty V(s,z,p)J_1(sr)s\,\mathrm{d}s, \quad (12)$$

where $J_1(\,)$ is the Bessel function of the first kind. Application of Eq. (11) to Eq. (10) yields

$$\frac{\mu_z(z)}{\mu_r(z)}\frac{\partial^2 V(s,z,p)}{\partial z^2}+\frac{\mu_z'(z)}{\mu_r(z)}\frac{\partial V(s,z,p)}{\partial z} - \left[s^2+\frac{\rho p^2}{\mu_r(z)}\right]V(s,z,p) = 0. \quad (13)$$

Substituting Eqs. (1) and (2) into Eq. (13), results in

$$\frac{\partial^2 V(s,z,p)}{\partial z^2}+\frac{2\alpha}{1+\alpha z}\frac{\partial V(s,z,p)}{\partial z} - \left[S^2+\frac{\rho p^2}{(\mu_z)_0(1+\alpha z)^2}\right]V(s,z,p) = 0, \quad (14)$$

where $S = s\sqrt{(\mu_r)_0/(\mu_z)_0}$. By defining

$$X = S(1+\alpha z), \quad Y = (1+\alpha z)^{1/2}V, \quad (15)$$

Eq. (14) can be rewritten as

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}X^2}+\frac{1}{X}\frac{\mathrm{d}Y}{\mathrm{d}X}-\left[\frac{1}{\alpha^2}+\frac{\beta^2}{X^2}\right]Y = 0, \quad (16)$$

where

$$\beta = \sqrt{\frac{1}{4}+\frac{\rho p^2}{(\mu_z)_0\alpha^2}}. \quad (17)$$

Eq. (16) is a modified Bessel differential equation. From the solution of Eq. (16) and considering the regularity condition at $z \to \infty$, the solution of Eq. (14) can be expressed as

$$V(s,z,p) = A(s,p)(1+\alpha z)^{-1/2}K_\beta\left[(1+\alpha z)\frac{S}{\alpha}\right], \quad (18)$$

where $K_\beta(\,)$ is the modified Bessel function of the second kind.

Substituting Eq. (18) into Eq. (12), the result is

$$u_\theta^*(r,z,p) = \int_0^\infty A(s,p)(1+\alpha z)^{-1/2}K_\beta\left[(1+\alpha z)\frac{S}{\alpha}\right] \times J_1(sr)s\,\mathrm{d}s. \quad (19)$$

From Eq. (19), the Laplace transform of the stresses $\tau_{\theta z}$ and $\tau_{r\theta}$ are obtained:

$$\tau_{\theta z}^*(r,z,p) = -\mu_z(z)\int_0^\infty A(s,p) \times \left\{\frac{\alpha}{2}(1+\alpha z)^{-3/2}K_\beta\left[(1+\alpha z)\frac{S}{\alpha}\right] - S(1+\alpha z)^{-1/2}K_\beta'\left[(1+\alpha z)\frac{S}{\alpha}\right]\right\}J_1(sr)s\,\mathrm{d}s, \quad (20)$$

$$\tau_{r\theta}^*(r,z,p) = \mu_r(z)\int_0^\infty A(s,p)(1+\alpha z)^{-1/2}K_\beta \times \left[(1+\alpha z)\frac{S}{\alpha}\right]\left[sJ_1'(sr)-\frac{J_1(sr)}{r}\right]s\,\mathrm{d}s, \quad (21)$$

where prime denotes derivative. In the Laplace transform domain, the boundary conditions on the plane $z = 0$ become

$$\tau_{\theta z}^*(r,0,p) = -\frac{\tau_0 r}{pa}, \quad 0 \leqslant r < a, \quad (22)$$

$$u_\theta^*(r,0,p) = 0, \quad r \geqslant a. \quad (23)$$

From Eqs. (19), (20), (22) and (23), a pair of dual integral equations are obtained as

$$\int_0^\infty B(s,p)J_1(sr)\,\mathrm{d}s = 0, \quad r \geqslant a, \quad (24)$$

