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Scaling relationships for indentation measurements

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Scaling relationships for indentation measurements

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ABSTRACT

Using dimensional analysis and finite-element calculations, several relationships that relate features of indentation loading and unloading curves to the hardness, the elastic modulus and the work of indentation are extended to conical indentation in elastic–plastic solids with various angles. These relationships provide new insights into indentation measurements. They may also be useful to the interpretation of results obtained from instrumented indentation experiments.

§ 1. INTRODUCTION

For over 100 years, indentation experiments have been performed to obtain the hardness of materials (Tabor 1951, 1996). Recent years have seen significant improvements in indentation equipment and a growing need to measure the mechanical properties of materials on small length scales. It is now possible to monitor, with high precision and accuracy, both the load and the displacement of an indenter during indentation experiments (Pethica *et al.* 1983, Stone *et al.* 1988, Bhushan *et al.* 1996). Several methods have been proposed to extract mechanical properties of materials from loading–unloading curves measured using instrumented indentation techniques, including materials hardness, elastic modulus and stress–strain relationships (Doerner and Nix 1986, Joslin and Oliver 1990, Oliver and Pharr 1992, Sakai 1993, 1999, Hainsworth *et al.* 1996, Rother 1996, Faulkner *et al.* 1998, Knapp *et al.* 1999, Giannakopoulos and Suresh 1999, Sun *et al.* 1999). However, questions remain, including the following: what properties can be obtained from instrumented indentation and what is hardness? In this paper, we shall summarize and extend our recent results (Cheng and Cheng 1997, 1998a, b, c, d, e, 1999a, b, 2000, 2001, Cheng and Li 2000) obtained using a scaling approach to indentation modelling, which provide some insights into instrumented indentation measurements.

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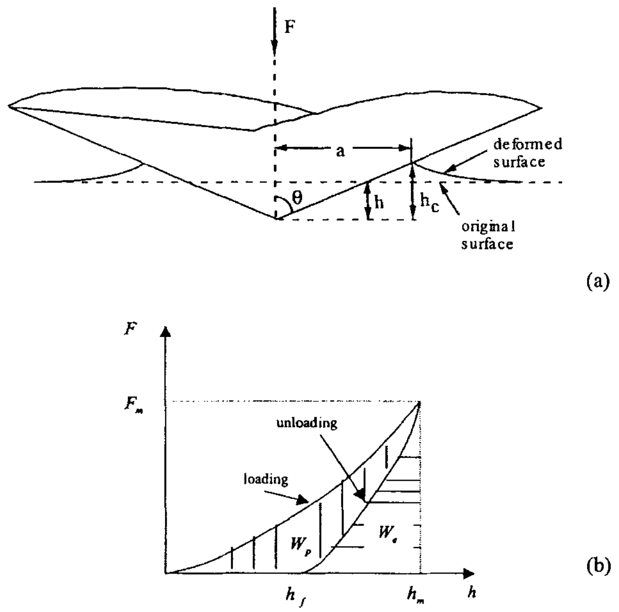

Figure 1. Illustration of (*a*) conical indentation and (*b*) loading and unloading curves, where W_{tot} is the total work, W_{p} is the irreversible work and W_{e} is the reversible work: $W_{\mathrm{tot}} = W_{\mathrm{p}} + W_{\mathrm{e}}$.

We consider a three-dimensional rigid conical indenter of a given half-angle θ indenting normally into a homogeneous solid. The friction coefficient at the contact surface between the indenter and the solid is assumed zero. The quantities of interest from the loading portion of indentation measurements include the force F and the contact depth h_{c} or the projected contact area $A_{\mathrm{c}} = \pi a^2$ (figure 1 (*a*)), from which the hardness under load, $H = F/A_{\mathrm{c}}$, can be evaluated. The initial unloading slope has been used to estimate the contact depth h_{c} and contact area A_{c}. The maximum indentation depth h_{m} and the residual final depth h_{f} at which the indenter detaches from the surface of materials during unloading (figure 1 (*b*)), have been used to characterize the properties of materials (Loubet *et al.* 1984, 1986). Finally, the total work W_{tot} of indentation and the irreversible work W_{p} of indentation, defined respectively as the area under the loading curve and that between the loading and unloading curves, have also been used for materials characterization (Sakai 1993, Hainsworth *et al.* 1996, Rother 1996, Cheng and Cheng 1998e, 1999b, Faulkner *et al.* 1998, Giannakopoulos and Suresh 1999). It is desirable, therefore, to investigate the relationships between these seemingly different measures of material properties obtained from indentation experiments.

