Heteroclinic cycles in lattice dynamical systems *

QIN Wenxin (秦文新)

(LNM, Institute of Mechanics, Chinese Academy of Sciences, Beijing 100080;

Department of Mathematics, Suzhou University, Suzhou 215006, China)

and QIAN Min (钱　敏)

(Department of Mathematics, Peking University, Beijing 100871, China)

Received April 28, 1997

Abstract A criterion of spatial chaos occurring in lattice dynamical systems ——heteroclinic cycle ——is discussed. It is proved that if the system has asymptotically stable heteroclinic cycle, then it has asymptotically stable homoclinic point which implies spatial chaos.

Keywords: lattice dynamical system, heteroclinic cycle, homoclinic point.

Lattice dynamical system is an efficient tool for analyzing space time chaos. So far there are more papers on numerical simulation (see ref. [1] and the references therein) than on rigorous mathematical analysis[2,3]. Ref. [2] proved for the first time the existence of space time chaos in terms of ergodic theory in lattice dynamical systems. In the point of view of dynamical theory, there are not any appropriate definitions for space time chaos. Some authors[4,5] discussed the lattice dynamical systems from the point of view of spatial complexity. A result obtained in ref. [5] indicates that an asymptotically stable homoclinic point implies spatial chaos. This paper is devoted to the relationship between the existence of heteroclinic cycle and spatial chaos in lattice dynamical systems. In the last section we discuss spatial chaos of a concrete example by using our results.

1 Main results

Let $B = \{ \mathrm{u} = \{ u_j \}, j \in \mathrm{Z} \}$, in which Z denotes the set of integers, $u_j \in \mathrm{R}^p$, R^p is p-dimensional Euclidean space with the standard inner produce $(\cdot, \cdot)$ and the norm $|\cdot| = \sqrt{(\cdot, \cdot)}$.

Let $l^\infty = \{ \mathrm{u} \mid \mathrm{u} \in B, \ \| u \|_{l^\infty} < \infty \}$, in which $\| u \|_{l^\infty} = \sup_{j \in Z} | u_j |$. In the sequel the subscript l^∞ in $\| \cdot \|_{l^\infty}$ is omitted. Let us denote by $(\mathrm{u})_j$ the coordinate of u at site j.

Consider in l^∞ the map $\mathbf{F}: l^\infty \to l^\infty$

$$(\mathbf{F}\mathrm{u})_j = F(\{ u_j \}^s), \tag{1.1}$$

in which $\{ u_j \}^s = \left(u_{j-s}, u_{j-s+1}, \cdots, u_j, u_{j+1}, \cdots, u_{j+s} \right) \in (\mathrm{R}^p)^{2s+1}$, F: is a $C^k (k \geq 2)$ map from $(\mathrm{R}^p)^{2s+1}$ to R^p. It is easy to show that $\mathbf{F}: l^\infty \to l^\infty$ is C^k if F is C^k. In this paper it is always supposed that F is $C^k (k \geq 2)$.

In space l^∞, we should consider spatial translations $S_{j_0} (j_0 \in Z)$ besides the time evolution $\mathbf{F}$

* Project supported by the National Natural Science Foundation of China.

$$(S_{j_0}\mathrm{u})_j = u_{j+j_0},\ j\in \mathrm{Z}. \tag{1.2}$$

It is obvious that each element S_{j_0} is a linear bounded operator and commutes with the evolution operator $\mathbf{F}$ for system (1.1):

$$S_j\circ \mathbf{F} = \mathbf{F}\circ S_j,\ j\in \mathrm{Z}. \tag{1.3}$$

Definition 1. u is called the steady-state solution of the system $\{\mathbf{F}^n\}_{n=1}^{+\infty}$ if it satisfies $\mathbf{F}\mathrm{u} = \mathrm{u}$.

Let $A = \{\mathrm{u} \mid \mathbf{F}\mathrm{u} = \mathrm{u}\}$. Obivously, $S_jA = A\ \forall j\in \mathrm{Z}$.

