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Analytical Calculations of Eulerian and Lagrangian Time Correlations in Turbulent Shear Flows

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Abstract The Taylor series expansion method is used to analytically calculate the Eulerian and Lagrangian time correlations in turbulent shear flows. The short-time behaviors of those correlation functions can be obtained from the series expansions. Especially, the propagation velocity and sweeping velocity in the elliptic model of space-time correlation are analytically calculated and further simplified using the sweeping hypothesis and straining hypothesis. These two characteristic velocities mainly determine the space-time correlations.

Key words: Time correlation, turbulent shear flows, elliptic model.

INTRODUCTION

A space-time correlation, or a two-time and two-point correlation, is in the center of turbulence statistic theory and has broad applications to engineering problems such as the prediction of sound radiated by turbulent flows. The well-known space-time correlation theory in isotopic turbulence is Direct Interaction Approximation (DIA) (Kraichnan, 1959). The DIA theory is based on a simple model for space-time correlation, which is called the sweeping hypothesis (Kraichnan 1964). The model indicates that the space-time correlation in isotropic turbulence is mainly determined by a sweeping velocity (root-mean-square of total energy) and energy spectra. Zhou and Rubinstein (1996) applied this model to predict the frequency spectra of sound generated by isotropic turbulence using Lighthill's acoustic analogy theory. This model shows that the sweeping velocity and energy spectrum are critical to the Large-eddy simulation (LES) prediction of sound frequency spectra (He, Wang and Lele, 2004).

The present study is devoted to develop a space-time correlation theory for turbulent shear flows. He and Zhang (2006) develop an elliptic model for space-time correlation in turbulent shear flows, which indicates that the space-time correlation is mainly determined by two characteristic velocities and spatial correlation. In this paper, we will use the Taylor series expansion technique to derive the two characteristic velocities. The Taylor series expansion technique has been used by Kaneda and Gotoh (Kaneda and Gotoh 1991, Gotoh and Kaneda 1991, Kaneda 1993) to calculate the space-time correlations in isotropic turbulence and will be used here to the ones in turbulent shear flows. We will first introduce the governing equations for space-time correlations and then calculate their two characteristic velocities using the Taylor series expansion technique. Finally, we will discuss the implication of the results obtained and draw the conclusions.

BASIC EQUATIONS

We consider the motion of an incompressible Newtonian fluid of unit density.

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} \tag{1}$$

$$\frac{\partial v_i}{\partial x_i} = 0 \tag{2}$$

Wherev, p, and ν are the velocity, the pressure and the kinetic viscosity respectively. The decomposition of (1) into the mean and fluctuating fields gives the governing equations of the fluctuating velocity.

$$\frac{\partial u_i}{\partial t} = -(U_j + u_j)\frac{\partial u_i}{\partial x_j} - u_j \frac{\partial U_i}{\partial x_j} + \frac{\partial \langle u_i u_j \rangle}{\partial x_j} - \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{3}$$

$$\frac{\partial U_i}{\partial x_i} = \frac{\partial u_i}{\partial x_i} = 0 \tag{4}$$

Where the repeated indices imply the summation and

$$U_i = \langle v_i \rangle , \quad u_i = v_i - \langle v_i \rangle , \quad p = \langle p \rangle + p' \tag{5}$$

Here the brackets denote ensemble average. We assume that the mean strain is constant, such as

$$U_i = \delta_{1i} U_1, \quad \frac{\partial U_i}{\partial x_j} = A_{ij} = S\delta_{i1}\delta_{j2} \tag{6}$$

Using the incompressible condition, we re-write the pressure equation as follows

$$\Delta p' = -\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} - 2\frac{\partial U_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \frac{\partial^2 \langle u_i u_j \rangle}{\partial x_i \partial x_j} \tag{7}$$

To facilitate the discussion on fluid particle, we use the so-called Lagrangian position function ψ and the generalized velocity field $\mathbf{v}$ defined as

$$\psi(y, s; x, t) = \delta^3 \left[y - r(x, t; s) \right] \tag{8}$$

$$v(x, t; s) = \int_V \psi(y, s; x, t) u(y, s) \mathrm{d}^3 y \tag{9}$$

where δ^3 is the 3-dimensional delta function, and the integration is carried out over the entire fluid region V. $r(x,t;s)$ is the velocity at times of the fluid particle that was at x at time t. The Lagrangian position velocity ψ obeys

$$\left(\frac{\partial}{\partial t} + (U_j + u_j)\frac{\partial}{\partial y_j}\right)\psi(y, s; x, t) = 0 \tag{10}$$

$$\psi(y, t; x, t) = \delta^3 \left(y - x \right) \tag{11}$$

Using (3) and (8)-(11), we obtain

$$\frac{\partial}{\partial s} v_i(x, t; s) = -\int \left(A_{ij} u_j + \frac{\partial p'}{\partial y_i} - \nu \frac{\partial^2 u_i}{\partial y_j \partial y_j} \right) \psi(y, s; x, t) \mathrm{d}y \tag{12}$$

