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The instability analysis of saturated soil under shear load

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Abstract

A theoretical description of shear instability is presented in a system of equations. It is shown that two types of instability may exist. One of them is dominated by pore pressure softening while the other by strain softening. A criterion combining pore pressure softening, strain hardening, and volume strain coefficient is obtained and practical implications are discussed. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Instability often leads to the failure of soil foundation. Although lot of work has been done trying to seek for the criterion of soil instability so as to evaluate the safety of foundations under vibration and to avoid instability and liquefaction, the mechanism of this problem has not been uncovered because of the complexity. The instability criterion of saturated soils has often adopted the Coulomb criterion [1]. On the viewpoint of instability, the criterion and factors of soil instability is discussed in this paper.

2. The mathematical model

2.1. Some assumptions

In order to clarify the problem, some assumptions are presented here.

1. The density of water and gains is constant, which means $\rho_s = \text{const.}$, $\rho_w = \text{const.}$
2. The moving acceleration can be neglected because it is smaller than the local acceleration.
3. The Darcy law is adopted.
4. The deformation can only occur in one direction but may have a gradient in the other direction.

   The geometrical configuration and the deformation can be expressed as follows:

$$x_1 = X_1,$$

$$x_2 = u(X_1, X_2) + X_2,$$

$$x_3 = X_3. \tag{1}$$

5. The constitutive equations can be expressed as follows under shear load:

$$\sigma_{ex} = f_1(\gamma, \dot{\gamma}, p),$$

$$\sigma_{ey} = f_2(\gamma, \dot{\gamma}, p),$$

$$\tau_{xy} = f_3(\gamma, \dot{\gamma}, p). \tag{2}$$

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The shear volume strain and the increment of pore pressure are adopted as follows:

$$\Delta\varepsilon_{v1} = C_1\tau\gamma,$$
$$\Delta p = E_r\Delta\varepsilon_{v2}, \tag{3}$$

where C_1 is a parameter and $C_1 = C_1(p)$, Δp is the increment in pore pressure, E_r is resilience module, $\Delta\varepsilon_{v2}, \Delta\varepsilon_{v2}$ are the volume strain caused by shear and recovered by water, respectively. The soil tries to shrink by shear but the pore water prevents the deformation, which leads to the increase in pore pressure.

2.2. The control equations

Since water cannot be compressed, $\varepsilon_{wv} = 0$. At the same time, it is obvious that the volume caused by shear is the sum of that caused by drainage and resilience, which is

$$C_1\tau\Delta\gamma = \frac{\Delta p}{E_r} + \Delta\varepsilon_{sv} \tag{4}$$

and

$$\frac{\partial\varepsilon_{v2}}{\partial t} = \frac{1}{K}\frac{\partial^2 p}{\partial x^2}, \tag{5}$$

in which p is the pore pressure and n is the pore ratio. K is the obstraction coefficient and $K = \mu/K$, where k is the penetration ratio and μ is the visco-coefficient [2,3].

Institute Eq. (5) of Eq. (4), we will get the first control equation

$$\frac{\partial p}{\partial t} - \frac{Er}{K}\frac{\partial^2 p}{\partial x^2} = C_1E_r\tau\frac{\partial\gamma}{\partial t}. \tag{6}$$

The equations of motion implies

$$\rho\frac{\partial^2\gamma}{\partial t^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\tau = \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}, \tag{7}$$

where $\rho = \rho_s - \rho_w$. Consider the assumption (4), Eq. (7) becomes

$$\rho\frac{\partial^2\gamma}{\partial t^2} - \frac{\partial^2\tau}{\partial x^2} = 0. \tag{8}$$

Now, the control equations can be rewritten as

$$\frac{\partial p}{\partial t} - \frac{Er}{K}\frac{\partial^2 p}{\partial x^2} = C_1E_r\tau\frac{\partial\gamma}{\partial t},$$
$$\rho\frac{\partial^2\gamma}{\partial t^2} = \frac{\partial^2\tau}{\partial x^2}, \tag{9}$$

where C_1, E_r are both functions of p. The solutions of these equations are difficult to seek for because of their non-linearity. It has been shown by experiments and computation that the soil deformation develops from slowly to fast [4,5], that means, from stable state to instability.

