

Rotation numbers of invariant manifolds around unstable periodic orbits for the diamagnetic Kepler problem

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Abstract

In this paper, a method to construct topological template in terms of symbolic dynamics for the diamagnetic Kepler problem is proposed. To confirm the topological template, rotation numbers of invariant manifolds around unstable periodic orbits in a phase space are taken as an object of comparison. The rotation numbers are determined from the definition and connected with symbolic sequences encoding the periodic orbits in a reduced Poincaré section. Only symbolic codes with inverse ordering in the forward mapping can contribute to the rotation of invariant manifolds around the periodic orbits. By using symbolic ordering, the reduced Poincaré section is constricted along stable manifolds and a topological template, which preserves the ordering of forward sequences and can be used to extract the rotation numbers, is established. The rotation numbers computed from the topological template are the same as those computed from their original definition.

1 Introduction

Many interesting nonlinear systems in experiments are well described by low-dimensional dynamical models. Their dynamical processes include transition of periodic orbits from stable to unstable and bifurcation to chaos. To make a global understanding of the system, some topological and geometrical methods based on periodic orbits are developed[1]. Symbolic dynamics, as a coarse-grained description of the dynamics, provides an effective tool to depict the topological dynamics[2]. In special, for a chaotic system, two curve families of stable and unstable manifolds intersect each other and decompose a Poincaré section[3,4]. Enumeration and existence of unstable periodic orbits (UPOs) in the Poincaré section are determined[5,6] and some numerical methods such as finding UPOs are proposed[7].

Besides the features of dynamics in a Poincaré section, evolution of manifolds in a phase space, which possibly gains a geometric insight into the dynamics, is another important characteristic of the system. For example, bifurcation to chaos can be identified by tracing the evolution of stable and unstable manifolds in the phase space with parameters. Recently, some algorithms for computing two-dimensional stable and unstable manifolds in a three-dimensional phase space are proposed[8,9,10] and applied to visualize the structure of chaos[11]. Since UPOs are closely related to invariant manifolds, a basic issue in their relations is describing how local invariant manifolds rotate around UPOs in the phase space. Moreover, how reducing their topological relation into a two-dimensional template is important for understanding the global organization of UPOs in chaos.

To motivate the visualization of rotation of invariant manifolds around UPOs and construction of topological template in reduction of stable manifolds, we consider the model for diamagnetic Kepler problem (DKP)[12]. In our previous works, two coordinate axes are chosen as a Poincaré section to form an annulus in a lifted space. The dynamics on the annulus can be reduced by considering the symmetry of system. In view of stretching and wrapping in the lifted space, symbolic dynamics without involving bounces has been established[13]. Due to the ordering of stable and unstable manifolds in the minimal domain (a reduced Poincaré section), a method to extract UPOs corresponding to short symbolic strings is proposed. A one-to-one correspondence between UPOs and symbolic sequences is shown under the system symmetry decomposition[14]. Although we only focused on the case of zero scaled energy, in our numerical experiences, the methods can be still used at the scaled energy ± 0.1 . In this paper, for the DKP, we calculate rotation numbers of invariant manifolds around UPOs and set up a connection between rotation numbers and symbolic sequences. Using symbolic

dynamics, we reduce stable manifolds to construct a topological template, which preserves the topological relation of UPOs.

2 Model system and Poincaré section

The Hamiltonian of a hydrogenic electron (with zero angular momentum) in a uniform magnetic field \mathbf{B} directed along the z -axis is given by

$$H = (1/2m)(p_\rho^2 + p_z^2) - e^2/(\rho^2 + z^2)^{1/2} + \frac{1}{2}m\omega^2\rho^2, \quad (1)$$

where $\omega = eB/2mc$ is half the cyclotron frequency. Converting to atomic units and transforming the cylindrical coordinates to semiparabolic ones, the Hamiltonian becomes

$$h = \frac{p_\mu^2}{2} + \frac{p_\nu^2}{2} - \epsilon(\mu^2 + \nu^2) + \frac{1}{8}\mu^2\nu^2(\mu^2 + \nu^2) \equiv 2, \quad (2)$$

where $\epsilon = E\gamma^{-2/3}$ is the scaled energy depending on the energy E and the dimensionless field strength parameter $\gamma = 2\omega$. The symmetry group of h consists of the identity e , two reflections σ_μ, σ_ν across the μ, ν axes, two diagonal reflections σ_{13}, σ_{24} , and three rotations C_4, C_2 and C_4^3 by $\pi/2, \pi$ and $3\pi/2$ around the center, respectively[15]. In the following, the symmetries C_4, C_2 and C_4^3 are denoted by ρ, π and $\bar{\rho}$, respectively. The time-reversal symmetry is denoted by T .