Definition 2. Suppose u is a steady-state solution of $\mathbf{F}$. If there are constants $c>0, 0<q<1$ and $\rho>0$ such that

$$\|\mathbf{F}^n\mathrm{u}' - \mathbf{F}^n\mathrm{u}''\| \le cq^n\,\|\mathrm{u}' - \mathrm{u}''\|,\ \forall n\in \mathrm{N}, \tag{1.4}$$

for any $\mathrm{u}', \mathrm{u}''\in l^\infty$, satisfying $\|\mathrm{u}' - \mathrm{u}\|\le\rho$, $\|\mathrm{u}'' - \mathrm{u}\|\le\rho$, then u is said to be asymptotically stable.

Definition 3. Suppose $\mathrm{u} = \{u_j\}\in A$. $\mathbf{U} = \left\{\mathrm{v} \mid \mathrm{v} = \{v_j\}\in l^\infty, |v_j - u_j| < \varepsilon, j\in \mathrm{Z}\right\}$ is called the homogeneous neighborhood of u with radius ε.

Definition 4. Suppose $\mathrm{u}^1, \mathrm{u}^2, \cdots, \mathrm{u}^n\in A$, $\mathrm{u}^i = \{u_j^i\}$. If there exist $a_1, a_2, \cdots, a_n\in \mathrm{R}^p$ such that

$$\begin{aligned}&\lim_{j\to-\infty} u_j^i = a_i,\ i = 1,2,\cdots,n,\\&\lim_{j\to+\infty} u_j^i = a_{i+1},\ a_{n+1} = a_1,\ i = 1,2,\cdots,n,\end{aligned} \tag{1.5}$$

then we say $\mathrm{u}^1, \mathrm{u}^2, \cdots, \mathrm{u}^n$ form heteroclinic n-cycle. Heteroclinic 1-cycle is called homoclinic point.

Remark 1. Definition 4 is derived from the definition of heteroclinic cycle in low-dimensional systems[6].

Remark 2. Let $\mathrm{v}^i = \{v_j^i\}$, in which $v_j^i = a_i, j\in \mathrm{Z}$. The continuity of $\mathbf{F}$ implies that $\mathrm{v}^i\ (i = 1,2,\cdots,n)$ are also the steady-state solution of $\mathbf{F}$, i.e. $\mathrm{v}^i\in A\ (i = 1,2,\cdots,n)$. In the sequel we always denote by $\mathrm{v}^1, \cdots, \mathrm{v}^n$ the n steady-state solutions corresponding to $\mathrm{u}^1, \cdots, \mathrm{u}^n$.

Theorem 1. *Suppose that* $\mathrm{u}^1, \cdots, \mathrm{u}^n\in A$ *form heteroclinic* n*-cycle, and that* $\mathrm{u}^1, \cdots, \mathrm{u}^n$ and $\mathrm{v}^1, \cdots, \mathrm{v}^n$ *are asymptotically stable. Then the lattice system* $\{\mathbf{F}^n\}$ *has asymptotically stable homoclinic point.*

Theorem 2. *Suppose* $\mathrm{u}^1\in A$ *is a homoclinic point, and* u^1 *and the corresponding* v^1 *are asymptotically stable. Then there exists a constant* $r>0$ *such that for any finite index set* $\{i\}\subset \mathrm{Z}$, *with* $|j' - j''|\ge r$ *for any* $j', j''\in\{i\}$, *the following holds:*

(i) *there exists a homoclinic point* $\mathrm{u}^* = \mathrm{u}^*(\{i\})$ *of* $\mathbf{F}$ *such that*

$$\prod \mathrm{u}^* \in \prod_i \mathbf{U}\left(\mathrm{u}^1 \cup \left(\cup_{j'\in\{i\}} S_{-j}\mathrm{u}^1\right)\right),\ j\in \mathrm{Z},$$

and

$$\prod{}' \mathrm{u}^* \notin \prod_i{}' \mathbf{U}\left(\mathrm{u}^1 \cup \left(\cup_{j''\in\{i\}\setminus j} S_{-j''}\mathrm{u}^1\right)\right),\ j'\in\{i\},$$

where $\prod_i$ *denotes the projection at site* j, $\mathbf{U}(\cdot)$ *is some neighborhood of the corresponding element.*