Now, three points should be emphasized: firstly, examining all the assumptions, Eq. (9) can deal with large shear deformation, because no limitation of the shear deformation has been introduced; secondly, the first part of Eq. (9) is a wave equation but the right-hand side of the second is a typical diffusion equation. In these two types different phenomena are coupled through the term $C_1E_r\tau\dot{\gamma}$. This is the distinctive feature of the phenomenon under consideration. Finally, Eq. (9) is obviously non-linear.

Since the aim of this paper is to seek for the condition under which a smooth deformation process changes into catastrophe. The perturbation method, which is widely used in the analysis of fluid dynamics, is adopted here. Hence, a smooth developing deformation state γ_0, τ_0, p_0 is taken as the base stae, which is a solution of Eq. (9). When perturbation has been acted on the control equations, we will be able to analyse the factors and condition of instability.

3. Perturbation analysis [6,7]

To seek for the point deformation from smoothness to catastrophe, we study the solutions in next form:

$$\gamma = \gamma_0 + \gamma', \quad |\gamma'| \ll |\gamma_0|,$$
$$p = p_0 + p', \quad |p'| \ll |p_0|, \tag{10}$$

where γ_0, p_0 is a solution of Eq. (9), and

$$\gamma' = \gamma^* e^{\alpha t + i\beta x},$$
$$p' = p^* e^{\alpha t + i\beta x}, \tag{11}$$

α, β are the frequency and the wave number, respectively.

Differentiating the constitutive relations (2), we get

$$\mathrm{d}\tau = R_0\,\mathrm{d}\gamma - Q_0\,\mathrm{d}\gamma + H_0\,\mathrm{d}\dot{\gamma}, \tag{12}$$

in which

$$R_0 = \left(\frac{\partial\tau}{\partial\gamma}\right)_0, \quad Q_0 = -\left(\frac{\partial\tau}{\partial p}\right)_0,$$
$$H_0 = \left(\frac{\partial\tau}{\partial\dot{\gamma}}\right)_0. \tag{13}$$

Therefore,

$$\tau^* = R_0\gamma^* - Q_0 p^* + \alpha H_0\gamma^*. \tag{14}$$

Institute Eqs. (10), (11), (14) of (9), the homogeneous system of equations is obtained as follows then:

$$[\rho\alpha^2 + \beta^2(R_0 + \alpha H_0)]\gamma^* - \beta^2 Q_0 p^* = 0$$
$$(C_1 E_r \tau_0\alpha + C_1 E_r H_0\alpha + C_1 E_r \dot{\gamma}_0 R_0)\gamma^*$$
$$- \left(C_1 E_r \dot{\gamma}_0 Q_0 + \alpha + \frac{E_r}{K}\beta^2\right)p^* = 0. \tag{15}$$

As we all know, the determinant of the coefficients should be equal to zero if the system has solutions, which leads to

$$\rho\alpha^3 + \left(\left(H_0 + \frac{\rho E_r}{K}\right)\beta^2 + \rho C_1 E_r \dot{\gamma}_0 Q_0\right)\alpha^2$$
$$+ A_1\alpha + \frac{E_r R_0}{K}\beta^4 = 0, \tag{16}$$

where $A_1 = (R_0 - Q_0 C_1 E_r \tau_0)\beta^2 + \dfrac{H_0 E_r}{K}\beta^4$.

It is a spectral equation. If α has a positive real root, instability is possible.

Now, we can give the dimensionless form of Eq. (16), using the next dimensionless variables

$$\alpha = \frac{1}{\rho k_1}\bar{\alpha}, \quad \beta^2 = \frac{1}{\rho R_0 k_1^2}\bar{\beta}^2, \quad k_1 = \frac{1}{K},$$
$$A = \frac{E_r}{R_0}, \quad B = \frac{H_0 K}{\rho R_0},$$
$$C = \frac{\rho C_1 E_r \dot{\gamma}_0 Q_0}{K}, \quad D = \frac{Q_0 C_1 E_r \tau_0}{R_0}. \tag{17}$$

Then, the spectral Eq. (16) can be reduced to the following form:

$$\bar{\alpha}^3 + [(A + B)\bar{\beta}^2 + C]\bar{\alpha}^2 + [(1 - D)\bar{\beta}^2$$
$$+ AB\bar{\beta}^4]\bar{\alpha} + A\bar{\beta}^4 = 0. \tag{18}$$

It is obvious that this equation has two extreme situations.

(i) For long wavelength ($\beta \to 0$), Eq. (18) has two solutions

$$\bar{\beta} = 0, \quad \bar{\alpha} = 0 \text{ or } \bar{\alpha} = -C \tag{19}$$

It shows that the deformation is always stable.