Figure 1 displays an orbit and boundary of the transformed potential for $\epsilon = 0$. A Poincaré section is chosen as follows. Imagine that the μ and ν axes are both of a finite width and length. A counter-clockwise contour is taken along the perimeter of the area forming by the two crossing imaginary rectangles. The Poincaré section is then obtained by recording the position and the tangent component of the momentum along the contour, i.e., the Birkhoff canonical coordinates[16] at intersecting points with the contour where an orbit enter the inside of the contour. The length of the contour is infinite. It is more convenient to transform the contour to one with a finite length. For example, in the first quadrant, the transformations $s = -\mu/(1+\mu)$ along the positive μ axis and $s = \nu/(1+\nu)$ along the positive ν axis convert the segment of the original contour in the first quadrant to interval of length 2 parametrized with $s \in [-1, 1)$. The variable corresponding to the momentum is taken as $v = -p_\mu/p$ at the positive μ axis, and $v = p_\nu/p$ at the positive ν axis, where $p = \sqrt{p_\mu^2 + p_\nu^2}$. In this way we may parametrize the whole contour with $s \in [-1, 7)$ and define corresponding v . The rotational symmetry under ρ, π and $\bar{\rho}$ in the original configurational space becomes the

translational symmetry of shifting s by a multiple of 2 in the $s - v$ plane. The dynamics on the Poincaré surface is then represented by a map on the annulus $s \in [-1, 7)$ and $v \in [-1, 1]$, which is taken as a fundamental domain (FD). When consider an image and preimage of the FD, we need extend to its lifted space. The partial image and preimage of the FD in the lifted space is given in Fig. 2. For example, in Fig. 2(a), zones 1 (2) and 1' (2') in the strip 1 (2) are mapped forward into zones +3 (+0) and +2 (+1), respectively. In the same way, the backward mapping of the strip 1 is given in Fig. 2(b). Since the rotational symmetry of the Hamiltonian (2) corresponds to the translational symmetry in lifted space, the annulus ($s \in [-1, 7)$, $v \in [-1, 1]$) on the Poincaré section can be reduced to a domain ($s \in [0, 2)$, $v \in [-1, 1]$).

In the conservative system, classical dynamics preserves an invariant volume in the phase space under constricting, stretching and folding. This behavior can be displayed in the Poincaré section. In Fig. 3(a), we draw the 9 lines ($s \in (0, 1)$, $v \in [-1, 1]$) in the reduced domain (RD) and their forward mapping in the lifted space. In the mapping, the original zone in the RD is stretched and folded, as well as wrapped. In order to display ordering of the lines in the forward mapping, we also plot connecting lines between two different strips in the lifted space. In the lifted space ($s \in (2, 3)$, $v \in [-1, 1]$), the ordering of lines in the top-left part preserves the same as the original one and the ordering in the bottom-right part is in reverse. In another lifted space ($s \in (1, 2)$, $v \in [-1, 1]$), the ordering of lines in the whole part is in reverse. It is clear that the wrapping of lines in the forward mapping is clockwise, if the lines in the strip ($s \in (1, 2)$, $v \in [-1, 1]$) are stuck to those in the strip ($s \in (2, 3)$, $v \in [-1, 1]$) in terms of their ordering. In the same way, in Fig. 3(b), the similar result can be obtained from the 9 lines ($s \in (1, 2)$, $v \in [-1, 1]$) in the RD and their forward mapping in the lifted space. So, the forward mapping illustrates the rotation of RD in the clockwise direction.

3 Stable and unstable invariant manifolds

In general, the invariant manifolds as a subset are contained in manifolds. The method for calculating stable and unstable manifolds (dynamical foliations) used in through the rest of our works is detailed in [2]. Using the same method, we can thus generate the stable and unstable invariant manifolds through unstable periodic points in a two-dimensional Poincaré section. Here we present a short introduction of the method.

(i) Unstable manifolds: Taking a circle around n steps backward mapping (x_{-n} , y_{-n}) of an unstable periodic point (x_0 , y_0), we get an ellipse centered at (x_0 , y_0) after the same steps forward mapping. Its long axis points to the

most stretching direction. When we fix the point (x_0, y_0) and increase n , the ellipse is stretched and rotated, as well as its most stretching direction changes slightly. When $n \rightarrow \infty$, the most stretching direction approaches a limit. This direction is the most stretching direction of the point (x_0, y_0) . After going a short distance along the direction, we get a new point (x_0, y_0) . Repeating the above process, we can get the new most stretching direction. Finally, an unstable invariant manifold is generated by connecting the points (x_0, y_0) . In fact, the deformation of ellipse is closely related to the dynamical matrix of Poincaré mapping.

