Roll Waves in Overland Flow

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Abstract: Roll waves are frequently observed in overland flow, especially in rill flow, which has an important effect on the development of soil erosion. Using one-dimensional St. Venant equations, this paper investigates the dynamics of periodic roll waves based on Dressler's and Brock's work. Under the assumption that the average flow depth equals the uniform flow depth, expressions of the roll-wave speed and roll-wave profile were obtained. Testing with the results observed by Brock in 1970 for wave properties shows that these expressions can approximately describe the characteristics of periodic permanent roll waves. Numerical solutions of roll waves under specific conditions are found. They show that when a roll wave appears, the shear stress of flow increases, and soil erosion accelerates.

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Introduction

Flow patterns in which the flow surface is not flat and its longitudinal profile undulates periodically are often observed in inclined open channels, as shown in Fig. 1. This kind of undulating flow movement is well known and is called roll waves. Roll waves were first observed in spillways and steep aqueducts in mountainous areas. When roll-wave trains occur, the water depth periodically increases, and the water may exceed the side wall and overflow. Also, the phenomenon of roll waves has also been observed in overland flow, especially in rill flow. Yoon and Wenzel (1971) and Emmett (1978), among others, referred to the existence of roll waves in overland flow. During the occurrence of roll waves, the flow is unsteady and nonuniform. The flow intensity increases spatially and temporally, augmenting the potential for soil erosion.

On the basis of limited observations, Cornish (1934) conjectured that roll waves depend primarily on the resistance to flow and that roll waves would not form if there were no resistance. Thomas (1937) analyzed roll waves and derived a necessary condition for formation of roll waves, based on the flow resistance. On the other hand, Rouse (1938) found that roll waves would not occur if the channel bed were sufficiently irregular, i.e., too much resistance would prevent the formation of roll waves. Dressler (1949) mathematically showed that a necessary condition for roll-wave formation was that the resistance must be less than a certain critical value.

Using the theory of flow instability, Benjamin (1957) and Yih (1963, 1977) investigated the linear stability of laminar flow down an inclined plane and determined the condition for instability: the critical Froude number was about 0.5. Starting with Jeffreys (1925), a number of criteria for unstable turbulent uniform flow have been developed (Craya 1952; Iwasa 1954; Koloseus and Davidian 1966). These criteria differ essentially in the assumptions regarding channel shape, velocity profile, and the resistance relation (Brock 1969). For a wide rectangular channel with constant friction factor, the critical Froude number is 2.0 (Liggett 1975). Although the stability condition is quite stringent, it can only be used to analyze ideal situations.

Thomas (1937, 1939) was probably the first to analytically describe large-amplitude roll waves. He considered a periodic wave with constant shape and velocity for unsteady flow and theoretically constructed a wave profile similar to observed roll waves. Using Thomas's basic idea, Dressler (1949) derived a closed-form solution for periodic permanent roll waves in a wide rectangular channel with constant friction. Explicit expressions for water height and shock height as a function of wave length, the static discharge rate as a function of the wave speed, and asymptotic formulas for wave speed in terms of the average discharge rate were derived. However, Dressler's solution contains some parameters that cannot be determined from known quantities. On the basis of Dressler's work, Brock (1970) developed a theory for periodic permanent roll waves using the shallow-water equations and tested them using experimental observations. He incorporated the normal depth into his theory and derived the expressions for wave shape, wave velocity, and wave period. Because of the complexity of the shallow-water equations, explicit solutions could not be found. If Brock's results are used, it becomes difficult to investigate the properties of shear stress distribution and the treatment of roll waves.

Roll waves, in general, are nonperiodic and nonpermanent. Brock (1969) even measured the spontaneously developed roll-wave trains in a laboratory channel. He described two portions of the roll-wave development: (1) the initial development phase, in which the waves do not overtake and combine with other waves; and (2) the final development, in which the wave period increases

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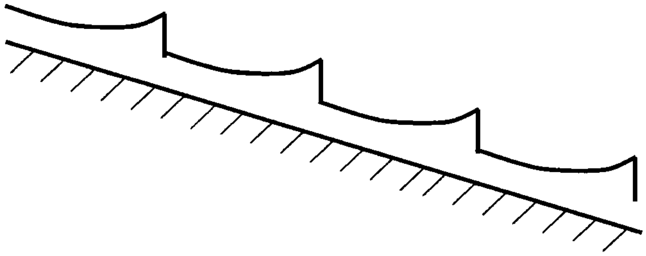

Fig. 1. Profile of roll wave

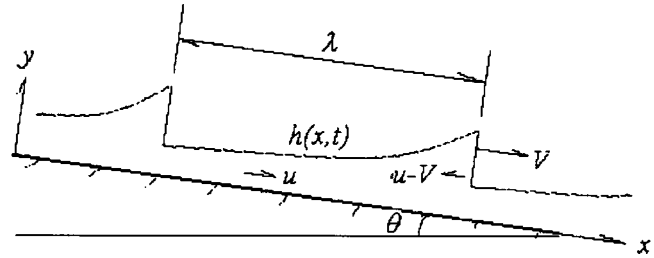

Fig. 2. Definition sketch

with distance as a result of wave overtaking. Zanutigh and Lamberti (2002) numerically simulated natural roll waves down an inclined rectangular channel under the same conditions tested by Brock (1969). The numerical solution accurately represented the evolution of roll waves caused by the uniform flow instability, as well as the increase in wave intensity because of the instability and wave coalescence. As in the experiments of Brock (1969), the computations of Zanuttigh and Lamberti (2002) also showed that the natural roll waves, in general, could not reach the final regime shape, but the wave period and height due to coalescence continuously increased along the channel.