(ii) For short wavelength ($\beta \to \infty$), Eq. (18) has only one solution, which is

$$\bar{\beta} \to \infty, \quad \bar{\alpha} = -\frac{1}{AB}. \tag{20}$$

It is again always stable.

But there is a negative term $1 - D$ which may lead to instability. It must occur at spectral wave numbers. Therefore, it is of interest to seek the wave number $\bar{\beta}_m$ for which the corresponding $\alpha_m > 0$ is a maximum. In addition to the spectral equation (18), $\bar{\alpha}_m$ and $\bar{\beta}_m$ must satisfy the next equation

$$\frac{\mathrm{d}\bar{\alpha}}{\mathrm{d}\bar{\beta}^2} = 0 \tag{21}$$

which is

$$\bar{\beta}_m^2 = -\frac{(A + B)\bar{\alpha}_m^2 + (1 - D)\bar{\alpha}_m}{2(AB\bar{\alpha}_m + A)}. \tag{22}$$

Keeping $\bar{\beta}_m^2 > 0$ in mind, we arrive at an important inequality to determine the limit of the $\bar{\alpha}_m$ value

$$0 < \bar{\alpha}_m < \frac{D - 1}{B + A} = \bar{\alpha}_m^*. \tag{23}$$

Combining both the spectral equation (18) and the extreme condition (23), the equation to determine $\bar{\alpha}_m$ can then be obtained

$$f_1 = f_2, \tag{24}$$

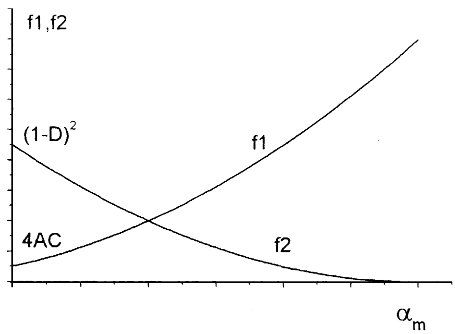

Fig. 1. Plots of the functions f_1, f_2, defined in Eqs. (25) and (26).

where

$$f_1 = 4(\bar{\alpha}_m + C)(AB\bar{\alpha}_m + A), \tag{25}$$

$$f_2 = [(A+B)\bar{\alpha}_m + (1-D)]^2. \tag{26}$$

From Fig. 1 for the region $\bar{\alpha}_m > 0$, it may be seen that the left branch of function f_2 and the right branch of f_1 must have an intersection between 0 and $\bar{\alpha}_m^*$ as long as

$$D - 1 > 2\sqrt{AC}. \tag{27}$$

This is the criterion for the existence of a solution $\bar{\alpha}_m$ and therefore what we desired. At most conditions $AC \approx 0$, then the criterion can be simplified to:

$$D = \frac{Q_0 C_1 E_r \tau_0}{R_0} > 1. \tag{28}$$

This means that the condition of instability refers to pore pressure softening overcoming the strain hardening. It is very interesting that occurrence of instability is not related to the penetration ratio K and the strain rate hardening H_0. However, these factors influence instability markedly in some other aspects which will be discussed later.

The intersection $\bar{\alpha}_m$ in Fig. 1 and the corresponding value of $\bar{\beta}_m$ represent the most probable unstable solution. The solution $\bar{\alpha}_m$ has the same order as $\bar{\alpha}_m^*$.

Hence, for qualitative discussion, the value of $\bar{\alpha}_m^*$ can be used to represent the point of intersection $\bar{\alpha}_m$. The characteristic time can be expressed as

$$t_c \sim \frac{1}{\alpha_m} \sim \frac{\rho k_1}{\bar{\alpha}_m} \sim \frac{\rho k_1 (A+B)}{D-1} = \frac{\rho k_1 E_r + H_0}{Q_0 C_1 E_r \tau_0 - R_0}.$$

It is obvious that the characteristic time is affected by resilence module, strain rate hardening, penetration, strain hardening and pore pressure softening.

The characteristic length l_c is related to t_c by

$$l_c^2/t_c \sim \alpha_m/\beta_m^2 \sim R_0 k_1 (\bar{\beta}_m^2/\bar{\alpha}_m).$$

Next, three interesting special cases: no penetration, no strain hardening and no pore pressure will be discussed.