(ii) Stable manifolds: Taking a circle around n steps forward mapping (x_n, y_n) of an unstable periodic point (x_0, y_0) , we get an ellipse centered at (x_0, y_0) after the same steps backward mapping. Following the above similar process, we can get the most stable direction of the point (x_0, y_0) and a stable invariant manifold.

4 Evolution of unstable manifolds in rotated Poincaré sections

In order to display the evolution of invariant manifolds around an UPO, we rotate counter-clockwise the (μ, ν) coordinates to the (μ_ϕ, ν_ϕ) coordinates with an angle $\phi \in [0, \pi/2]$. By using the same transformations in Section II, the Poincaré map (s_ϕ, v_ϕ) is obtained from the (μ_ϕ, ν_ϕ) coordinates and reduced to a domain $(s_\phi \in [0, 2), v_\phi \in [-1, 1])$. Since h has C_{4v} and time-reversal symmetries, we take 4 UPOs (4)(5)(14)(15) with different symmetries in the Table I as examples to investigate the evolution of invariant manifolds. Their plots in the configuration space with 10 rotation coordinate axes and periodic points with unstable invariant manifolds in the RDs are drawn in Figs. 4(a)-(d), respectively. From the initial point in the $+\nu_{\phi=0}$ coordinate axis corresponding to the point 1 in the RD with $\phi = 0$, each UPO goes into the second quadrant as displayed in the configuration space by an arrow. Its time process is recorded in the RDs with 10 rotation angles ϕ . So, the figure in the RD with $\phi = \pi/2$ is the same as that in the RD with $\phi = 0$ besides the first point in the former RD is the second point in the later one. In a periodic process, we will calculate the advanced phase θ of an unstable manifold in rotation around the periodic orbits to determine rotation number $\theta/2\pi$. The phase θ is counted positive (negative) when the rotation of unstable manifold around the periodic orbits is counter-clockwise (clockwise).

The UPOs with 4 different symmetries are described as follows:

(i) The UPO (4) displayed in the configuration space of Fig. 4(a) has σ_μ and T symmetries, but not ρ , π , $\bar{\rho}$, σ_ν , σ_{13} and σ_{24} symmetries. In the second quadrant of configuration space, the orbit starting from the $+\nu_{\phi=0}$ coordinate axis goes to the $-\mu_{\phi=0}$ ($+\nu_{\phi=\pi/2}$) coordinate axis. It corresponds to that the point 1 moves to 2 in the RD with $\phi = 0$, i.e. 1 in the RD with $\phi = \pi/2$. In the process, the orbit passes through the $+\nu_{\phi=\pi/18}$, $+\nu_{\phi=\pi/9}$, \dots , $+\nu_{\phi=4\pi/9}$ coordinate axes, while the point 1 moves in the RDs with $\phi = \pi/18$, $\phi = \pi/9$, \dots , $\phi = 4\pi/9$. At the same time, an unstable manifold passing through the point 1 evolves in the RDs. The phase of unstable manifold advances by an angle close to $-\pi$. In the third quadrant of configuration space, the orbit starting from the $-\mu_{\phi=0}$ coordinate axis goes to the $-\nu_{\phi=0}$ ($-\mu_{\phi=\pi/2}$) coordinate axis. It corresponds to that the point 2 moves to 3 in the RD with $\phi = 0$, i.e. 2 in the RD with $\phi = \pi/2$. In the process, the orbit passes through the $-\mu_{\phi=\pi/18}$, $-\mu_{\phi=\pi/9}$, \dots , $-\mu_{\phi=4\pi/9}$ coordinate axes, while the point 2 moves in the RDs with $\phi = \pi/18$, $\phi = \pi/9$, \dots , $\phi = 4\pi/9$. The phase of unstable manifold advances by an angle close to $-\pi$. The orbit starting from the $-\nu_{\phi=0}$ coordinate axis goes into the fourth quadrant of configuration space and back to the $-\nu_{\phi=0}$ coordinate axis. It corresponds to that the point 3 moves to 4 in the RD with $\phi = 0$. In the process, the orbit passes through the $-\nu_{\phi=\pi/18}$ coordinate axis two times, while the point 3 goes into the RD with $\phi = \pi/18$, then leaps to the point 4 and comes back the RD with $\phi = 0$. In the moving processes of the point 3 from the RD with $\phi = 0$ into the RD with $\phi = \pi/18$ and of the point 4 from the RD with $\phi = \pi/18$ into the RD with $\phi = 0$, the directions of unstable manifolds are almost invariant. However, in the leaping process of the orbit from the point 3 to 4 in the RD with $\phi = \pi/18$, an unstable direction indicated as the arrow from the periodic point to its neighboring point on the unstable manifold is approximately reversed. The phase of unstable manifold advances by an angle close to $-\pi$. In the third quadrant of configuration space, the orbit starting from the $-\nu_{\phi=0}$ ($-\mu_{\phi=\pi/2}$) coordinate axis goes to the $-\mu_{\phi=0}$ coordinate axis. It corresponds to that the point 4 moves to 5 in the RD with $\phi = 0$, i.e., the point 3 in the RD with $\phi = \pi/2$ moves to 5 in the RD with $\phi = 0$. In the process, the orbit passes through the $-\mu_{\phi=4\pi/9}$, $-\mu_{\phi=7\pi/18}$, \dots , $-\mu_{\phi=\pi/18}$ coordinate axes, while the point 3 moves in the RDs with $\phi = 4\pi/9$, $\phi = 7\pi/18$, \dots , $\phi = \pi/9$ and then the point 5 moves in the RD with $\phi = \pi/18$. Since the $-\nu_{\phi=\pi/18}$ coordinate axis intersects the orbit in the fourth quadrant, two points 3 and 4 are added in the RD with $\phi = \pi/18$. The phase of unstable manifold advances by an angle close to $-\pi$. In the second quadrant of configuration space, the orbit starting from the $-\mu_{\phi=0}$ ($+\nu_{\phi=\pi/2}$) coordinate axis goes to the $+\nu_{\phi=0}$ coordinate axis. It corresponds to that the point 5 moves to 6 in the RD with $\phi = 0$, i.e., the point 4 in the