Studies on the formation of roll waves have been conducted in steep channels and spillways. However, roll waves also form in overland flow, especially in rills formed by erosion. The dynamic properties of roll waves are important because the hydrodynamic force distribution becomes nonuniform, which, after the occurrence of waves, has a significant effect on the development of soil erosion. Therefore, the objective of this paper is to further develop an approach for analytical treatment of periodic permanent roll waves on steep hillslopes and to discuss the hydrodynamic force distribution when roll waves form in overland flow to understand their influence on soil erosion, especially rill erosion.

Mathematical Analysis of Roll Waves

Overland flow over a steep slope can be described by using a one-dimensional form of the St. Venant equations expressed as

$$\begin{aligned}&\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+h\frac{\partial u}{\partial x}=q_*\\&\frac{\partial u}{\partial t}+g\cos\theta\frac{\partial h}{\partial x}+u\frac{\partial u}{ux}=g(\sin\theta-S_f)\end{aligned}\qquad(1)$$

where h=water depth; u=flow velocity; g=gravitational acceleration; θ=slope angle; S_f=energy slope; t=time; x=coordinate along the slope; q_* is a source item and represents the rainfall excess, $q_*=p\cos\theta-i$, where p=rainfall intensity (m/s), and i is the infiltration rate (m/s).

In overland flow, the roll wave motion is more complex because of the existence of rainfall over the land surface and infiltration into soil. At present, no theory seems to exist for analytically describing a nonperiodic and nonpermanent roll wave that develops in unsteady flow. It is quite difficult to develop an analytical treatment of roll waves if Eqs. (1) are employed, primarily because of $q_*=p\cos\theta-i$. Since the objective of this study is to better comprehend the hydrodynamic force distribution when roll waves form in overland flow in order to understand their influence on soil erosion, it is assumed that a first approximation of these properties can be obtained by assuming that the source item, q_*, has a negligible effect. Therefore, Eqs. (1) become the same as the equations for open channel flow with a steep slope:

$$\begin{aligned}&\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+h\frac{\partial u}{\partial x}=0\\&\frac{\partial u}{\partial t}+g\cos\theta\frac{\partial h}{\partial x}+u\frac{\partial u}{ux}=g(S_0-S_f)\end{aligned}\qquad(2)$$

where $S_0=\sin\theta$. The channel-flow equation and the sheet-flow equation have essentially no difference. The other assumption made here is that the roll wave is a periodic, traveling wave as well as a periodic hydraulic jump (surge). In the roll-wave flow, the crest of one wave just connects with the trough of another wave through a discontinuity, i.e., a hydraulic jump, as shown in Fig. 2.

For a periodic permanent wave train, Dressler (1949) and Brock (1970) analytically developed the expressions for wave shape, wave speed, and wave period. However, their solutions are not applicable in practical situations. Similar to the treatment of Dressler and Brock, time can be eliminated from Eqs. (2) by introducing $\xi=x-Vt$, in which ξ=a coordinate system moving at the roll-wave velocity, c. The system of Eqs. (2) can be transformed as

$$\begin{aligned}&(u-c)\frac{dh}{d\xi}+h\frac{du}{d\xi}=0\\&(u-c)\frac{du}{d\xi}+g\cos\theta\frac{dh}{d\xi}=g(S_0-S_f)\end{aligned}\qquad(3)$$

From the first equation of Eqs. (3), one gets

$$(u-c)h=\text{const}=q_r\qquad(4)$$

where q_r=discharge relative to the moving coordinate.

Substituting $hdu/d\xi=-(u-c)dh/d\xi$ into the second equation of Eqs. (3) and simplifying, one obtains

$$\frac{dh}{d\xi}=\frac{g(S_0-S_f)}{g\cos\theta-\dfrac{(u-c)^2}{h}}=\frac{S_0-S_f}{\cos\theta-\dfrac{q_r^2}{gh^3}}\qquad(5)$$