(ii) u^* *is asymptotically stable.*

Remark. Given an index set $\{i\} = \{j_1, j_2, \cdots, j_n\}$ satisfying the conditions of Theorem 2, without loss of generality, we assume that the homoclinic point u^1 has only one "hump", and the "hump" appears at site $j = 0$. According to Theorem 2, we may construct a new steady-state solution u^* with "n-hump". Furthermore, the shape of u^* at site $j_k (k = 1, 2, \cdots, n)$ is similar to that of u^1 at site $j = 0$. To be precise, the coordinate of u^* at site j_k is contained in a small neighborhood of the coordinate of $S_{-j_k} u^1$ at site j_k.

If the conditions of Theorem 1 are satisfied, then the lattice system has an asymptotically stable homoclinic point, hence the complicated spatial structure appears. If the homoclinic point is viewed as a "one-hump" steady-state solution, then we may construct a steady-state solution with arbitrary finite "humps" as long as the distance between one "hump" and another "hump" is greater than some constant $r > 0$. Then the lattice system is said to have spatial chaos. In fact, the action of the translational group $\{S_j\}$ on the constructed steady-state solution set can be related to symbolic dynamical system. Denote by Λ_r the steady-state solution set constructed according to Theorem 2. Let $w = \{w_j\}$, $w_j \in \{0, 1\}$. $w \in \Sigma_r$ if and only if the equalities $w_{j'} = 1$ and $w_{j''} = 1$ imply the condition $|j' - j''| \geq r$. The set Σ_r is endowed with the metrices $\| w' - w'' \| = \sum_{j \in Z} \frac{1}{2^{|j|}} | w_j' - w_j'' |$. The shift map σ_{j_0} acting on Σ_r is defined as follows:

$$(\sigma_{j_0} w)_j = w_{j + j_0}.$$

Then the symbolic dynamical system $\{\sigma_j\}$ on the set Σ_r describes exactly the action of the translational system $\{S_j\}$ on the set Λ_r of the steady-state solution set.

2 Proofs of theorems

Proof of Theorem 1. We only prove the case $n = 2$, because when $n > 2$, we can construct in a similar way the heteroclinic $(n-1)$-cycle $\cdots$, up to heteroclinic 2-cycle. The proof is divided into three steps. 1. By making use of contraction principle construct a new steady-state solution u^*. 2. Prove u^* is a homoclinic point. 3. Prove u^* is asymptotically stable.

Step 1. Since u^1 is asymptotically stable, from definition 2, it follows that there exist constants $c_1 > 0$, $0 < q_1 < 1$, $\rho_1 > 0$ such that for any two points u and v in the homogeneous neighborhoods of u^1 with radius ρ_1, the following holds:

$$\| \mathbf{F}^n u - \mathbf{F}^n v \| \leq c_1 q_1^n \| u - v \|, \quad n \in N.$$

Take $m_1 > 0$ such that $c_1 q_1^{m_1} = a < 1/6$. Then $\| \mathbf{F}^m u - \mathbf{F}^m v \| \leq a \| u - v \|$ for $m \geq m_1$, i.e. $\mathbf{F}^m$ is contraction in the homogeneous neighborhood of u^1 with radius ρ_1. This homogeneous neighborhood is positively invariant with respect to $\mathbf{F}^m$. In fact, $\mathbf{F}^m$ is contraction in the homogeneous neighborhood of u^1 with radius less than ρ_1, and these neighborhoods are positively invariant for $\mathbf{F}^m$. Since u^1, u^2, v^1 and u^2 are all asymptotically stable, we may take common $m > 0$, $\rho > 0$ such that $\mathbf{F}^m$, $\mathbf{F}^{m-1}$ are contractions in the homogeneous neighborhoods of u^1 (or u^2 or v^1 or v^2) with radius ρ, the contraction factor $a < 1/6$.