$$\int_0^\infty sB(s,p)G(s,p)J_1(sr)\,\mathrm{d}s = \frac{\tau_0 r}{(\mu_z)_0 pa}, \quad 0 \leqslant r < a, \quad (25)$$

where

$$B(s,p) = sA(s,p)K_\beta\left(\frac{S}{\alpha}\right), \tag{26}$$

$$G(s,p) = \frac{\frac{\alpha}{2}K_\beta\left(\frac{S}{\alpha}\right) - SK'_\beta\left(\frac{S}{\alpha}\right)}{sK_\beta\left(\frac{S}{\alpha}\right)}. \tag{27}$$

The dual integral Eqs. (24) and (25) can be solved by applying the method in [6], which has been extensively used for the case of homogeneous materials with cracks [7] and for the case of layered composites with cracks [8]. The solution of Eqs. (24) and (25) is given by

$$B(s,p) = \frac{4\tau_0 a^{5/2}}{3(\mu_z)_0 p\sqrt{2\pi}}\sqrt{s} \times \int_0^1 \sqrt{\xi}\Phi^*(\xi,p)J_{3/2}(sa\xi)\mathrm{d}\xi, \tag{28}$$

where $\Phi^*(\xi,p)$ should satisfy the Fredholm integral equation of the second kind

$$\Phi^*(\xi,p) + \int_0^1 \Phi^*(\xi,p)M(\xi,\eta,p)\mathrm{d}\eta = \xi^2. \tag{29}$$

The kernel function $M(\xi,\eta,p)$ in Eq. (29) is computed from

$$M(\xi,\eta,p) = \sqrt{\xi\eta}\int_0^\infty s\left[G\left(\frac{s}{a},p\right) - 1\right] \times J_{3/2}(s\xi)J_{3/2}(s\eta)\,\mathrm{d}s. \tag{30}$$

The Fredholm integral equation of the second kind in Eq. (29) can be solved numerically.

5. Dynamic stress intensity factor

The Laplace transform of the dynamic stress intensity factor can be extracted from the asymptotic expansion of the stresses around the crack tip in the Laplace transform domain.

Integration of $B(s,p)$ in Eq. (28) by parts gives

$$B(s,p) = \frac{4\tau_0 a^{3/2}}{3(\mu_z)_0 p\sqrt{2\pi}}\frac{1}{\sqrt{s}}\left\{ -\Phi^*(1,p)J_{1/2}(sa) + \int_0^1 \frac{1}{\sqrt{\xi}}J_{1/2}(sa\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}[\xi\Phi^*(\xi,p)]\mathrm{d}\xi\right\}. \tag{31}$$

From Eqs. (31), (26), (20) and (21), it is found that

$$\tau^*_{\theta z}(r,z,p) = \frac{4\tau_0 a^{3/2}\mu_z(z)}{3(\mu_z)_0\sqrt{2\pi}}\sqrt{\frac{(\mu_r)_0}{(\mu_z)_0}}\frac{\Phi^*(1,p)}{p} \times \int_0^\infty -\sqrt{s}\frac{(1+\alpha z)^{-1/2}K'_\beta\left[(1+\alpha z)\frac{S}{\alpha}\right]}{K_\beta\left(\frac{S}{\alpha}\right)} \times J_{1/2}(sa)J_1(sr)\mathrm{d}s + \cdots, \tag{32}$$

$$\tau^*_{r\theta}(r,z,p) = \frac{4\tau_0 a^{3/2}\mu_r(z)}{3(\mu_z)_0\sqrt{2\pi}}\frac{\Phi^*(1,p)}{p} \times \int_0^\infty -\sqrt{s}\frac{(1+\alpha z)^{-1/2}K_\beta\left[(1+\alpha z)\frac{S}{\alpha}\right]}{K_\beta\left(\frac{S}{\alpha}\right)} \times J_{1/2}(sa)J'_1(sr)\mathrm{d}s + \cdots \tag{33}$$