If the turbulent fluctuation velocity is statistically homogeneous and stationary, the Eulerian and Lagrangian space-time correlation can be represented as

$$R_E(r, \tau) = \langle u_i(x, t) u_i(x + r\mathrm{e}_1, t + \tau) \rangle \tag{13}$$

$$R_L(r, \tau) = \langle u_i(x, t) v_i(x + r\mathrm{e}_1, t; t + \tau) \rangle \tag{14}$$

LAGRANGIAN AND EULERIAN SPACE-TIME CORRELATIONS

For small$\tau$and$r$, we may expand $R_E(r,\tau)$ and $R_L(r,\tau)$ in Taylor series about $\tau = 0$ and $r = 0$;

$$\begin{aligned} R_E(r, \tau) = R_E(0, 0) &+ \frac{\partial R_E}{\partial r}(0, 0) r + \frac{\partial R_E}{\partial \tau}(0, 0)\tau + \frac{1}{2}\frac{\partial^2 R_E}{\partial r^2}(0, 0) r^2 + \frac{\partial^2 R_E}{\partial r \partial \tau}(0, 0) r\tau \\ &+ \frac{1}{2}\frac{\partial^2 R_E}{\partial \tau^2}(0, 0)\tau^2 + O(r^3, r^2\tau, r\tau^2, \tau^3) \end{aligned} \tag{15}$$

$$\begin{aligned} R_L(r, \tau) = R_L(0, 0) &+ \frac{\partial R_L}{\partial r}(0, 0) r + \frac{\partial R_L}{\partial \tau}(0, 0)\tau + \frac{1}{2}\frac{\partial^2 R_L}{\partial r^2}(0, 0) r^2 + \frac{\partial^2 R_L}{\partial r \partial \tau}(0, 0) r\tau \\ &+ \frac{1}{2}\frac{\partial^2 R_L}{\partial \tau^2}(0, 0)\tau^2 + O(r^3, r^2\tau, r\tau^2, \tau^3) \end{aligned} \tag{16}$$

We assume that the velocity fluctuation filed is statistically homogeneous and stationary, which yields

$$\frac{\partial}{\partial r}R_E(0,0) = \frac{\partial}{\partial \tau}R_E(0,0) = \frac{\partial}{\partial r}R_L(0,0) = \frac{\partial}{\partial \tau}R_L(0,0) = 0 \tag{17}$$

We further assume that the velocity fluctuation field is quasi-normal, which implies that the odd order moments are equal to zero and the even order moments can be represented in terms of second order moment. We can obtain

$$R_{E,rr} = R_{L,rr} = -\frac{2}{3}\int_0^\infty k^2 E(k)\mathrm{d}k \tag{18}$$

$$R_{E,r\tau} = \frac{2}{3}U_1 \int_0^\infty k^2 E(k)\mathrm{d}k, \quad R_{L,r\tau} = 0 \tag{19}$$

$$R_{E,\tau\tau} = -\frac{2}{3}U_1^2 \int_0^\infty k^2 E(k)\mathrm{d}k - \frac{2}{3}S^2 \int_0^\infty E(k)\mathrm{d}k - \int_0^\infty \mathrm{d}q \int_0^\infty \mathrm{d}p E(q)E(p) f_E(p,q) \tag{20}$$

$$R_{L,\tau\tau} = -\frac{2}{3}S^2 \int_0^\infty E(k)\mathrm{d}k - \int_0^\infty \mathrm{d}q \int_0^\infty \mathrm{d}p E(q)E(p) f_L(p,q) \tag{21}$$

Where

$$f_E(p,q) = \int_{|p-q|}^{p+q} \frac{k^3}{pq} a(k,p,q)\mathrm{d}k, \quad f_L(p,q) = \int_{|p-q|}^{p+q} \frac{k^3}{pq} a'(k,p,q)\mathrm{d}k \tag{22}$$

$$a(k,p,q) = \frac{1}{2}(1 - xyz - 2y^2z^2), \quad a'(k,p,q) = \frac{1}{2}(1 - z^2 - y^2 + y^2z^2) \tag{23}$$

$$x = \cos(p \wedge q),\ y = \cos(k \wedge q),\ z = \cos(k \wedge p) \tag{24}$$

The viscous term $2\nu \int_0^\infty k^4 E(k)\mathrm{d}k$ is neglected from (20) and (21) due to infinitely large Reynolds number.