The map $\mathbf{F}^m$ in space l^∞ may be written as

$$(\mathbf{F}^m u)_j = F^{(m)} \left(\{u_j\}^{ms} \right), \tag{2.1}$$

where $F^{(m)}$ is a map from $(R^P)^{2ms+1}$ to R^p. From definition 4, it follows that

$$\lim_{j \to +\infty} u_j^1 = a_2, \quad \lim_{j \to -\infty} u_j^2 = a_2.$$

Thus for $\rho > 0$, there exists $j^* > 0$ such that $| u_j^1 - a_2 | < \frac{\rho}{3}$ for $j > j^*$, and $| u_j^2 - a_2 | < \frac{\rho}{3}$ for $j < -j^*$. Take $j_0 \geq j^* + 2ms$. Denote respectively by $\mathbf{U}^1$ and $\mathbf{U}^2$ the homogeneous neighborhood of u^1 and u^2 with radius $\frac{\rho}{3}$. From (1.3) it follows that $S_{j_0}\mathbf{U}^1$ and $S_{-j_0}\mathbf{U}^2$ are invariant for $\mathbf{F}^m$, i.e. $(\mathbf{F}^m(S_{j_0}\mathbf{U}^1) \subset S_{j_0}\mathbf{U}^1, \mathbf{F}^m(S_{-j_0}\mathbf{U}^2) \subset S_{-j_0}\mathbf{U}^2)$ and $\mathbf{F}^m$ is a contraction in these neighborhoods with contraction factor $a < 1/6$. Take

$$V_i = \begin{cases} \prod_i (S_{j_0}\mathbf{U}^1), & i < -ms, \\ \prod_i (S_{j_0}\mathbf{U}^1) \cap \prod_i (S_{-j_0}\mathbf{U}^2), & |i| \leq ms, \\ \prod_i (S_{-j_0}\mathbf{U}^2), & i > ms, \end{cases}$$

where $\prod_i$ is the natural projection on site i. Evidently $V_i \neq \emptyset$, for $|i| \leq ms$. Let $\mathbf{U} = \prod_{i\in Z} V_i$ be an open set in space l^∞. We prove that $\mathbf{F}^m\mathbf{U} \subset \mathbf{U}$, and $\mathbf{F}^m$ is contraction in $\mathbf{U}$. For any $u \in \mathbf{U}$, $u = \{u_j\}$, we have $u_j \in V_j$. For $|j| > ms$, since $(\mathbf{F}^m u)_j$ is determined only by $2ms + 1$ neighbors of u_j, by making use of the invariance of $S_{j_0}\mathbf{U}^1$ and $S_{-j_0}\mathbf{U}^2$ with respect to $\mathbf{F}^m$ we have $(\mathbf{F}^m u)_j \in V_j$ due to the construction of V_j. For $|j| \leq ms$ (without loss of generality, we assume $-ms \leq j \leq 0$) we have $(\mathbf{F}^m u)_j \in \prod_j (S_{j_0}\mathbf{U}^1)$. We now prove $(\mathbf{F}^m u)_j \in \prod_j (S_{-j_0}\mathbf{U}^2)$. Note that $(\mathbf{F}^m u)_j$ is dependent on $\{u_j\}^{ms}$. Take an element $u' = \{u_j'\}$ in $S_{-j_0}\mathbf{U}^2$ such that $u_i' = u_i$ $(-ms \leq i \leq j + ms)$. Then the following holds: $\left|F^{(m)}(\{u_j'\}^{ms}) - (S_{-j_0}u^2)_j\right| = \left|(\mathbf{F}^m u')_j - (\mathbf{F}^m(S_{-j_0}u)^2)_j\right| \leq \|\mathbf{F}^m u' - \mathbf{F}^m(S_{-j_0}u^2)\| \leq a\|u' - S_{-j_0}u^2\| \leq a\frac{\rho}{3}$. Similarly, take two points u' and u'' in the homogeneous neighborhood of v^2 with radius ρ such that their coordinates at site i $(j - ms \leq i \leq j + ms)$ are u_i' and u_i'', respectively. Therefore, $\left|F^{(m)}(\{u_j'\}^{ms}) - F^{(m)}(\{u_j\}^{ms})\right| \leq \|\mathbf{F}^m u' - \mathbf{F}^m u''\| \leq a\rho$. $\left|(\mathbf{F}^m u)_j - (S_{-j_0}u^2)_j\right| = \left|F^{(m)}(\{u_j\}^{ms}) - (S_{-j_0}u^2)_j\right| \leq \left|F^{(m)}(\{u_j\}^{ms}) - F^{(m)}(\{u_j'\}^{ms})\right| + \left|F^{(m)}(\{u_j'\}^{ms}) - (S_{-j_0}u^2)_j\right| \leq a\left(\frac{\rho}{3} + \rho\right) \leq \frac{1}{6}\left(\frac{\rho}{3} + \rho\right) \leq \frac{\rho}{3}$, i.e. $(\mathbf{F}_u^m)_j \in \prod_j (S_{-j_0}\mathbf{U}^2)$.