Note that the infinite integrals are convergent everywhere except at the singular points which occupy the crack edge. Since the solution near the edge is desired it is necessary to evaluate the unbounded portions of these integrals in the neighbourhood of the singular points. Note that the integrals are finite and continuous for any given values of s, the divergence of the integrals at the crack edge corresponds to $s \to \infty$. Carrying out the expansion for large s and considering the following asymptotic behavior of $K_\beta(x)$ and $K'_\beta(x)$ when $x \to \infty$,

$$K_\beta(x) = \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}\left[1 + \mathrm{O}\left(\frac{1}{x}\right)\right], \tag{34}$$

$$K'_\beta(x) = -\sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}\left[1 + \mathrm{O}\left(\frac{1}{x}\right)\right], \tag{35}$$

the lower-order terms of the stresses are

$$\tau^*_{\theta z}(r,z,p) = \frac{4\tau_0 a\mu_z(z)}{3\pi(\mu_z)_0}\sqrt{\frac{(\mu_r)_0}{(\mu_z)_0}}\frac{\Phi^*(1,p)}{p}(1+\alpha z)^{-1} \times \int_0^\infty \sin(sa)\exp(-Sz)J_1(sr)\,\mathrm{d}s = \frac{4\tau_0 a}{3\pi}\sqrt{\frac{(\mu_r)_0}{(\mu_z)_0}}\frac{\Phi^*(1,p)}{p}(1+\alpha z) \times \int_0^\infty \sin(sa)\exp(-Sz)J_1(sr)\,\mathrm{d}s, \tag{36}$$

$$\tau^*_{r\theta}(r,z,p) = \frac{4\tau_0 a\mu_r(z)}{3\pi(\mu_z)_0}\frac{\Phi^*(1,p)}{p}(1+\alpha z)^{-1} \times \int_0^\infty \sin(sa)\exp(-Sz)J_0(sr)\,\mathrm{d}s = -\frac{4\tau_0 a}{3\pi}\frac{(\mu_r)_0}{(\mu_z)_0}\frac{\Phi^*(1,p)}{p}(1+\alpha z) \int_0^\infty \sin(sa)\exp(-Sz)J_0(sr)\,\mathrm{d}s. \tag{37}$$

Define $\gamma = \sqrt{(\mu_r)_0/(\mu_z)_0}$, the integrals in Eqs. (36) and (37) are evaluated:

$$\int_0^\infty \sin(sa)\exp(-Sz)J_1(sr)\,\mathrm{d}s = \mathrm{Im}\left[\int_0^\infty \exp[is(a+i\gamma z)]J_1(sr)\,\mathrm{d}s\right] = \mathrm{Im}\left[\frac{1}{r}\left(1+\frac{i(a+i\gamma z)}{\sqrt{r^2-(a+i\gamma z)^2}}\right)\right], \tag{38}$$

$$\int_0^\infty \sin(sa)\exp(-sz)J_0(sr)\,\mathrm{d}s = \mathrm{Im}\left[\int_0^\infty \exp[is(a+i\gamma z)]J_0(sr)\mathrm{d}s\right] = \mathrm{Im}\left[\frac{1}{\sqrt{r^2-(a+i\gamma z)^2}}\right]. \tag{39}$$

Note that $r = a + r_1\cos\theta_1,\ z = r_1\sin\theta_1$ and near the crack tip $r_1 \ll a$, this results in

$$\int_0^\infty \sin(sa)\exp(-Sz)J_1(sr)\,\mathrm{d}s = \frac{1}{\sqrt{2r_1 a}}\mathrm{Re}\left[\frac{1}{\sqrt{\cos\theta_1 - i\gamma\sin\theta_1}}\right] + \mathrm{O}(r_1^0), \tag{40}$$

$$\int_0^\infty \sin(sa)\exp(-Sz)J_0(sr)\,\mathrm{d}s = \frac{1}{\sqrt{2r_1 a}}\mathrm{Im}\left[\frac{1}{\sqrt{\cos\theta_1 - i\gamma\sin\theta_1}}\right] + \mathrm{O}(r_1^0). \tag{41}$$

The polar coordinates r_1 and θ_1 are defined in Fig. 1.