RD with $\phi = \pi/2$ moves to 6 in the RD with $\phi = 0$. In the process, the orbit passes through the $+\nu_{\phi=4\pi/9}$, $+\nu_{\phi=7\pi/18}$, \dots , $+\nu_{\phi=\pi/18}$ coordinate axes, while the point 4 moves in the RDs with $\phi = 4\pi/9$, $\phi = 7\pi/18$, \dots , $\phi = \pi/9$ and then the point 6 moves in the RD with $\phi = \pi/18$. The phase of unstable manifold advances by an angle close to $-\pi$. The orbit starting from the $+\nu_{\phi=0}$ ($+\mu_{\phi=\pi/2}$) coordinate axis goes into the first quadrant of configuration space and back to the $+\nu_{\phi=0}$ ($+\mu_{\phi=\pi/2}$) coordinate axis. It corresponds to that the point 6 moves to 1 in the RD with $\phi = 0$, i.e., the point 5 moves to 6 in the RD with $\phi = \pi/2$. In the process, the orbit passes through the $+\mu_{\phi=4\pi/9}$ coordinate axis two times, while the point 5 goes into the RD with $\phi = 4\pi/9$, then leaps to 6 and comes back the RD with $\phi = \pi/2$. In the moving processes of the point 5 from the RD with $\phi = \pi/2$ into the RD with $\phi = 4\pi/9$ and of the point 6 from the RD with $\phi = 4\pi/9$ into the RD with $\phi = \pi/2$, the directions of unstable manifolds are almost invariant. However, in the leaping process from the point 5 to 6 in the RD with $\phi = 4\pi/9$, an unstable direction is approximately reversed. The phase of unstable manifold advances by an angle close to $-\pi$.

So, in the periodic process, the unstable manifold returns to its original position, as well as the phase of unstable manifold advances by -6π . The rotation number of UPO (4) is -3.

(ii) The UPO (5) displayed in the configuration space of Fig. 4(b) passes through the origin. Its right limit orbit has σ_{13} symmetry, but not ρ , π , $\bar{\rho}$, T , σ_{μ} , σ_{ν} and σ_{24} symmetries. Similarly, in the periodic process, the phase of unstable manifold advances by -6π . The rotation number of UPO (5) is -3.

(iii) The UPO (14) displayed in the configuration space of Fig. 4(c) has σ_{ν} symmetry, but not ρ , π , $\bar{\rho}$, T , σ_{μ} , σ_{13} and σ_{24} symmetries. Similarly, in the periodic process, the phase of unstable manifold advances by -8π . The rotation number of UPO (14) is -4.