So far we have proved $\mathbf{F}^m\mathbf{U} \subset \mathbf{U}$. In a similar way we have $\mathbf{F}^{m-1}\mathbf{U} \subset \mathbf{U}$. In the following we will prove $\mathbf{F}^m$ is contraction in $\mathbf{U}$.

Suppose $u' = \{u_j'\}$, $u'' = \{u_j''\} \in \mathbf{U}$. For any $i \in Z$ (assume $i \leq 0$), take two points $v' = \{v_j'\}$, $v'' = \{v_j''\}$ in the homogeneous neighborhood of $S_{j_0}u^1$ with radius ρ such that

$$\begin{cases} v_j' = u_j', & i - ms \leq j \leq i + ms, \\ v_j' = (S_{j_0}u^1)_j, & \text{otherwise}. \end{cases}$$

$$\begin{cases} v_j'' = u_j'', & i - ms \leq j \leq i + ms \\ v_j'' = (S_{j_0}u^1)_j, & \text{otherwise}. \end{cases}$$

Then $\left|(\mathbf{F}^m u')_i - (\mathbf{F}^m u'')_i\right| = \left|F^{(m)}(\{u_i'\}^{ms}) - F^{(m)}(\{u_i''\}^{ms})\right| \leq \|\mathbf{F}^m v' - \mathbf{F}^m v''\| \leq a\|v' - v''\|$, $\forall i \in Z$, and hence $\|\mathbf{F}^m u' - \mathbf{F}^m u''\| \leq a\|u' - u''\|$. Consequently $\mathbf{F}^m$ is a contraction in $\mathbf{U}$.

By contraction principle, there is a unique point $\mathbf{u}^* \in \mathbf{U}$, such that $\mathbf{F}^m \mathbf{u}^* = \mathbf{u}^*$. Due to the fact that $\mathbf{F}^{m-1}\mathbf{u}^* \in \mathbf{U}$ and $\mathbf{F}^m(\mathbf{F}^{m-1}\mathbf{u}^*) = \mathbf{F}^{m-1}(\mathbf{F}^m\mathbf{u}^*) = \mathbf{F}^{m-1}\mathbf{u}^*$, i.e. $\mathbf{F}^{m-1}\mathbf{u}^*$ is also a fixed point of $\mathbf{F}^m$ in $\mathbf{U}$, we have $\mathbf{F}^{m-1}\mathbf{u}^* = \mathbf{u}^*$ from the uniqueness. Consequently, $\mathbf{F}\mathbf{u}^* = \mathbf{F}(\mathbf{F}^{m-1}\mathbf{u}^*) = \mathbf{F}^m\mathbf{u}^* = \mathbf{u}^*$. Then $\mathbf{u}^*$ is a steady-state solution of $\mathbf{F}$.

Step 2. Let $\mathbf{u}^* = \{u_j^*\}$. From definition 4 it follows that $\lim_{j\to+\infty} u_j^2 = a_1$. For sufficiently large j we have $|u_j^2 - a_1| < \frac{2}{3}\rho$ and $|u_j^* - u_j^2| < \frac{\rho}{3}$ due to the construction of $\mathbf{U}$. Then $|u_j^* - a_1| < \rho$ for sufficiently large j. We now prove $\lim_{j\to+\infty} u_j^* = a_1$.