Substituting Eqs. (40) and (41) into Eq. (30), the local stress field is obtained:

$$r^*_{\theta z}(r_1,\theta_1,p) = \frac{K^*_{\mathrm{III}}(p)}{\sqrt{2\pi r_1}}\mathrm{Re}\left[\frac{1}{\sqrt{\cos\theta_1 - i\gamma\sin\theta_1}}\right] + \mathrm{O}(r_1^0), \tag{42}$$

$$r^*_{r\theta}(r_1,\theta_1,p) = \frac{K^*_{\mathrm{III}}(p)}{\sqrt{2\pi r_1}}\mathrm{Re}\left[\frac{i\gamma}{\sqrt{\cos\theta_1 - i\gamma\sin\theta_1}}\right] + \mathrm{O}(r_1^0). \tag{43}$$

The Laplace transform of the dynamic stress intensity factor $K^*_{\mathrm{III}}(p)$ in Eqs. (42) and (43) is

$$K^*_{\mathrm{III}}(p) = \sqrt{\frac{(\mu_r)_0}{(\mu_z)_0}}\frac{4}{3\pi}\tau_0\sqrt{\pi a}\frac{\Phi^*(1,p)}{p}, \tag{44}$$

in which $\Phi^*(1,p)$ is the value of $\Phi^*(\xi,p)$ evaluated at the crack edge corresponding to $\xi = 1$.

The dynamic stress intensity factor in the time domain is

$$K_{\mathrm{III}}(t) = \sqrt{\frac{(\mu_r)_0}{(\mu_z)_0}}\frac{4}{3\pi}\tau_0\sqrt{\pi a}\frac{1}{2\pi i}\int_{\mathrm{Br}}\frac{\Phi^*(1,p)}{p}\mathrm{e}^{pt}\,\mathrm{d}p. \tag{45}$$

6. Results and discussion

The functional dependence of the stresses on r_1 and θ_1 as shown in Eqs. (42) and (43) reveals that the local dynamic stresses in orthotropic functionally graded materials also possess the inverse square root singularity in terms of r_1 while the angular distribution in θ_1 is the same as that for the orthotropic layered composite [8].

The dynamic stress intensity factor shown in Eq. (45) is different than that of the homogeneous orthotropic material. The factor $\sqrt{(\mu_r)_0/(\mu_z)_0}$ appears for the orthotropic FGM; it could have a significant effect.

Fig. 2 shows the numerical results of $\gamma\Phi^*(1,p)$ as a function of the dimensionless Laplace transform wave number $(c_z)_{20}/pa$ for several different values of α and $\gamma = \sqrt{(\mu_r)_0/(\mu_z)_0}$. Here, $(c_z)_{20} = \sqrt{(\mu_z)_0/\rho}$. The influences of the nonhomogeneity