In this paper, we shall first review the scaling relationships for indentation loading and unloading curves, contact area, and hardness. This is followed by extensive elastic–plastic finite-element calculations to investigate the correlations further (Cheng and Cheng 1998e, 1999b) between $W_{\mathrm{p}}/W_{\mathrm{tot}}$ and $h_{\mathrm{f}}/h_{\mathrm{m}}$, and between H/E^* and $h_{\mathrm{f}}/h_{\mathrm{m}}$, where the reduced modulus $E^* = E/(1-\nu^2)$, and E and ν are Young's modulus and Poisson's ratio respectively. We show that the relationships established earlier hold for indentation in elastic–plastic solids with work hardening over a wide range of indenter semicone angles. Furthermore, we shall provide explicit expressions for these relationships so that they can be readily available for future applications.

§2. DIMENSIONAL ANALYSIS AND FINITE-ELEMENT CALCULATIONS

The stress σ–strain ε curves of the solids under uniaxial tension are assumed to be given by

$$\begin{aligned} \sigma &= E\varepsilon \quad \text{for } \varepsilon \leqslant \frac{Y}{E}, \\ \sigma &= K\varepsilon^n \quad \text{for } \varepsilon \geqslant \frac{Y}{E}, \end{aligned} \tag{1}$$

where E is Young's modulus, Y is the initial yield stress, K is the strength coefficient and n is the work-hardening exponent (Lubliner 1990). To ensure continuity, we note that $K = Y(E/Y)^n$. Consequently, either E, Y and K or E, Y and n are sufficient to describe the uniaxial stress–strain relationship. We use the latter set of parameters extensively in the following discussions. When n is zero, equation (1) becomes the model for elastic–perfectly plastic solids. For most metals, n has a value between 0.1 and 0.5 (Dieter 1976).

During loading, F and h_c can be written, according to dimensional analysis (Cheng and Cheng 1998c, 1999a), as

$$F = Eh^2\Pi_\alpha\left(\frac{Y}{E}, \nu, n, \theta\right), \tag{2}$$

$$h_c = h\Pi_\beta\left(\frac{Y}{E}, \nu, n, \theta\right), \tag{3}$$

where Π_α and Π_β are dimensionless functions of four dimensionless parameters Y/E, ν, n and θ. Clearly, the force F on the indenter is proportional to the square of the indenter displacement h. The contact depth h_c is proportional to h. Consequently, the hardness under load is independent of indenter displacement h or indenter load F for the class of materials with their constitutive relationships given by equation (1).

Although the square dependence of the loading curves for indentation using conical and pyramidal indenters has been known from previous analyses and experiments, the dimensional analysis makes clear that this square dependence is a consequence of the absence of a length parameter in the above problem. A deviation from the square dependence is therefore an indication of the existence of a length parameter l. In fact, dimension analysis shows that the corresponding loading curve is then given by

$$F = Eh^2\Pi_\alpha\left(\frac{Y}{E}, \frac{h}{l}, \nu, n, \theta\right), \tag{2'}$$

which is, in general, not proportional to the square of indenter displacement. For example, loading curves can depart from the square dependence if the conical or pyramidal indenter has a finite tip radius R (Cheng and Cheng, 1998b). In this case, the tip radius R enters the analysis as a length parameter (in place of l in equation (2′)). A less obvious example is indentation in materials that have strain-rate sensitivity. It has been shown that the loading curves can be different from the square dependence for indentation in power-law creep solids using conical or pyramidal indenters (Cheng and Cheng, 2001). As a consequence, the hardness value takes on a depth dependent, that is an indentation size effect. Other mechanisms exist that can introduce a length parameter and make the loading curves more complex, including inhomogeneity in the materials (e.g. composition variation), fracture or strain-gradient plasticity effects (Nix and Gao 1998). The shape of indentation loading curves can, therefore, be used to probe intrinsic (due to materials properties) or extrinsic (due to geometry) length scale in indentation problems.