(iv) The UPO (15) displayed in the configuration space of Fig. 4(d) has ρ , π , $\bar{\rho}$ symmetries, but not T , σ_{μ} , σ_{ν} , σ_{13} and σ_{24} symmetries. Similarly, in the periodic process, the phase of unstable manifold advances by -16π . The rotation number of UPO (15) is -8.

Thus, in the rotation of unstable invariant manifolds around UPOs, we have determined rotation numbers. At the same time, we have also obtained that the advanced phase $|\theta|$ of unstable invariant manifold in a Poincaré mapping does not exceed π . In the Sect. V, we will present a method to calculate the rotation numbers in a Poincaré section.

5 Rotation of unstable manifolds in a Poincaré section

According to the natural ordering in the lifted space and the occurrence of tangencies of manifolds, we have the region partition in the RD with symbols (L_0, R_0, R_1, R_2 and L_2) and the ordering for forward sequences[13]

$$\bullet L_0 < \bullet R_0 < \bullet R_1 < \bullet R_2 < \bullet L_2. \quad (3)$$

The forward mapping preserves the ordering in regions of $\bullet L_0$ and $\bullet L_2$, but reverses the ordering in regions of $\bullet R_0, \bullet R_1$ and $\bullet R_2$.

In the RD, some symmetries of the Hamiltonian (2) are reduced, it can be reflected by the relation of orbit periods to sequence ones. So, we can firstly calculate the advanced phases of unstable directions in rotation around UPOs in the RD and then add the contribution of symmetries to determine rotation numbers. The 4 UPOs with different symmetries in the Sect. IV are still taken as examples.

(i) In Fig. 5(a), we draw the periodic points encoded by $R_0^2 R_1 R_2^2 R_1$ and stable and unstable invariant manifolds passing through the points. The periodic points are denoted by circles. In order to illustrate the evolution of unstable direction around the periodic points, we take another initial point near the periodic point 1 on the unstable invariant manifold. The forward mapping of the point is also drawn in the figure and denoted by crosses. The arrows from periodic points to their neighboring points on the unstable invariant manifolds display unstable directions. In the forward mapping from the periodic point 1 to 2, the symbolic sequence $\bullet R_0^2 R_1 R_2^2 R_1$ is shifted to $\bullet R_0 R_1 R_2^2 R_1 R_0$ and the original unstable direction is approximately reversed. Since the rotation is clockwise, we obtain $-\pi$ rotation of the unstable direction. In the forward mapping from the periodic point 2 to 3, the symbolic sequence $\bullet R_0 R_1 R_2^2 R_1 R_0$ is shifted to $\bullet R_1 R_2^2 R_1 R_0^2$ and the unstable direction is approximately reversed. We also obtain $-\pi$ rotation of the unstable direction, i.e. -2π rotation of the original unstable direction. In the same way, $-\pi$ rotation of the unstable direction is obtained in the forward mapping from the periodic point 3 to 4. In the periodic point 4, we take another neighboring point denoted by a triangle to replace the point denoted by a cross. In the same way, $3 \times (-\pi)$ rotation of the unstable direction is obtained in the forward mapping from the periodic point 4 to 5, from the periodic point 5 to 6 and from the periodic point 6 to 1. Thus, during the mapping in the sequence period, the original unstable direction goes back and the total advance of phase is -6π . Since the orbit period is equal to the sequence one, i.e., the UPO (4) has not the $\rho, \pi, \bar{\rho}$ symmetries, the rotation number of

UPO encoded by $R_0^2 R_1 R_2^2 R_1$ is -3.

(ii) In Fig. 5(b), the periodic points with the right limit encoded by $L_0 R_0^2 R_1 L_2 R_2^2 R_1$ and the stable and unstable invariant manifolds passing through the points are drawn. In the forward mapping from the periodic point 1 to 2, the symbolic sequence $\bullet L_0 R_0^2 R_1 L_2 R_2^2 R_1$ is shifted to $\bullet R_0^2 R_1 L_2 R_2^2 R_1 L_0$ and the original unstable direction is approximately preserved. In the forward mapping from the periodic point 5 to 6, the same result is obtained. In other forward mappings, $6 \times (-\pi)$ rotation of the unstable direction is added. Thus, during the mapping in the sequence period, the original unstable direction goes back and the total advance of phase is -6π . Since the orbit period is equal to the sequence one, the rotation number of UPO (5) encoded by $L_0 R_0^2 R_1 L_2 R_2^2 R_1$ is -3.