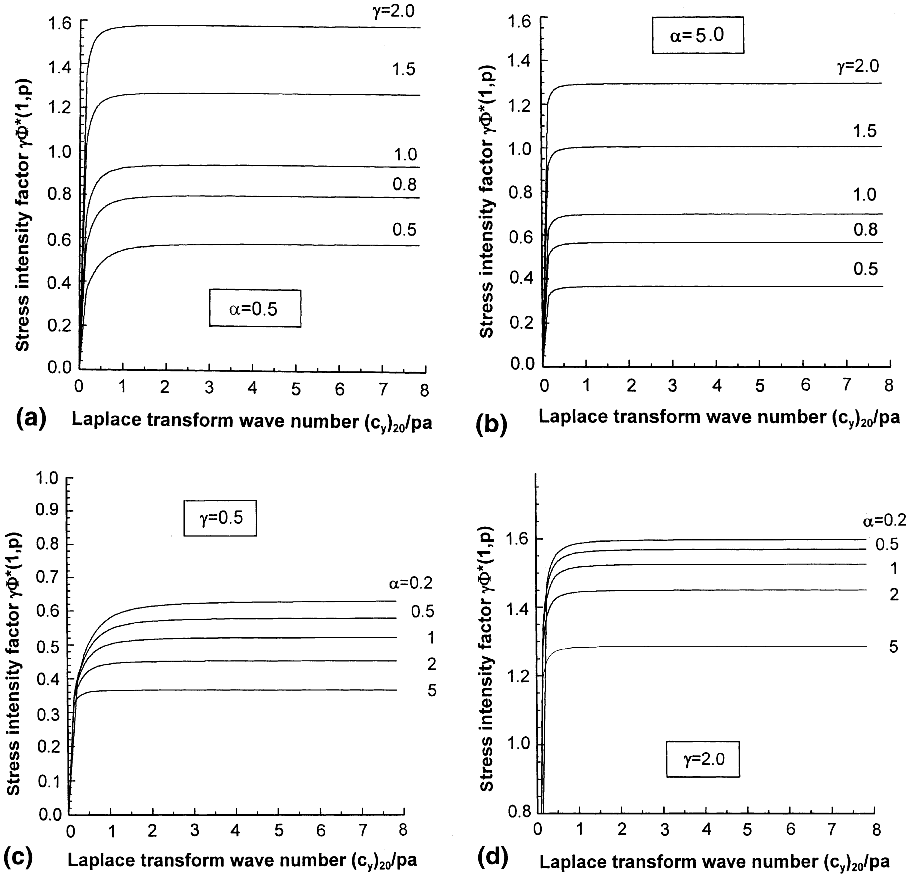

Fig. 2. Variations of stress intensity factor in Laplace transform plane for (a) $\alpha = 0.5$, (b) $\alpha = 5.0$, (c) $\gamma = 0.5$ and (d) $\gamma = 2.0$.

parameter α and the orthotropy parameter γ on $\gamma\Phi^*(1,p)$ are obvious. For fixed γ, the magnitude of $\gamma\Phi^*(1,p)$ decreases when α increases. For fixed α, the magnitude of $\gamma\Phi^*(1,p)$ increases when γ increases.

The numerical Laplace transform inverse scheme [9] is used to give the dynamic stress intensity factor given by Eq. (45). Fig. 3 displays the normalized dynamic stress intensity factor $3\pi K_{\rm III}(t)/4\tau_0\sqrt{\pi a}$ as a function of $(c_z)_{20}t/a$. The influence of α and γ on $K_{\rm III}(t)$ can be seen in Fig. 3. All the curves reach a peak and then oscillate about the static value with decreasing magnification. For fixed α, the dynamic stress intensity factors are larger for larger values of γ. For fixed γ, the dynamic stress intensity factor is less for larger values of α.

From Fig. 3, it is also seen that the peak value of the dynamic stress intensity factor can be suppressed by increasing the shear moduli's gradients or increasing the shear modulus normal to the crack. Fig. 3 also shows that the time required to reach the peak decreases with increasing α or γ.

7. Conclusion

The transient response is examined for an orthotropic FGM with a penny-shapped crack under torsinal impact. The local stress field around the crack edge is determined. The dynamic stress intensity factor shows that the nonhomogeneity and orthotropy of FGM has a significant influence on the local stresses. The peak value of the dynamic