(By contradiction) if this is not true, then $\exists \varepsilon_0 > 0$ and $\exists$ a sequence $j_n \to +\infty$ such that $|u_{j_n}^* - a_1| > \varepsilon_0$, $n = 1, 2, \cdots$. Take an integer k such that $a^k\rho < \varepsilon_0$. Let j_n be sufficiently large. Then we have $|u_i^* - a_1| < \rho$, for $j_n - kms \le i \le j_n + kms$. Take an element $\mathbf{u}'$ in l^∞ such that the coordinate of $\mathbf{u}'$ at site i is $u_i^*(j_n - kms \le i \le j_n + kms)$, and the other coordinates are a_1. Then $|u_{j_n}^* - a_1| = |(\mathbf{F}^{mk}\mathbf{u}^*)_{j_n} - a_1| = |F^{(mk)}(\{u_{j_n}\}^{mks}) - a_1| \le \|\mathbf{F}^{mk}\mathbf{u}' - \mathbf{v}^1\| \le a^k \|\mathbf{u}' - \mathbf{v}^1\| \le a^k\rho < \varepsilon_0$. A contradiction! Consequently $\lim_{j\to+\infty} u_j^* = a_1$ $\lim_{j\to-\infty} u_j^* = a_1$ is proved in a similar way. The above discussion leads to the conclusion that $\mathbf{u}^*$ is a homoclinic point.

Step 3. So far we have proved that there exists a neighborhood $\mathbf{U}$ of $\mathbf{u}^*$ such that for any $\mathbf{u}', \mathbf{u}'' \in \mathbf{U}$, we have $\|\mathbf{F}^m\mathbf{u}' - \mathbf{F}^m\mathbf{u}''\| < a\|\mathbf{u}' - \mathbf{u}''\|$. For any n, $\exists t \in \mathbf{N}$ such that $n = tm + r$, $0 \le r \le m$. Hence $\|\mathbf{F}^n\mathbf{u}' - \mathbf{F}^n\mathbf{u}''\| = \|\mathbf{F}^{mt}(\mathbf{F}^r\mathbf{u}') - \mathbf{F}^{mt}(\mathbf{F}^r\mathbf{u}'')\| \le a^t \|\mathbf{F}^r\mathbf{u}' - \mathbf{F}^r\mathbf{u}''\| \le ca^t\|\mathbf{u}' - \mathbf{u}''\|$. We conclude that $\exists \rho' > 0$, $c_1 = \frac{c}{a} > 0$, $q = \sqrt[m]{a} < 1$, for any $\mathbf{u}', \mathbf{u}'' \in l^\infty$ and $\|\mathbf{u}' - \mathbf{u}^*\| z\rho'$, $\|\mathbf{u}'' - \mathbf{u}^*\| < \rho'$. The following inequality holds: $\|\mathbf{F}^n\mathbf{u}' - \mathbf{F}^n\mathbf{u}''\| \le c_1 q^n \|\mathbf{u}' - \mathbf{u}''\|$. Consequently $\mathbf{u}^*$ is asymptotically stable.

Theorem 1 is proved by combining steps 1—3.

Proof of Theorem 2. The proof of Theorem 2 is exactly similar to that of Theorem 1. Taking $\{i\} = \{i_1, i_2\}$ for an example, we sketch the proof. Suppose the "hump" of $\mathbf{u}^1$ appears at sites $j = 0$. $\mathbf{F}^m$ is contraction in some neighborhood $\mathbf{U}^1$ of $\mathbf{u}^1$. Shifting the "hump" of $\mathbf{u}^1$ to site i_1 and i_2, we get two new steady-state solutions $S_{-i_1}\mathbf{u}^1$ and $S_{-i_2}\mathbf{u}^1$. Then by combining $S_{-i_1}\mathbf{U}^1$ and $S_{-i_2}\mathbf{U}^1$ we obtain a new neighborhood $\mathbf{U}$. By proving $\mathbf{F}^m$ is a contraction in $\mathbf{U}$, we get a steady-state solution $\mathbf{u}^*$ whose shapes at sites i_1 and i_2 are roughly the same as $\mathbf{u}^1$ at site $j = 0$. The asymptotical stability of $\mathbf{u}^*$ is proved in the same way as in Theorem 1.