We return to ideally sharp conical indenters indenting solids that obey equation (1) and examine unloading. Because unloading begins at the maximum indentation depth h_m, the length parameter h_m enters the expression for the unloading curve (Cheng and Cheng 1998c, 1999a):

$$F = Eh^2 \Pi_\gamma\left(\frac{Y}{E}, \frac{h}{h_m}, \nu, n, \theta\right), \tag{4}$$

where Π_γ is a dimensionless function of five parameters Y/E, h/h_m, ν, n and θ. In contrast with loading, equation (4) shows that the force F is, in general, *not* proportional to the square of the indenter displacement h. It also depends on the ratio h/h_m, through the dimensionless function Π_γ. This is consistent with well-known experimental observations that the force during unloading is not simply proportional to the square of indenter displacement.

From equations (2)–(4), we can obtain the following scaling relationships (Cheng and Cheng 1998e, 1999b):

$$\frac{h_f}{h_m} = \Pi_\phi\left(\frac{Y}{E}, \nu, n, \theta\right), \tag{5}$$

$$\frac{W_p}{W_{tot}} = \Pi_\omega\left(\frac{Y}{E}, \nu, n, \theta\right), \tag{6}$$

$$\frac{H}{E^*} = \Pi_h\left(\frac{Y}{E}, \nu, n, \theta\right), \tag{7}$$

where the dimensionless functions Π_ϕ, Π_ω and Π_h can be derived from Π_α, Π_β and Π_γ. Using these relationships, Cheng and Cheng (1998e, 1999b) showed, for a particular cone angle (e.g. $\theta = 68°$), that there is an approximate one-to-one correspondence between H/E^* and W_p/W_{tot}, and between W_p/W_{tot} and h_f/h_m.

In this work, extensive finite-element computations were carried out using ABAQUS (*ABAQUS User's Manual* 1998) to investigate these relationships further for various angles θ. In the calculations, the work-hardening exponent n was assumed to be 0.0, 0.1 and 0.5 respectively. For each work-hardening exponent, 15 different values of Y/E were used varying from 0.0002 to 0.1. For each combination of n and Y/E, indentations were simulated for semicone angles of 45, 60, 70.3 and 80° respectively. Poisson's ratio was assumed to be 0.3 for all the calculations. The hardness values were obtained from load divided by the projected contact area under load from finite-element outputs. A total of 180 cases were studied. The finite-element model has been discussed in detail previously (Cheng and Cheng 1998a, Cheng and Li 2000). Once the numerical results for equations (5)–(7) were obtained, they were used to explore the interdependence of W_p/W_{tot}, h_f/h_m and H/E^* for semicone angles of 45, 60, 70.3 and 80° respectively.

§ 3. RESULTS AND DISCUSSION

3.1. *Relationship between* W_p/W_{tot} *and* h_f/h_m

A single one-to-one relationship is observed between W_p/W_{tot} and h_f/h_m, which is shown in figure 2. A remarkable feature of this relationship is that it contains all finite-element results for different values of Y/E, ν, n and θ. Furthermore, this relationship is approximately linear for $h_f/h_m > 0.4$ or for $W_p/W_{tot} > 0.2$. The linear relationship between W_p/W_{tot} and h_f/h_m, obtained using a least-squares curve fitting, is given by (figure 2)

$$\frac{W_p}{W_{tot}} = (1+\gamma)\frac{h_f}{h_m} - \gamma \qquad \text{for} \quad \frac{h_f}{h_m} > 0.4, \tag{8}$$

where $\gamma = 0.27$. This one-to-one correspondence suggests that W_p/W_{tot} and h_f/h_m contain the same information about the mechanical property of materials so far as indentation tests are concerned. The measurement of one leads to the measurement of the other. In practice, however, the determination of W_p and W_{tot} can be made more accurately than the determination of h_f, since the former is from the integration of loading–unloading curves and the latter is from the estimate of a single point on the unloading curve.