DISCUSSION

In the derivation of the equations (18)$\sim$(21), we make the three assumptions: (1). the velocity fluctuation is statistically homogeneous and stationary; (2). the velocity fluctuation is quasi-norm; (3). the energy spectra of velocity fluctuation are statistically isotropic. The first two assumptions are reasonable for velocity fluctuations in turbulent shear flows, but the third assumption is under verification for turbulent shear flows.

The equations (20) and (21) have two parts: one is corresponding to the anisotropic part and another corresponding to the isotropic part. The latter is exactly as same as the ones obtained by Kaneda and Gotoh for isotropic turbulence. To further simplify the second parts, we have already used the results from the sweeping hypothesis for the Eulerian time correlation and the straining hypothesis for Lagrangian time correlation. For the isotropic turbulence, the sweeping hypothesis (Kraichnan, 1964) suggests:

$$R_E(k,\tau) = 2E(k)\exp(-0.5V^2k^2\tau^2), \quad V = \sqrt{\langle u_i^2 \rangle / 3} \tag{25}$$

The straining hypothesis (Gotoh et al., 1993) suggests:

$$R_L(k,\tau) = 2E(k)\exp(-0.5(\textstyle\int_0^k p^2 E(p)\mathrm{d}p)\tau^2) \tag{26}$$

$R_E(k,\tau)$ and $R_L(k,\tau)$are the Fourier coefficients of the space-time correlations $R_E(r,\tau)$ and $R_L(r,\tau)$ respectively.

Finally, we can use these results to simplify (20) and (2)

$$R_{E,\tau\tau} = -\frac{2}{3}\int_0^\infty k^2 E(k)\mathrm{d}k(U_1^2 + S^2\lambda^2 + \langle u_i^2 \rangle) \tag{27}$$

$$R_{L,\tau\tau} = -\frac{2}{3}\int_0^\infty k^2 E(k)\mathrm{d}k(S^2\lambda^2 + \langle \tilde{u}_i^2 \rangle) \tag{28}$$

Here λ is the differential length scale, defined as

$$\lambda^2 = \int_0^\infty E(k)\mathrm{d}k / \int_0^\infty k^2 E(k)\mathrm{d}k \tag{29}$$

and

$$\left\langle \tilde{u}_i^2 \right\rangle = 2\int_0^\infty E(k)F(k)\mathrm{d}k \tag{30}$$

The filter function is defined as

$$F(k) = \int_0^k p^2 E(p)\mathrm{d}p / \int_0^\infty p^2 E(p)\mathrm{d}p \tag{31}$$

He and Zhang (2006) propose an elliptic model for Eulerian space-time correlations in turbulent shear flows. In the elliptic model, there are two characteristic velocities: the propagation velocity and the sweeping velocity. These two characteristic velocities can be calculated from above results:

$$V_c = R_{E,r\tau}/R_{E,rr} = U_1 \tag{32}$$

$$V_s = R_{E,\tau\tau}/R_{E,rr} - V_c = S^2\lambda^2 + \left\langle u_i^2 \right\rangle \tag{33}$$

CONCLUSION

The Taylor series expansion method is used to carry out a short-time and short-distance analysis of the Lagrangian and Eulerian space-time correlation. In the Eulerian space-time correlation, the first-order coefficients are zero and the second-order coefficients are determined by mean velocity (mean shear rate), the differential scales and energy spectra of fluctuation velocity.
The sweeping hypothesis and the straining hypothesis are applied to simplify the representations of the second order coefficients. The Lagrangian description intrinsically excludes the convection effects induced by both the mean velocity and fluctuating velocity. The exact forms including the effects of the mean velocity and shear rate are formulated in this study.
This results obtained implies that to an accurate prediction of LES on the space-time correlation in turbulent shear flows needs the accurate prediction on enstrophy spectrum in additional to the mean velocities, mean shear rates and energy spectra. The present results can be further improved via the consideration on the non-Gaussian statistics effect and the anisotropic effect on the fluctuating velocity.

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