(iii) In Fig. 5(c), the periodic points encoded by $L_0 R_1 R_2 R_1^2 L_2 R_1 R_0 R_1^2$ and stable and unstable invariant manifolds passing through the points are drawn. In the forward mapping from the periodic point 1 to 2, the symbolic sequence $\bullet L_0 R_1 R_2 R_1^2 L_2 R_1 R_0 R_1^2$ is shifted to $\bullet R_1 R_2 R_1^2 L_2 R_1 R_0 R_1^2 L_0$ and the original unstable direction is approximately preserved. In the forward mapping from the periodic point 6 to 7, the same result is obtained. In other forward mappings, $8 \times (-\pi)$ rotation of the unstable direction is added. Thus, during the mapping in the sequence period, the original unstable direction goes back and the total advance of phase is -8π . Since the orbit period is equal to the sequence one, the rotation number of UPO (14) encoded by $L_0 R_1 R_2 R_1^2 L_2 R_1 R_0 R_1^2$ is -4.

(iv) In Fig. 5(d), the periodic points encoded by $L_0 R_1^2 R_0^2$ and stable and unstable invariant manifolds passing through the points are drawn. In the forward mapping from the periodic point 1 to 2, the symbolic sequence $\bullet L_0 R_1^2 R_0^2$ is shifted to $\bullet R_1^2 R_0^2 L_0$ and the original unstable direction is approximately preserved. In other forward mappings, $4 \times (-\pi)$ rotation of the unstable direction is added. Thus, during the mapping in the sequence period, the original unstable direction goes back and the total advance of phase is -4π . Since the orbit period is 4 times of the sequence one, i.e., the UPO (15) has the ρ , π , $\bar{\rho}$ symmetries, the rotation number of the UPO encoded by $L_0 R_1^2 R_0^2$ is -8.

In the above examples describing the rotation of unstable directions around periodic points, the forward map corresponding to the shift with L_0 or L_2 (R_0 or R_1 or R_2) approximately preserves (reserves) the original unstable direction. So, we can multiply the numbers of R_0 , R_1 and R_2 in 5-letter symbolic sequences by one half of the ratios of orbit periods to sequence ones to determine rotation numbers of UPOs.

Since the RD has the π -rotation symmetry, the 5-letter symbolic dynamics can be reduced to the 3-letter one in the minimal domain (MD)

($s \in [0, 1)$, $v \in [-1, 1]$)[13]. The MD is partitioned and denoted by symbols L_0 , R_0 and R_1 . The correspondence of 5-letter symbolic sequences with 3-letter ones is $L_0 \rightarrow L_0$, $R_0 \rightarrow R_0$, $R_1 \rightarrow R_1$, $R_2 \rightarrow R_0$ and $L_2 \rightarrow L_0$. In general, the number of R_0 , R_1 and R_2 in 5-letter symbolic sequences is twice of the number of R_0 and R_1 in 3-letter ones. The ratios of orbit periods to sequence ones for the former are one half of those for the later. Of course, the simple repeating of 3-letter symbolic sequences in 5-letter ones will be removed. For example, the 3-letter symbolic sequences $R_0^2 R_1$ and $L_0 R_0^2 R_1$ correspond to the 5-letter ones $R_0(R_2)R_0(R_2)R_1 R_2(R_0)R_2(R_0)R_1$ and $L_0(L_2)R_0(R_2)R_0(R_2)R_1 L_2(L_0)R_2(R_0)R_2(R_0)R_1$, respectively. Rotation numbers of two UPOs can be determined by calculating total numbers of letters R_0 and R_1 in 3-letter symbolic sequences and multiplying them by one half of the ratios of orbit periods to sequence ones. The same rotation numbers of the UPOs can be also obtained by using the method for 5-letter symbolic sequences. Thus, using the method, we extract rotation numbers of 38 UPOs from symbolic sequences as given in Table I.

6 Topological Template

After the region partition and symbolic ordering are introduced, the families of stable and unstable manifolds constitute curve coordinates in the RD. Each stable (unstable) manifold has the same forward (backward) symbolic sequence. The ordering on stable (unstable) manifolds is described by that of forward (backward) symbolic sequences[14]. In Fig. 6, two families of sub-manifolds divided by the partition line $\bullet C_0$ or $\bullet C_2$ have the opposite ordering. The ordering of stable (unstable) manifolds increases monotonically from the left-bottom (left-top) to right-top (right-bottom) along each unstable (stable) manifold.