Several workers have previously explored correlations between W_p/W_{tot} and h_f/h_m. Lawn and Howes (1981) studied the elastic recovery effect in the indentation of several ceramic materials and steels. By assuming that the respective loading and unloading curves are given by

$$F = Ah^2 \qquad \text{and} \qquad F = B(h^2 - h_f^2), \tag{9}$$

where A and B are functions of materials properties and indenter geometry, a relationship between W_p/W_{tot} and h_f/h_m was obtained:

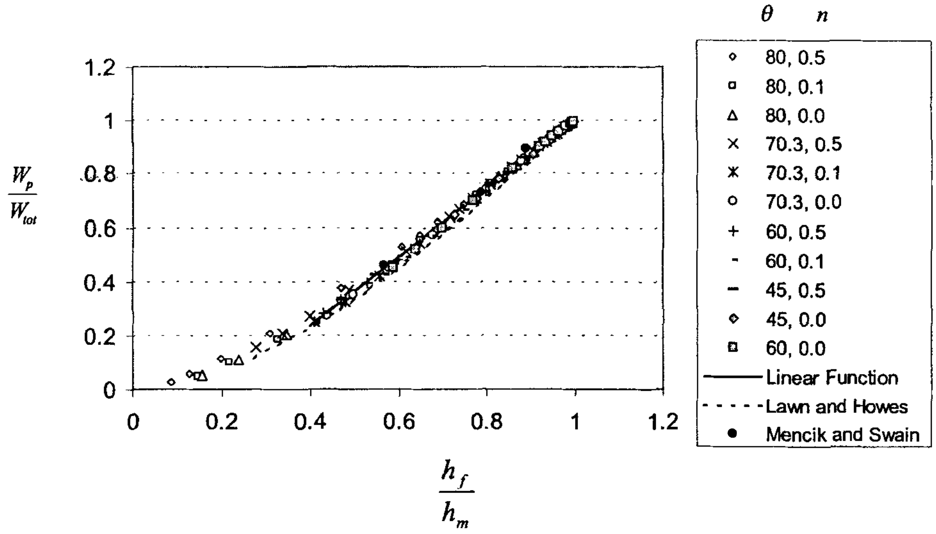

Figure 2. A relationship between h_f/h_m and W_p/W_{tot}.

$$\frac{W_p}{W_{tot}} = 1 - \left\{ \left[1 - 3\left(\frac{h_f}{h_m}\right)^2 + 2\left(\frac{h_f}{h_m}\right)^3 \right] \Big/ \left[1 - \left(\frac{h_f}{h_m}\right)^2 \right] \right\}. \qquad (10)$$

Equation (10) is plotted in figure 2. The finite-element calculations and equation (10) agree well with each other.

Menčik and Swain (1994) have also explored the relationship between W_p/W_{tot} and h_f/h_m. Their experimental results for two types of steel and glass are also included in figure 2. By assuming that the respective loading and unloading curves are given by

$$F = ah^m \qquad \text{and} \qquad F = b(h - h_f)^l, \qquad (11)$$

where a and b are functions of materials properties and indenter geometry, a relationship between W_p/W_{tot} and h_f/h_m is then given by

$$\frac{W_p}{W_{tot}} = \frac{m+1}{l+1}\frac{h_f}{h_m} - \frac{m-l}{l+1}. \qquad (12)$$

Equations (8) and (12) are the same if $m = 2.0$ and $l = 1.362$. The above discussion shows that a general relationship between W_p/W_{tot} and h_f/h_m exists and this relationship is explicitly independent of indenter geometry. This relationship is also explicitly independent of the details of the materials properties, and the stress distribution under the self-similar indenters.