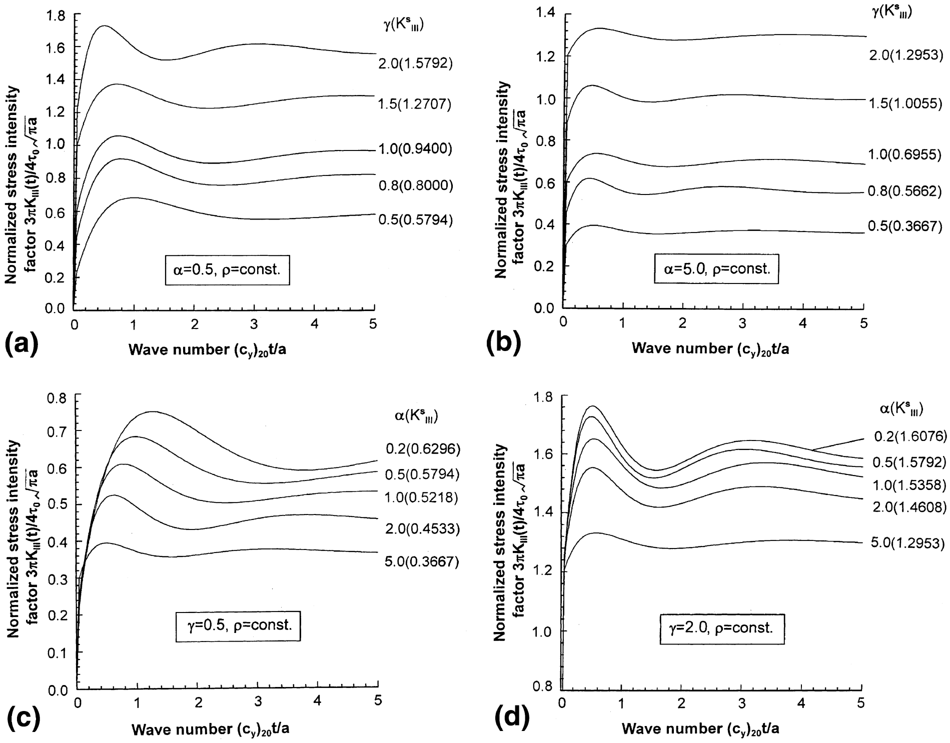

Fig. 3. Normalized stress intensity factor versus wave number for (a) $\alpha = 0.5$, (b) $\alpha = 5.0$, (c) $\gamma = 0.5$ and (d) $\gamma = 2.0$.

stress intensity factor can be suppressed by increasing the shear moduli's gradient. Increasing the shear modulus normal to the crack can also reduce the peak value of the dynamic stress intensity factor.

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References

[1] M.K. Kassir, G.C. Sih, Three-dimensional crack problems, Mechanics of Fracture, 2, Noordhoff International Publsihing (Now Kluwer Academic Publishers, Boston), 1975.

[2] M. Ozturk, F. Erdogan, Mode I crack problem in an inhomogeneous orthotropic medium, Int. J. Engng. Sci. 35 (9) (1997) 869–883.

[3] F. Erdogan, The crack problem for bonded nonhomogeneous materials under antiplane shear loading, ASME J. Appl. Mech. 52 (1985) 823–828.

[4] N. Konda, F. Erdogan, The mixed-mode crack problem in a nonhomogeneous elastic plane, Engng. Frac. Mech. 47 (1994) 533–545.

[5] A. Gerasoulis, R.P. Srivastav, A Griffith crack problem for a nonhomgenous medium, Int. J. Engng. Sci. 18 (1980) 239–247.

[6] E.T. Copson, On certain dual integral equations, Proc. Glasgow Math. Association 5 (1961) 19–24.

[7] E.P. Chen, G.C. Sih, Transient response of cracks to impact loads, in: G.C. Sih (Ed.), Mechanics of Fracture, vol. 4, Noordhoff International Publishing (Now Kluwer Academic Publishers, Boston), 1977.

[8] G.C. Sih, E.P. Chen, Cracks in composite materials, Mechanics of Fracture 6, Martinus nijhoff publishers (Now Kluwer Academic Publishers, Boston), 1981.

[9] M.K. Miller, W.T. Guy, Numerical inversion of the Laplace transform by use of Jacobi polynomials, SIAM J. Numer. Anal. 3 (1966) 624–635.