Along each stable manifold in zones $\bullet L_0$, $\bullet R_0$ and $\bullet R_1$ of Fig. 6, we construct all points in the curve to a point. The point preserves the forward symbolic sequence and the ordering of stable manifold. So, in the left region of Fig. 6, the points in three zones are reduced to three lines. Connecting the three lines, we obtain a belt partitioned by the symbols $\bullet C_0$ and $\bullet B_0$, and denoted by the symbols $\bullet L_0$, $\bullet R_0$ and $\bullet R_1$ as given in the top of Fig. 7(a). From the left to right along the belt, the ordering of forward sequences increases monotonically. The forward mapping of the left region ($s \in (0, 1)$ and $v \in [-1, 1]$) in Fig. 6, i.e. the right region ($s \in (2, 3)$, $v \in [-1, 1]$) in Fig. 3(a), can be reduced in the RD. So, in the forward mapping, the zones $\bullet L_0$ and $\bullet R_0$ still keep in the region ($s \in (0, 1)$, $v \in [-1, 1]$) encoded by $L_0 \bullet$ and $R_0 \bullet$, respectively, but the zone $\bullet R_1$ moves in the region ($s \in (1, 2)$,

$v \in [-1, 1]$) encoded by $R_1 \bullet$. In Fig. 6, we again partition the RD and encode it by corresponding backward symbols. The two zones $L_0 \bullet$ and $R_0 \bullet$ are partitioned to five zones by the lines $\bullet B_0$ and $\bullet C_0$, as well as the zone $R_1 \bullet$ is partitioned to three zones by the lines $\bullet B_2$ and $\bullet C_2$. In the stretching and folding processes of forward mapping, the original three zones are mapped into the eight zones. We still constrict all points along stable manifolds in each zone. Thus, the points in eight zones are reduced to five lines in the left region and three lines in the right region. Connecting the eight lines, we obtain a belt partitioned by the symbols $L_0 \bullet B_0$, $C_0 \bullet R_1$, $R_0 \bullet B_0$, $R_0 \bullet C_0$, $R_1 \bullet C_2$ and $R_1 \bullet B_2$, and encoded by the strings $L_0 \bullet R_0$, $L_0 \bullet R_1$, $R_0 \bullet R_1$, $R_0 \bullet R_0$, $R_0 \bullet L_0$, $R_1 \bullet L_2$, $R_1 \bullet R_2$ and $R_1 \bullet R_1$ in the bottom of Fig. 7(a). The ordering of the bottom belt is the same as that of the top one. So, the forward mapping of the region ($s \in (0, 1)$, $v \in [-1, 1]$) can be described by a twisting part of topological template as given in Fig. 7(a).

In the same way, along each stable manifold in zones $\bullet R_1$, $\bullet R_2$ and $\bullet L_2$ of Fig. 6, all points in the curve are constricted to a point. A belt containing the point is partitioned by the symbols $\bullet B_2$ and $\bullet L_2$, and encoded by the symbols $\bullet R_1$, $\bullet R_2$ and $\bullet L_2$ as given in the top of Fig. 7(b). From the left to right along the belt, the ordering of forward sequences increases monotonically. The forward mapping of the right region ($s \in (1, 2)$ and $v \in [-1, 1]$) in Fig. 6, i.e. the left region ($s \in (-1, 0)$, $v \in [-1, 1]$) in Fig. 3(b), can be reduced in the RD. So, in the forward mapping, the zones $\bullet L_2$ and $\bullet R_2$ still keep in the region ($s \in (1, 2)$, $v \in [-1, 1]$) encoded by $L_2 \bullet$ and $R_2 \bullet$, but the zone $\bullet R_1$ moves in the region ($s \in (0, 1)$, $v \in [-1, 1]$) encoded by $R_1 \bullet$. We still constrict all points along stable manifolds in each zone and obtain the belt partitioned by the symbols $R_1 \bullet B_0$, $R_1 \bullet C_0$, $R_2 \bullet C_2$, $R_2 \bullet B_2$, $C_2 \bullet R_1$ and $L_2 \bullet B_2$, and encoded by the strings $R_1 \bullet R_1$, $R_1 \bullet R_0$, $R_1 \bullet L_0$, $R_2 \bullet L_2$, $R_2 \bullet R_2$, $R_2 \bullet R_1$, $L_2 \bullet R_1$ and $L_2 \bullet R_2$ in the bottom of Fig. 7(b). So, the forward mapping of the region ($s \in (1, 2)$, $v \in [-1, 1]$) can be described by a twisting part of topological template as given in Fig. 7(b).