3.2. *Relationship between h_f/h_m and H/E^**

Finite-element calculations reveal the relationships between final depth, hardness and elastic modulus and they are shown in figure 3. An approximately linear function exists between h_f/h_m and H/E^* for each indenter angle. The relationships can be summarized as

$$\frac{h_f}{h_m} = 1 - \lambda \frac{H}{E^*}, \qquad (13)$$

where

$$\lambda = 1.50 \tan(\theta) + 0.327 \qquad \text{for} \quad 60° \leqslant \theta \leqslant 80°. \qquad (14)$$

The angular dependence of λ is shown in figure 4. We note that, while equation (13) still holds for indenters with sharper angles (e.g. 45°), the numerical value of λ does not follow equation (14) for small angles. Unlike the relationship between W_p/W_{tot} and h_f/h_m, the relationship between h_f/h_m and H/E^* depends on the indenter angle, especially when the angle is large.

The approximately linear relationship between h_f/h_m and H/E^* has also been observed experimentally and numerically. Also shown in figure 3 are the experimental results of Menčik and Swain (1994) for two types of steel and glass. Their data points fall approximately on a straight line, although the line is offset from the present finite-element results. Using finite-element calculations, Marx and Balke (1997) have shown a linear relationship for conical indentation, with a semicone angle of 70.3°, in elastic–plastic solids with bilinear uniaxial stress–strain relationships. Cheng and Cheng (1998e, 1999b) have shown a similar relationship for conical indentation in elastic–plastic solids with power-law work hardening for a particular semicone angle of 68°.

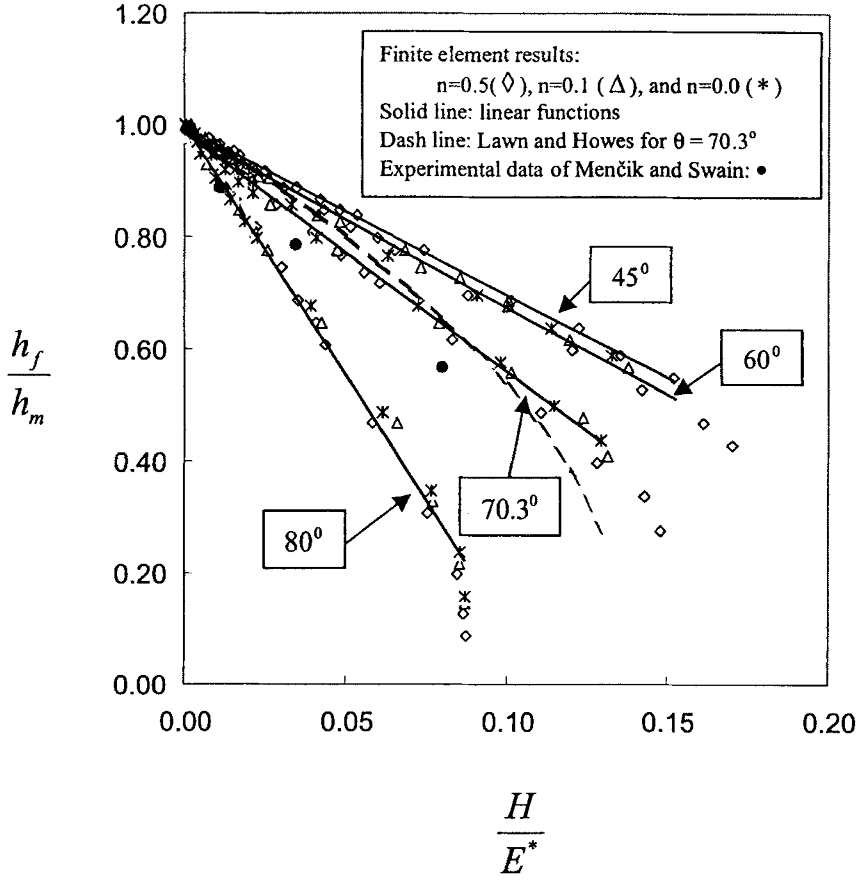

Figure 3. Relationships between h_f/h_m and H/E^*.