In the coordinate system fixed to the wave, the roll wave will be static and the flow velocity will be $u-c$. The actual water velocity u with respect to the fixed inclined plane will be greater in the crest region than in the trough region, but nowhere will the water velocity be smaller than the wave speed c, i.e., $u-c<0$. The roll-wave flow is connected by the hydraulic jump; the flow will turn from the relative subcritical flow into the relative supercritical flow in one wavelength. Therefore, there must be a critical point in each wavelength. At this critical point the gradient, $dh/d\xi$, should be a finite positive value, and the flow depth can be expressed as

$$h_c = (u_c - c)^2/g\cos\theta, \tag{6}$$

where u_c = flow velocity at the critical point. The denominator of $dh/d\xi$ in Eq. (5) is $\cos\theta - q_r^2/gh_c^3 = 0$. This demands that the numerator of Eq. (5) should also be 0. Thomas (1939) was the first to find that this condition must exist, that is to say, at the critical point, there is

$$S_f = S_0 \tag{7}$$

Periodic Condition and Discontinuity Condition

Although the periodic and discontinuity conditions of periodic roll waves are well known, we give their expressions for the sake of completeness and to do the analysis here. For periodic roll waves, the average values of dynamic parameters in one wavelength are equal to the average values in one period. The average unit discharge in one period at a point $x = x_0$ is

$$\overline{q(t)} = \frac{1}{T}\int_0^T q(t,x_0)dt = \frac{1}{T}\int_0^T q(x_0 - Vt)dt = \frac{1}{T}\int_0^T q\left(x_0 - \frac{\lambda}{T}t\right)dt \tag{8}$$

Let $x = x_0 - (\lambda/T)t, dx = -(\lambda/T)dt$. Then, one obtains

$$\overline{q(t)} = \frac{1}{T}\int_\lambda^0 q(x)dx\left(-\frac{T}{\lambda}\right) = \frac{1}{\lambda}\int_0^\lambda q(x - Vt_0)dx = \overline{q(x)} \tag{9}$$

where t_0 is any time. This equation shows that the average unit discharge in one period of the roll-wave flow is equal to the average value in one wavelength. The rest may be deduced by analogy.

If $\bar{q}$ is used to represent the average unit discharge, it certainly equals the unit discharge of uniform flow under undisturbed conditions q_0, or, say, the steady incoming flow discharge, i.e., $\bar{q} = q_0$. Within one wave length there should be a point where the flow discharge equals q_0. Suppose that the velocity at this point is u_0 and that the water depth is h_0. Then, from Eq. (4), one gets

$$q_r = (u_0 - c)h_0 = q_0 - ch_0 \tag{10}$$

Thus, one has

$$q_0 = \bar{q} = \frac{1}{\lambda}\int_0^\lambda uhdx = \frac{c}{\lambda}\int_0^\lambda hdx + q_r \tag{11}$$

Eq. (11) leads to

$$\frac{1}{\lambda}\int_0^\lambda hdx = h_0 \tag{12}$$

which indicates that the average flow depth in one wavelength is equal to h_0.

The two adjoining waves of periodic permanent roll waves can be seen to connect through a discontinuity, or say, a hydraulic jump. The water depths at the two sides of the discontinuity in the moving coordinate system satisfy the hydraulic jump relation that can be expressed as

$$h_1h_2(h_1 + h_2) = \frac{2q_r^2}{g} = \frac{2h_c^3r_2^2}{g} = 2h_c^3\cos\theta \tag{13}$$

where $h_1 > h_2$, i.e., h_2 = depth before the jump and h_1 = depth after the jump. Solving Eq. (13) yields

$$h_1 = \frac{1}{2}\left(\sqrt{h_2^2 + 8h_c^3\cos\theta/h_2} - h_2\right) \tag{14}$$

Roll-Wave Speed

According to the hydraulic jump condition, at the critical point there is $q_r^2 = gh_c^3\cos\theta$. Because of Eq. (4) and because $u - c < 0$, the value of q_r is negative. Thus, one has

$$q_r = -(gh_c^3\cos\theta)^{1/2} \tag{15}$$

If the velocity is calculated using the Chézy formula, then

$$u_c = C\sqrt{h_cS_f} = C\sqrt{h_cS_0} \tag{16}$$

where C = Chézy coefficient. Therefore,

$$c = u_c - \frac{q_r}{h_c} = [CS_0^{1/2} + (g\cos\theta)^{1/2}]h_c^{1/2} = (r_1 + r_2)h_c^{1/2} \tag{17}$$

in which $r_1 = CS_0^{1/2}$ and $r_2 = (g\cos\theta)^{1/2}$.

The average discharge during a roll wave equals the uniform flow discharge under steady state. For an approximate treatment, it is reason to assume that the average depth equals the uniform flow depth. Thus, the uniform flow formulas are applicable; and therefore

$$\begin{aligned} u_0 &= C\sqrt{RS_f} = C\sqrt{h_0S_0} \\ q_0 &= u_0h_0 = Ch_0^{3/2}S_0 \end{aligned} \tag{18}$$

Considering that $q_r = q_0 - ch_0 = q_0 - cu_0^2/r_1^2$, $q_r = -h_c^{3/2}r_2$, and $q_0 = u_0h_0 = u_0^3/r_1^2$, in addition to the critical flow depth condition $h_c = [q_r^2/(g\cos\theta)]^{1/3} = [(cu_0^2/r_1^2 - q_0)/r_2]^{2/3}$, one obtains a cubic equation for V

$$c^3 = (r_1 + r_2)^3\left(\frac{u_0^2c}{r_1^2r_2} - \frac{u_0^3}{r_1^2r_2}\right) \tag{19}$$