4. Some other conditions

4.1. No penetration, $K \to \infty$

In this case, the spectral Eq. (16) becomes

$$\rho\alpha^2 + (H_0\beta^2 + \rho C E_r \dot{\gamma}_0 Q_0)\alpha + (R_0 - Q_0 E_r C_1 \tau_0)\beta^2 = 0. \tag{29}$$

We can see, if $R_0 - Q_0 E_r C_1 \tau_0 < 0$, namely $D > 1$, α will have a positive real root and instability will occur. It is important to appreciate that the same formal criterion (28) can be used whether the instability is penetrational or not.

4.2. No strain hardening, $R_0 = 0$

Now, the spectral equation (16) becomes

$$\rho\alpha^2 + \left[\left(H_0 + \frac{\rho E_r}{K}\right)\beta^2 + \rho C_1 E_r \dot{\gamma}_0 Q_0\right]\alpha + \left[\frac{E_r H_0}{K}\beta^4 - C_1 E_r Q_0 \tau_0 \beta^2\right] = 0. \tag{30}$$

The condition α has positive real root as given below:

$$\frac{H_0}{K}\beta^2 - C_1 Q_0 \tau_0 < 0. \tag{31}$$

In this criterion, penetration ratio K, strain rate hardening H_0 and pore pressure softening play the role.

4.3. No pore pressure softening, $Q_0 = 0$

Now, we turn to discuss the second mode of instability which there is no pore pressure softening. In this case, we can formulate the spectral equation

$$\rho\alpha^3 + \left(H_0 + \frac{\rho E_r}{K}\right)\beta^2\alpha^2 + \left(R_0\beta^2 + \frac{E_r H_0}{K}\beta^4\right)\alpha + \frac{E_r R_0}{K}\beta^4 = 0. \quad (32)$$

Though H_0, K must be positive, R_0 may be negative. Therefore, $R_0 < 0$ may be another possible cause of instability. Eq. (32) can be rewritten as

$$\rho\alpha^3 + \left(H_0 + \frac{\rho E_r}{K}\right)\beta^2\alpha^2 + \frac{E_r H_0}{K}\beta^4\alpha = \frac{E_r|R_0|}{K}\beta^4 + |R_0|\beta^2. \quad (33)$$

It is easy to see that there must be a solution $\alpha > 0$; therefore, deformation must be unstable. It is very simple to show that no maximum exists in α and α is a monotonically increasing function of β, with

$$\lim_{\beta\to 0} \alpha \to 0 \quad \text{and} \quad \lim_{\beta\to\infty} \alpha \to \frac{|R_0|}{H_0},$$

$$\lim_{\beta\to\infty} t = t_{\min} = \frac{H_0}{|R_0|}.$$

This implies the shorter the wavelength, the earlier the occurrence of instability. Nevertheless, it is a totally different instability mode. There is no further criterion except $R_0 < 0$, with which, as we have seen, no characteristic length and time are associated but there exists a minimum time $t_{\min}$.

4.4. Practical criterion

Now we concentrate on the instability mode dominated by pore pressure softening, by turning towards practical considerations.

It is especially useful that criterion (28) implies a pore pressure criterion. Recalling $[Q_0] = \text{stress/(pore pressure)}$, we can easily deduce that the inequality (28) is equivalent to a pore pressure criterion. It is desirable to establish a criterion connecting state parameters and material constants on each side of the inequality.

If the constitutive relation of the soil concerned is formulated explicitly, the critical pore pressure is easy to obtain. Suppose, for instance,

$$\tau = G_0\left(\frac{\sigma_{e0} - p}{\sigma_{e0}}\right)^b \gamma, \quad (34)$$

where G_0 is the initial shear module, and b is a constant, σ_{e0} is initial effective stress and p is pore pressure. Then, the criterion pore pressure is obtained as follows by the inequality (28):

$$\frac{p}{\sigma_{e0}} > 1 - \frac{C_1 E_r \tau\gamma b}{\sigma_{e0}}. \quad (35)$$

5. Conclusions

It has been shown that two types of possible instability of saturated soil under vibration shear load may exist. One is dominated by pore pressure softening while the other by strain softening.

The criterion for the first mode of instability combines pore pressure softening, strain hardening and resilience module, which can be expressed simply as Eq. (28). This mode of instability may lead to failure of soils. The other one requests $R_0 < 0$.

The above criterion implies a practical critical pore pressure. The critical pore pressure can be obtained simply when the constitutive relation has an explicit expression.

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