3 Application

Let $\mathbf{u} = \{u_j\}$. Consider in l^∞ sine-lattice:

$$u_j(n+1) = u_j(n) + \alpha \sin u_j(n) + \beta(u_{j-1}(n) - 2u_j(n) + u_{j+1}(n)), \tag{3.1}$$

in which $\alpha > 0$ and $\beta > 0$. Suppose $\mathbf{F}$ is the corresponding map of (3.1).

To get the steady-state solution, let $\alpha \sin u_j + \beta(u_{j-1} - 2u_j + u_{j+1}) = 0$. Denote $u_j = v_{j-1}$. Then

$$v_j = u_{j+1} = -\frac{\alpha}{\beta}\sin u_j + 2u_j - u_{j-1} = -\frac{\alpha}{\beta}\sin v_{j-1} + 2v_{j-1} - u_{j-1}.$$

We get a map on R^2 by changing the symbols:

$$T:\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} y \\ -\frac{\alpha}{\beta}\sin y + 2y - x \end{pmatrix}.$$

Given a point (x_0, y_0), by iterating T we get the trajectory

$$\cdots,(x_{-n}, y_{-n}), \cdots,(x_{-1}, y_{-1}), (x_0, y_0), \cdots,(x_n, y_n), \cdots.$$

If $\sup_{i\in Z}|x_i| < +\infty$, then $u = (\cdots, x_{-n}, \cdots, x_{-1}, x_0, x_1, \cdots, x_n, \cdots)$ is a steady-state solution of (3.1). In the following we discuss the properties of the steady-state solutions of (4.1) by investigating the map T on R^2.

Evidently, (x, x) is a fixed point of T if x satisfies $\sin x = 0$, and $u = \{u_j\}(u_j = x, j \in Z)$ is a steady-state solution of (3.1). Consider a fixed point of $T(-\pi, -\pi)$, where the Jacobian matrix is

$$\begin{pmatrix} 0 & 1 \\ -1 & 2+\frac{\alpha}{\beta} \end{pmatrix} = A.$$

The characteristic equation is $\lambda^2 - \left(2+\frac{\alpha}{\beta}\right)\lambda + 1 = 0$ which has two positive roots $\lambda_1\lambda_2$, $\lambda_1 < 1$, $\lambda_2 > 1$. Then point $P = (-\pi, -\pi)$ has a saddle type. Similarly the other fixed point $Q = (\pi, \pi)$ has also a saddle type. Both of them have stable manifold W^s and unstable manifold W^u.

Denote by R the symmetric map with respect to line $y = -x$ on R^2. The symmetric map with respect to line $y = x$ is denoted by R'. Then $R(x, y) = (-y, -x)$, $R'(x, y) = (y, x)$. It is easy to verify

$$T\circ R\circ T = R,\ T\circ R'\circ T = R'.$$

Then $R(W^u(P)) = W^s(Q)$ and $R'(W^u(P)) = W^s(P)$. To show $W^u(P) \cap W^s(Q) \neq \emptyset$, $W^s(P) \cap W^u(Q) \neq \emptyset$, it is sufficient to prove the intersection of $W^u(P)$ and line $y = x$ is not empty. Then T has heteroclinic cycle, which implies that (3.1) has heteroclinic cycle in the sense of definition 4.

Denote by (ξ_0, η_0) the eigenvector corresponding to eigenvalue $\lambda_2 > 1$ at P. Then

$$A\begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} = \lambda_2\begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix}, \quad \frac{\eta_0}{\xi_0} = \lambda_2 > 1.$$

Let $E = (-\pi, -\pi/2)$, $G = (-\pi/2, -\pi/2)$. In the following we prove that $W^u(P)$ must intersect line segment EG when $W^u(P)$ leaves triangle PEG. Suppose that a vector (ξ, η) satisfies $1 < \eta/\xi < +\infty$. For $-\pi < y_0 < -\pi/2$, we have

$$\begin{pmatrix} 0 & 1 \\ -1 & 2-\frac{\alpha}{\beta}\cos y_0 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \eta \\ -\xi + \left(2-\frac{\alpha}{\beta}\cos y_0\right)\eta \end{pmatrix},$$

and

$$\frac{\left(2-\frac{\alpha}{\beta}\cos y_0\right)\eta - \xi}{\eta} \geq 2 - \frac{\xi}{\eta} > 1.$$