Eq. (19) has two real roots, which are

$$c_1 = \frac{r_1 + r_2}{r_1}u_0, \qquad c_2 = \frac{-r_2(r_1 + r_2) + \sqrt{r_2}(r_1 + r_2)\sqrt{4r_1 + r_2}}{2r_1r_2}u_0 \tag{20}$$

An analysis of these two roots shows that when $\theta \to 90°$, u_0 does not approach zero. So $u_2 \to \infty$ is not reasonable. Hence, the wave speed c of a roll wave is

$$c = c_1 = \frac{r_1 + r_2}{r_1}u_0 \tag{21}$$

Therefore, with the assumption that the average depth equals the uniform flow depth, one can easily find the value of the roll-wave speed. Moreover, one can also obtain the critical depth h_c

$$h_c = \frac{c^2}{(r_1 + r_2)^2} = \frac{u_0^2}{r_1^2} = h_0 \tag{22}$$

Water Surface Profile

With the assumption that the average depth equals the uniform flow depth, one may derive an expression for the roll-wave surface profile. From Eq. (5), the water-surface profile of a roll wave flow can be recast as

$$\frac{dh}{d\xi}=\frac{S_0-S_f}{\cos\theta-\frac{q_r^2}{gh^3}}=\frac{gh^3\left(S_0-\frac{u^2}{C^2h}\right)}{gh^3\cos\theta-g\cos\theta h_c^3}=\frac{C^2S_0h^3-h^2u^2}{C^2\cos\theta(h^3-h_c^3)} \tag{23}$$

$$\begin{aligned}\text{numerator}&=C^2S_0h^3-[(u-c)h+ch]^2\\&=\frac{u_c^2h^3}{h_c}-[(u_c-c)h_c+ch]^2\\&=\frac{h-h_c}{h_c}[h_c^2(c-u_c)^2+h^2u_c^2+hh_c(u_c^2-c^2)]\end{aligned} \tag{24}$$

where the velocity critical point u_c is

$$u_c=Ch_c^{1/2}S_0^{1/2}=c+\frac{q_r}{h_c} \tag{25}$$

From Eq. (23), the relationship between c,u_c and r_1,r_2 can be simplified as

$$\frac{dh}{d\xi}=\tan\theta\frac{h^2-(2k+k^2)h_ch+k^2h_c^2}{(h^2+hh_c+h_c^2)} \tag{26}$$

where $k=r_2/r_1$. In Eq. (26), the numerator on the right side can be factored as $(h-h_a)(h-h_b)$, where h_a and h_b are

$$h_a,h_b=\frac{k[(2+k)\pm\sqrt{k^2+4k}]}{2}h_c \tag{27}$$

The condition for the occurrence of roll waves that has been derived requires $k<0.5$, which leads to $0<h_b<h_a<h_c$. Thus, the implicit solution of h and ξ can be obtained from Eq. (26)

$$\xi=\cot\theta\left(\text{const}+h+A\ln\frac{h-h_a}{h_c-h_a}-B\ln\frac{h-h_b}{h_c-h_b}\right) \tag{28}$$

in which

$$A=\frac{h_a^2+h_ah_0+h_0^2}{h_a-h_b},\qquad B=\frac{h_b^2+h_ah_0+h_0^2}{h_a-h_b}$$

Choosing the constant to satisfy $\xi=0$ when $h=h_0$, and substituting h_c with h_0, one has

$$\xi=\cot\theta\left(h-h_0+A\ln\frac{h-h_a}{h_0-h_a}-B\ln\frac{h-h_b}{h_0-h_b}\right) \tag{29}$$

Because $\xi(h_1)-\xi(h_2)=\lambda$, Eq. (29) yields

$$\lambda=\cot\theta\left(h_1-h_2+A\ln\frac{h_1-h_a}{h_2-h_a}-B\ln\frac{h_1-h_b}{h_2-h_b}\right) \tag{30}$$

Hence, the roll wave can be summed up as follows:

$$\begin{aligned}&\lambda=\cot\theta\left(h_1-h_2+A\ln\frac{h_1-h_a}{h_2-h_a}-B\ln\frac{h_1-h_b}{h_2-h_b}\right)\\&\frac{1}{\lambda}\int_0^\lambda h(\xi)d\xi=h_0\\&h_1=\frac{\sqrt{h_2^2+8h_0^3\cos\theta/h_2}-h_2}{2}\end{aligned} \tag{31}$$

in which h_0, h_a, and h_b are all known.

