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# Variational approach to the Thomas–Fermi equation

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**Abstract**

By the semi-inverse method, a variational principle is obtained for the Thomas–Fermi equation, then the Ritz method is applied to solve an analytical solution, which is a much simpler and more efficient method.



*Keywords:* Variational principle; Ritz method; Thomas–Fermi equation

Recently, Bender et al. [1] proposed a new perturbation technique based on an artificial parameter $\delta$, the method is often called $\delta$-method [1,2]. Consider the Thomas–Fermi equation [1–4]

$$u''(x) = x^{-1/2}u^{3/2}, \quad u(0) = 1, \quad u(\infty) = 0. \tag{1}$$

The basic idea of the $\delta$-method is to replace the right-hand side of the Thomas–Fermi equation by one which contains the parameter $\delta$, i.e.

$$u''(x) = u^{1+\delta}x^{-\delta}. \tag{2}$$

The solution is assumed to be expanded in a power series in $\delta$

$$u = u_0 + \delta u_1 + \delta^2 u_2 + \cdots. \tag{3}$$

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To solve $u_n$ for $n > 1$, we need some unfamiliar functions, so it might meet some difficulties in promoting this method.

Khuri [4], Adomian [5] and Wazwaz [6,7] applied the decomposition method to the Thomas–Fermi equation, a review on some new recently developed nonlinear analytical techniques can be found in detail in Ref. [8]. In this paper, we will use the Ritz's method to obtain an analytical solution of the problem. By the semi-inverse method [9–11], we can easily obtain the following functional:

$$J(u) = \int_0^\infty \left( \frac{1}{2} u'^2 + \frac{2}{5} x^{-1/2} u^{5/2} \right) \mathrm{d}x, \tag{4}$$

whose Euler equation satisfies the Thomas–Fermi equation.

We choose the trial function in the form

$$u = \mathrm{e}^{-\beta x}, \tag{5}$$

where $\beta$ is an unknown constant.

Submitting (5) into (4), the functional turns out to be a function of $\beta$

$$J(\beta) = \int_0^\infty \left( \frac{1}{2} \beta^2 \mathrm{e}^{-2\beta x} + \frac{2}{5} x^{-1/2} \mathrm{e}^{-5\beta x/2} \right) \mathrm{d}x = \frac{1}{4}\beta + \Gamma(1/2)\sqrt{\frac{2}{5\beta}}. \tag{6}$$

The stationary condition of the functional (5) can be now approximately obtained by

$$\frac{\partial J}{\partial \beta} = \frac{1}{4} - \sqrt{\frac{\pi}{10}} \beta^{-3/2} = 0, \tag{7}$$

which leads to the result

$$\beta = \left( \frac{8\pi}{5} \right)^{1/3}. \tag{8}$$

A highly accurate numerical solution of the Thomas–Fermi equation has been provided by Kobayashi et al. who give the initial slop

$$u'_{\mathrm{exact}}(0) = -1.5880710. \tag{9}$$

From Eq. (5), we have

$$u(0) = -1.71299. \tag{10}$$

The 7.8% accuracy of the initial slop is remarkably good in view of the crudeness of the trial function. We can obtain a much better result by using a trial function involving few parameters, such as $u = \mathrm{e}^{-\beta x}(1 + c_1 x + c_2 x^2 + \cdots)$, where $c_i$ are unknown constants to be further determined by the Ritz method.

Hereby we propose a simple variational approach to the Thomas–Fermi equation, which reveals much better than Adomian's decomposition method,

$\delta$-method, and others. The present method has the rapidity of convergence, the first-order approximation obtained by the present technology is always of high accuracy.

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