Then $W^u(P)$ intersects line segment EG. Note that $T(EG) = E'G'$, where $E' = (-\pi/2, \alpha/\beta)$, $G' = \left(-\pi/2, \alpha/\beta - \pi/2\right)$. The line segment $E'G'$ is above the line $y = -x$ if $\alpha/\beta > \pi$. Then the intersection of $W^u(P)$ and the line $y = -x$ is not empty, which leads to the conclusion that system (3.1) has heteroclinic cycle.

Suppose that $W^u(P)$ intersects line $y = -x$ at $H(x_0, y_0)$, and the symmetric point H

with respect to $y = x$ is $H' = (y_0, x_0)$. By iterating H we get a trajectory

$$\cdots, (x_{-n}, y_{-n}), \cdots, (x_0, y_0), \cdots, (x_n, y_n), \cdots,$$

with the property that $x_n \to -\pi, y_n \to -\pi$ as $n \to -\infty$ and $x_n \to \pi, y_n \to \pi$ as $n \to +\infty$. Let $u^1 = \{u_j^1\}$ $u_j^1 = y_j, j \in Z$. Then u^1 is a steady-state solution of (3.1) satisfying $u_j^1 \to -\pi (j \to -\infty)$, $u_j^1 \to \pi (j \to +\infty)$. Similarly, by iterating H' we get another steady-state solution of (4.1) $u^2 = \{u_j^2\}$, satisfying $u_j^2 \to \pi (j \to -\infty)$, $u_j^2 \to -\pi (j \to +\infty)$. According to definition 4, u^1 and u^2 form heteroclinic 2-cycle. Let $v^1 = \{v_j^1\}$, $v_j^1 = -\pi, j \in Z$, $v^2 = \{v_j^2\}$, $v_j^2 = \pi, j \in Z$. It is easy to prove that when $4\beta + \alpha < 1$ and $\alpha/\beta > \pi$, the radius of the spectrum of the differential $D\mathbf{F}(u^1)$ (or $D\mathbf{F}(u^2)$ or $D\mathbf{F}(v^1)$ or $D\mathbf{F}(v^2)$) is less than 1, then u^1, u^2, v^1, v^2 are all asymptotically stable.

The above discussion leads to the conclusion.

Theorem 3. *For system* (3.1), *when* $\alpha/\beta > \pi$ *and* $\alpha + 4\beta < 1$, *there exists a heteroclinic* 2-*cycle* u^1 *and* u^2. u^1, u^2 *and the corresponding steady-state solutions* v^1 *and* v^2 *are asymptotically stable, hence the system has an asymptotically stable homoclinic point.*

According to Theorem 2, we may construct a steady-state solution set of system (3.1), denoted by A' such that the action of the translational group $\{S_j\}$ on A' is chaotic.

References

1 Kaneko, K., *Theory and Applications of Coupled Map Lattices*, Chichester: John Wiley, 1993.
2 Bunimovich, L. A., Sinai, Ya, G., Space time chaos in coupled map lattices, *Nonlinearity*, 1988, 1: 491.
3 Afraimovich, V. S., Pesin, Ya., Travelling waves in lattice models of multidimensional and multi-component media: I General hyperbolic properties, *Nonlinearity*, 1993, 6: 429.
4 Chow, S. N., Shen, W., Dynamics in a discrete Nagumo equation ——Spatial chaos SIAM, *J. Appl. Math.*, 1995, 55 (6): 1764.
5 Afraimovich, V. S., Chow, S. N., Topological spatial chaos and homoclinic points of Z^d-action in lattice dynamical systems, *Japan, J. Indust. Appl. Math.*, 1995, 12: 1.
6 Yan, Y., Qian, M., Transversal cycle and application to Hénon map, *Chinese Science Bulletin*, 1985, 30(13): 961.