Although Eq. (31) is in closed form, its solution is difficult because of the existence of the implicit integral. To solve Eq. (31), it may be noted that for any monotonic function $y=f(x)$ defined in an interval $[a,b]$ and its inverse function $x=g(y)$, the definite integral $F=\int_a^b f(x)dx$ has the following relation with $G=\int_{f(a)}^{f(b)}g(y)dy$

$$G+F=bf(b)-af(a) \tag{32}$$

Eq. (32) can be easily obtained by integration by parts. Thus, the implicit integral in Eq. (31) can be converted as

$$\lambda h_0=\int_0^\lambda h(\xi)d\xi=\int_a^b h(\xi)d\xi=bh_1-ah_2-\int_{h_2}^{h_1}\xi(h)dh \tag{33}$$

where a and b are values of ξ corresponding to h_2 and h_1, respectively, within one wave length ($\xi=0$, $h=h_0$, $a<0<b$).

Then, with the relation of ξ and h, one can have

$$\begin{aligned}b&=h_1-h_0+A\ln\frac{h_1-h_a}{h_0-h_a}-B\ln\frac{h_1-h_b}{h_0-h_b}\\a&=h_2-h_0+A\ln\frac{h_2-h_a}{h_0-h_a}-B\ln\frac{h_2-h_b}{h_0-h_b}\end{aligned} \tag{34}$$

and

$$\begin{aligned}\lambda h_0&=bh_1-ah_2-\cot\theta\int_{h_2}^{h_1}\left(h-h_0+A\ln\frac{h-h_a}{h_a-h_b}-B\ln\frac{h-h_b}{h_a-h_b}\right)dh\\&=bh_1-ah_2-\cot\theta\left[-\frac{1}{2}(2A-2B+2h_0-h_1-h_2)(h_1-h_2)+A(h_1-h_a)\ln\frac{h_1-h_a}{h_0-h_a}\right.\\&\quad\left.-A(h_2-h_a)\ln\frac{h_2-h_a}{h_0-h_a}-B(h_1-h_b)\ln\frac{h_1-h_a}{h_0-h_a}-B(h_2-h_b)\ln\frac{h_2-h_a}{h_0-h_a}\right]\end{aligned} \tag{35}$$

In this manner, once the average discharge q_0, the slope angle θ, and the resistance coefficient C are known, the average water depth h_0, the average velocity u_0, the wave speed c, as well as h_a,h_b,A,B,a,b can be obtained, Then, the set of Eqs. (31) can be solved; and the wave length λ, the largest water depth h_1, the smallest water depth h_2 and the wave height h_1-h_2 can be obtained.

It may be useful to write Eqs. (31) in full form

Table 1. Pertinent Characteristics of Six Experiments (Cases)

Characteristic	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
Slope gradient	5°	5°	15°	15°	25°	25°
Discharge (mL/s)	100	196	100	203	101	192
Average flow depth at outlet of slope (mm)	0.74	1.29	0.68	0.89	0.32	0.61

$$\lambda h_0 = bh_1 - ah_2 - \cot\theta\left[-\frac{1}{2}(2A-2B+2h_0-h_1-h_2)(h_1-h_2) + A(h_1-h_a)\ln\frac{h_1-h_a}{h_0-h_a} - A(h_2-h_a)\ln\frac{h_2-h_a}{h_0-h_a} - B(h_1-h_b)\ln\frac{h_1-h_a}{h_0-h_a} - B(h_2-h_b)\ln\frac{h_2-h_a}{h_0-h_a}\right]\lambda$$
$$= \cot\theta\left(h_1-h_2+A\ln\frac{h_1-h_a}{h_2-h_a} - B\ln\frac{h_1-h_b}{h_2-h_b}\right)h_1$$
$$= \frac{\sqrt{h_2^2+8h_0^3\cos\theta/h_2}-h_2}{2} \tag{36}$$

The term u_0^2/g can be used as a reference length for nondimensionalizing the above equations. The resulting equations will have the same form as the original equations. Thus, one can directly regard them as dimensionless equations. From these equations one can find that when $\lambda\to 0$, h_1, $h_2\to h_0$ when $\lambda\to\infty$, $h_2\to h_a$, $h_1\to[(h_a^2+8h_0^3\cos\theta/h_a)^{1/2}-h_a]/2$, and $h_0=u_0^2/r_1^2=u_0^2k^2/r_2^2=u_0^2k^2/(g\cos\theta)$.

Experimentation and Comparison with Brock's Observation

To understand the characteristics of roll waves in overland flow on a hillslope, roll waves were observed in a laboratory flume in the Artificial Rainfall Hall at the Northwestern Institute of Water and Soil Conservation of the Chinese Science Academy in Yingliang, China. The test flume was 130 cm long, 50 cm wide, and 10 cm deep, with a smooth bottom made up of organic glass, as shown in Fig. 3. The slope of the flume was adjustable at different slopes. The water flow was supplied at the top inlet of the flume instead of artificial rainfall. The runoff discharge was controlled at the inlet of the flume for different slope gradients. For a series of experiments corresponding to different slopes (5,15,25°) with different discharges, the roll-wave phenomenon was observed in overland flow. Pertinent characteristics of six experiments (cases) for different slopes and discharges are given in Table 1.