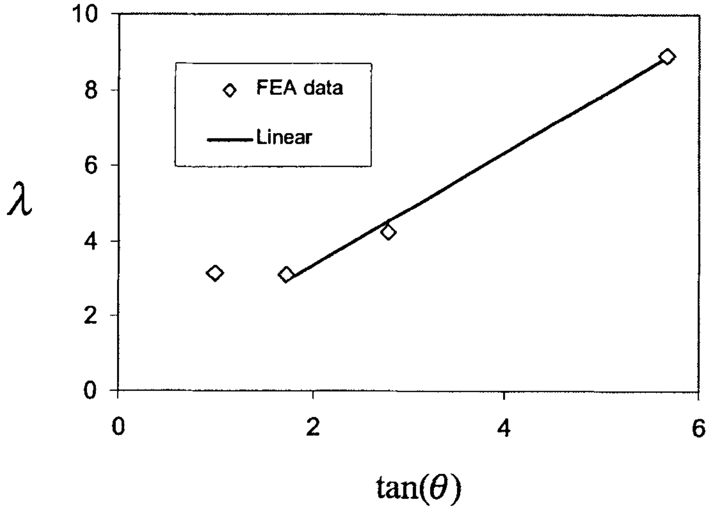

Figure 4. The dependence of λ on $\tan(\theta)$ with a linear function: $\lambda = 1.50\tan(\theta) + 0.327$ for $60° \leqslant \theta \leqslant 80°$: FEA, finite-element analysis.

Although approximately linear relationships between h_f/h_m and H/E^* were observed experimentally and numerically, there is no known analytical derivation of the linear relationship. However, several workers have proposed models that link the two quantities (Lawn and Howes 1981, Sakai 1993). In these models, the degrees of piling-up and sinking-in of surface profiles were treated as adjustable parameters. The relationships between h_f/h_m and H/E^* are influenced by these parameters. An example is the relationship proposed by Lawn and Howes (1981):

$$\left(\frac{h_f}{h_m}\right)^2 = 1 - \left[2\left(\frac{\gamma_E}{\gamma_H}\right)^2 \tan(\theta)\right]\frac{H}{E^*}, \tag{15}$$

where γ_E and γ_H are parameters that are affected by the degree of surface sinking in or piling up. In figure 3, this relationship for an indenter with a half-angle of 70.3° is illustrated. A reasonable agreement between the Lawn–Howes model and the finite-element results was found when the parameter γ_E/γ_H is about 1.18. However, equation (15) does not predict the approximately linear relationship. Future research is needed to understand better the relationship between h_f/h_m and H/E^*.

3.3. *Relationship between W_e/W_{tot} and H/E^**

Substituting equation (13) into equation (8) and using $W_e = W_{tot} - W_p$, we obtain

$$\frac{H}{E^*} = \kappa \frac{W_e}{W_{tot}}, \tag{16}$$

where $\kappa = 1/[\lambda(1+\gamma)]$ for $60° \leqslant \theta \leqslant 80°$. The value for κ obtained using this expression cannot be applied to sharp cones (e.g. 45°) owing to the validity range of λ in equation (14). From this relationship, the ratio of H/E^* can be obtained readily by measuring the work of indentation to obtain W_e/W_{tot}. Furthermore, since the ratio of H/E^{*2} can be obtained from the initial slopes of the unloading curves (Joslin and Oliver 1990, Hainsworth *et al.* 1996, Cheng and Cheng 1998e) the values of H and E^* can, in principle, be obtained from the work of indentation and the initial unloading slope. Although equation (16) was obtained for indentation in homogeneous solids, recent work by Malzbender and de With (2000) suggested that it may also be applicable to thin films and coatings deposited on substrates, provided that fracture and delamination events do not influence the energy dissipation significantly.

§4. SUMMARY

We have derived scaling relationships for the loading and unloading curves, contact depth, and hardness for indentation in elastic–plastic solids with work hardening. Using these relationships and extensive finite-element calculations, we found three approximately linear relationships between H/E^*, W_p/W_{tot} and h_f/h_m. Thus, the measurement of one leads to the measurement of the other. Specifically, the ratio of hardness and reduced modulus can be obtained readily from the work of indentation by integrating the loading and unloading curves. Together with the analysis of initial unloading slopes, the values for hardness and reduced modulus may also be obtained. In addition to assisting the interpretation of indentation measurements, we believe that the relationships may have other applications in areas of surface engineering and tribology.

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