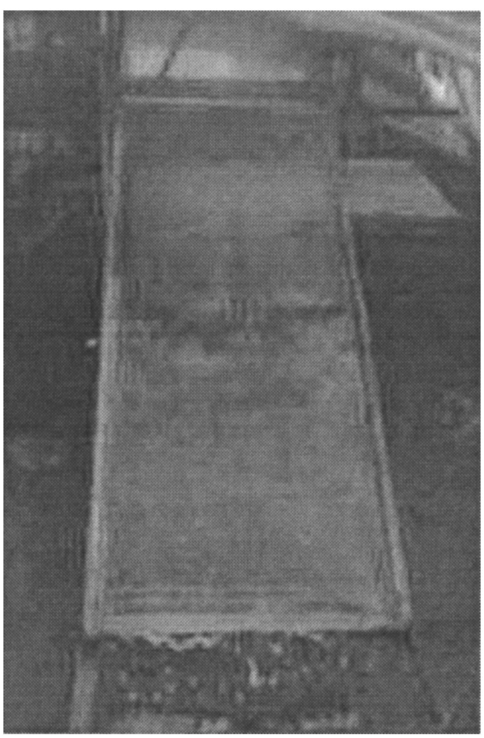

Fig. 3. Experimental flume

Because of the limitations of the measurement instruments, actual measurements could, unfortunately, not be made but roll waves were observed qualitatively. Although quantitative observations were not made, it is thought that these qualitative observations are important for understanding the properties of roll waves in overland flow. Experimental observations showed that roll waves occurred in overland flow. Usually, the roll wave is not stable and cannot remain periodically stable. It often occurs such that the back wave catches up with the front wave and then the two waves merge into one wave. The occurrence of roll waves led to the variation of the hydrodynamic force distribution in overland flow.

Brock (1970) observed in the laboratory the wave properties of periodic permanent roll waves. To compare the theory with Brock's experimental results, the dimensionless quantities $c'(=c/(gh_0)^{1/2})$, $h'_{\max}(=h_1/h_0)$, and $h'_{\min}(=h_2/h_0)$ were be used. Table 2 contains the experimental and theoretical results of Brock (1970) for the 17 experimental runs made. The Froude number F ranged from 2.65 to 5.90 whereas S_0 varied from 0.01942 to 0.1192. In particular, the predicted values of c' are smaller than those observed. In addition, the wave speed c in the present theory only related to the normal depth of flow and Froude number, which cannot reflect its variation along with the wave period well. Nevertheless, Table 2 shows that the proposed theory is suitable for both small and large slopes and can approximately describe the periodic permanent roll waves. In addition, the wave properties of roll waves can be easily obtained by the proposed

Table 2. Comparison of Theoretical Results and Experimental Results of Brock (1970)

Run	Channel length (m)	Channel width (cm)	Channel surface	S_0	h_0 (cm)	Froude number	S_0T'	Experiments c'	Experiments h'_{max}	Experiments h'_{min}	Brock's theory, small S_0 c'	Brock's theory, small S_0 h'_{max}	Brock's theory, small S_0 h'_{min}	Brock's theory, large S_0 c'	Brock's theory, large S_0 h'_{max}	Brock's theory, large S_0 h'_{min}	Present theory c'	Present theory h'_{max}	Present theory h'_{min}
1	40	110	smooth	0.01942	1.98	2.65	1.30	3.94	1.22	0.86	3.70	1.21	0.86	3.75	1.19	0.87	3.73	1.20	0.85
2	39	11.7	smooth	0.05011	0.800	3.71	1.08	5.18	1.30	076	4.80	1.43	0.72	5.05	1.34	0.76	5.06	1.38	0.73
3	39	11.7	smooth	0.05011	0.800	3.71	1.64	5.27	1.46	0.68	4.90	1.66	0.66	5.15	1.50	0.71	5.06	1.56	0.65
4	39	11.7	smooth	0.05011	0.800	3.71	2.14	5.36	1.63	0.63	5.00	1.83	0.63	5.30	1.62	0.69	5.06	1.66	0.62
5	39	11.7	smooth	0.08429	0.530	4.63	1.63	6.46	1.54	0.66	5.90	1.97	0.55	6.45	1.62	0.63	6.63	1.71	0.58
6	39	11.7	smooth	0.08429	0.800	4.96	2.50	7.01	1.91	—	6.65	2.56	—	7.30	2.06	—	6.96	2.13	0.55
7	39	11.7	smooth	0.08429	0.530	4.63	2.89	6.79	2.00	0.56	6.25	2.54	0.50	6.85	2.03	0.60	6.63	2.10	0.59
8	39	11.7	smooth	0.08429	0.530	4.63	4.07	7.15	2.35	0.54	6.55	2.94	0.51	7.15	2.32	0.60	6.63	2.19	0.52
9	39	11.7	smooth	0.08429	0.530	4.63	4.53	7.24	2.49	0.53	6.65	3.06	0.52	7.25	2.40	0.61	6.63	2.28	0.50
10	24	11.7	smooth	0.1192	0.534	5.60	2.25	7.74	1.78	0.52	7.30	2.69	0.43	8.20	2.06	0.53	8.32	2.17	0.54
11	24	11.7	smooth	0.1192	0.534	5.60	3.55	8.24	2.31	0.47	7.75	3.31	0.42	8.65	2.53	0.52	8.32	2.47	0.49
12	24	11.7	smooth	0.1192	0.784	5.90	4.25	8.82	2.65	0.44	8.40	3.78	0.41	9.30	2.90	0.51	8.47	2.61	0.42
13	24	11.7	smooth	0.1192	0.534	5.60	5.19	8.76	2.82	0.45	8.25	3.93	0.44	9.15	2.94	0.55	8.32	2.65	0.43
14	24	11.55	rough	0.1192	0.506	3.74	1.98	5.43	1.34	0.70	4.95	1.79	0.64	5.50	1.45	0.73	5.24	1.65	0.65
14	24	11.55	rough	0.1192	0.506	3.74	3.73	5.73	1.54	0.67	5.30	2.26	0.61	5.80	1.69	0.73	5.24	1.71	0.62
16	24	11.55	rough	0.1192	0.954	4.04	4.19	6.14	1.55	0.75	5.75	2.57	0.58	6.35	1.88	0.70	5.74	1.78	0.67
17	24	11.55	rough	0.1192	0.506	3.74	5.64	5.95	1.68	0.62	5.55	2.61	0.65	6.00	1.85	0.76	5.24	1.79	0.60

theory, which contains only three known parameters: average discharge, bed slope, and resistance coefficient.

Actually the experiments of Brock (1970) and the computations of Zanuttigh and Lamberti (2002) have shown that natural roll waves are nonperiodic and nonpermanent and that the wave period and height of roll waves continuously increase along the channel due to the wave irregularity and coalescence. Therefore, the proposed theory is only an approximate description for periodic permanent roll waves, or for the stage of roll waves before wave coalescence occurs.

Analysis of Water Surface Profile

Fig. 4 plots curves of periodic and dynamic relationships for several k values on a 15° slope. Here, the periodic condition and the dynamic condition refer to the first two equations in the final set of Eqs. (36). The former is obtained from the periodic property, which is derived from implicit integration, and the latter is found from the water surface profile. Fig. 4 shows that if λ is greater than a certain value, then these two curves approach each other. On this 15° slope, when λ exceeds 40 times h_1, the difference between the curves is less than 10^{-4}. The corresponding h_1 is extremely close to the maximum value that it can reach when $\lambda \to \infty$. This outcome indicates that when the wave length exceeds a certain value, the periodic relationship and the dynamic relationship can be satisfied simultaneously. The appearance of roll waves is caused by the instability of primary flow. If a roll wave appears, the wave height will not be close to 0. Hence, it is inferred that the solution of h_1 approaches its extreme value as

$$h_1 = \frac{1}{2}\left(\sqrt{h_a^2 + 8h_0^3 \cos\theta / h_a} - h_a\right)$$

On the other hand, a very small variation of h_1 can correspond to a large variation of λ. This relation indicates that the strict periodic roll-wave flow is unstable, or in other words, the roll wave cannot remain stable periodically. This conclusion is in accord with the practical observations. The actual roll wave is not

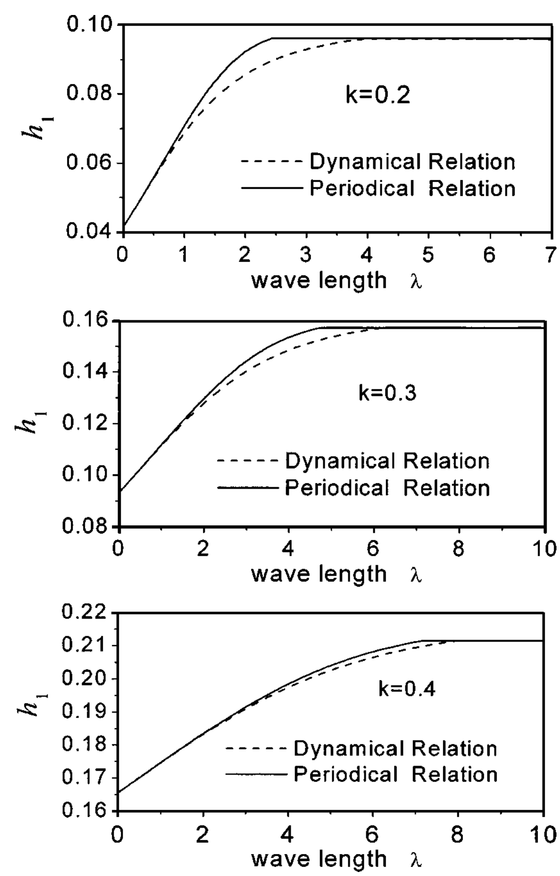

Fig. 4. Periodic condition and dynamic condition of roll waves

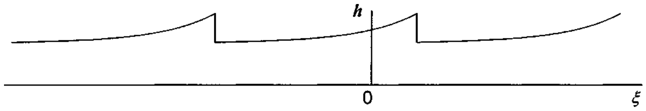

Fig. 5. Water surface curve of roll wave on a 15° slope; $k=0.4$, the ratio of h/ξ is enlarged 25 times

Fig. 7. Sketch of rill bottom

stable and does not usually have a strict periodicity. It often occurs such that the back wave catches up with the front wave and they merge into one wave. A plot of the water- surface curve is given in Fig. 5 for related parameters, $\theta=15°$, $k=0.4$. For purposes of clarity, the ratio of the flow depth and the flow-direction coordinate h/ξ is enlarged 25 times in Fig. 5.

Distribution of Flow Shear Stress

If the flow shear stress is calculated with $\tau=\gamma hS_f$, the ratio of the actual shear stress at one point to the shear stress of uniform flow is hS_f/h_0S_0. Using Chézy's formula, $S_f=u^2/(C^2h)$, along with $u=c+q_r/h$, $q_r=-h_0^{3/2}r_2$, and $c=(r_1+r_2)h_0^{1/2}$, one obtains

$$\frac{hS_f}{h_0S_0}=\frac{u^2}{u_0^2}=\left[1+k\left(1-\frac{h_0}{h}\right)\right]^2 \tag{37}$$

Eq. (37) indicates that the maximum shear stress appears at the place of the maximum water depth. The ratio of the maximum shear stress to the shear stress of uniform flow for varying k on a 15° slope is shown in Fig. 6. It shows that the maximum shear stress can increase 10 – 30% when the roll wave appears. This finding is helpful in understanding soil erosion on a hillslope. It shows that when a roll wave occurs in overland flow on a slope, the overland flow exhibits different hydraulic characteristics. One of the characteristics is to increase the flow-erosion potential; that is, soil erosion can occur more easily on the slope when a roll wave forms, i.e., the roll wave existing in overland flow can accelerate soil erosion on slopes.

In addition, it is thought that at least two other aspects are linked with roll waves in overland flow:

- In the Loess Plateau area of China, most rill erosion forms a kind of typical geometric configuration with sidestep, as shown in Fig. 7.
- Although many factors influence the rill generation, roll waves may be one of the important factors, and it is reasoned that roll waves increase the probability of rill generation.

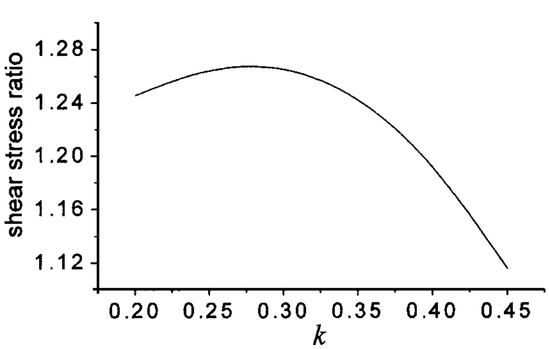

Fig. 6. The ratio of maximum shear stress to uniform-flow shear stress

Conclusions

The occurrence of roll waves in overland flow has an important effect on the development of soil erosion on hillslopes. For discussing the main influence of roll waves on soil erosion, the roll wave in overland flow is simplified as a periodic traveling wave. Then, on the basis of Dressler's (1949) and Brock's (1970) work, an analytical treatment for the dynamic properties of roll waves is developed using the St. Venant equations and the Chézy resistance formula. Under the assumption that the average depth equals the uniform-flow depth, the closed explicit expressions for the wave speed, the profile of the roll wave, and entire treatment of roll waves were derived. The wave properties of roll waves can be easily obtained from only three known parameters: average discharge, bed slope, and resistance coefficient. The equations are solved numerically under some specific parameter values. The results show that the water-flow shear stress increases once the roll wave occurs, and it can accelerate the soil erosion process on slopes and increase the probability of rill generation.

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Notation

The following symbols are used in this paper:

A = parameter;
a = parameter;
B = parameter;
b = parameter;
C = Chézy coefficient;
c = wave speed of roll wave;
g = gravity acceleration;
h = water depth;
h_a = parameter;
h_b = parameter;
h_c = water depth at the critical point;
h_0 = water depth at the point where the flow discharge equals the average discharge;
h_1 = water depth after the jump;
h_2 = water depth before the jump;
i = infiltration rate of soil;
k = parameter $k=r_2/r_1$;
p = rainfall intensity;
q_r = discharge relative to the moving coordinate;
q_0 = discharge of steady incoming flow;
q_* = rainfall excess $q_*=p\cos\theta-i$;
$\overline{q}$ = average unit discharge in one period of roll wave;
r_1 = parameter $r_1=S_0^{1/2}c$;
r_2 = parameter $r_2=(g\sin\theta)^{1/2}$;

S_f = energy slope;
S_0 = bed slope $S_0 = \sin\theta$;
T = wave period of roll wave;
t = time;
u = average flow velocity;
u_c = flow velocity at the critical point;
u_0 = flow velocity at the point where the flow discharge equals the average discharge;
x = coordinate along the slope;
γ = specific gravity of water;
θ = slope angle;
λ = wave length of roll wave;
ξ = coordinate of moving coordinate system fixed to the roll wave, $\xi = x - Vt$; and
τ = flow shear stress.

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