In the two twisting parts of topological template, the belts reflect approximately the direction of unstable manifolds. Using the twisting parts of topological templates, we can easily calculate rotation numbers for given symbolic sequences. In the same way, the 4 UPOs with different symmetries in Sect. II are still taken as examples. For the sequence $R_0^2 R_1 R_2^2 R_1$ encoding the UPO(4), firstly, $R_0 \bullet R_0 R_1 R_2^2 R_1$ is obtained by shifting $R_1 \bullet R_0^2 R_1 R_2^2$. When an arrow is put on the $\bullet R_0$ zone of top belt in Fig. 7(a), after a forward mapping, the arrow is moved on the $R_0 \bullet R_0$ zone of bottom belt in Fig. 7(a). Since the arrow rotates clockwise to its opposite direction, we count the process as -1. Then, $R_0 \bullet R_1 R_2^2 R_1 R_0$ is obtained by shifting $R_0 \bullet R_0 R_1 R_2^2 R_1$. Since an arrow on the $\bullet R_0$ zone of top belt in Fig. 7(a) rotates clockwise

to its opposite direction, we count the forward mapping as -1. Repeating the above process, we get the total number -6 counting the forward mapping in sequence period. Since the orbit period is equal to the sequence one, we can thus obtain rotation number of the UPO encoded by $R_0^2 R_1 R_2^2 R_1$ is -3. For the sequence $L_0 R_0^2 R_1 L_2 R_2^2 R_1$ encoding the UPO(5), after a forward mapping for $\bullet L_0 R_0^2 R_1 L_2 R_2^2 R_1$, an arrow on the top belt in Fig. 7(a) moves parallelly on the zone $L_0 \bullet R_0$ of bottom belt. We count the forward mapping as 0. Following the same process, we get the total number -6 counting the forward mapping in the sequence period. Since the orbit period is equal to the sequence one, we can thus obtain rotation number of the UPO encoded by $L_0 R_0^2 R_1 L_2 R_2^2 R_1$ is -3. Similarly, for the sequence $L_0 R_1 R_2 R_1^2 L_2 R_1 R_0 R_1^2$ ($L_0 R_1^2 R_0^2$) encoding the UPO(14) (UPO(15)), we get the total number -8 (-4) counting the forward mapping in the sequence period. Since the orbit period is equal to (4 times of) the sequence one, we can thus obtain rotation number of the UPO encoded by $L_0 R_1 R_2 R_1^2 L_2 R_1 R_0 R_1^2$ ($L_0 R_1^2 R_0^2$) is -4 (-8). By comparing with the former computation for the 4 UPOs from the definition and in a Poincaré section, the same results are extracted from the topological template.

By combining the two twisting parts in Figs. 7(a)(b), suspension of the Poincaré mapping, which displays the relative position of zones in the forward mapping from the top belt to bottom one, is obtained. A global topological template of the RD is constructed by connecting the suspension with a flow corresponding to the Poincaré mapping in Fig. 8. The template preserves the ordering of forward sequences encoding stable manifolds in a belt and the same or inverse ordering of symbolic encoding in the forward mapping of all parts in the belt.

7 Conclusion and discussion

In summary, we have presented the systematic study of the evolution of invariant manifolds around unstable periodic orbits and the reduction of them to construct a topological template in terms of symbolic dynamics for the diamagnetic Kepler problem. To confirm the topological template, rotation numbers of invariant manifolds around unstable periodic orbits in a phase space, which quantify the evolution, are determined from the definition and connected with symbolic sequences encoding the periodic orbits. Only symbolic codes, which correspond to the forward mapping with inverse ordering, can contribute to the rotation of invariant manifolds. By using symbolic ordering, the reduced Poincaré section is constricted along stable manifolds and a topological template, which preserves the ordering of forward sequences

and can be used to extract the rotation numbers, is established. The rotation numbers computed from the topological template are the same as those computed from the original definition.

Since unstable periodic orbits in phase space are the skeleton of the chaotic system, the local evolution of manifolds near unstable periodic orbits can present basic features of the global evolution of manifolds in phase space. One of the basic features can be quantified by the rotation number and embedded in the topological template.

In the semiclassical Green's function, the phase correction is related to Maslov indices of the UPOs[17], which has been connected with symbolic sequences of unstable periodic orbits due to boundary coding[18]. The relation of Maslov indices to rotation numbers of unstable periodic orbits remains to be determined.

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9 FIGURE CAPTION

Fig. 1. A typical orbit and boundary of the transformed potential for the diamagnetic Kepler problem at $\epsilon = 0$.

Fig. 2. A image (a) and preimage (b) of the strips 1 and 2 of the fundamental domain ($s \in [-1, 7)$, $v \in [-1, 1]$) in the lifted space.

Fig. 3. 9 lines in (a) ($s \in (0, 1)$, $v \in [-1, 1]$) or (b) ($s \in (1, 2)$, $v \in [-1, 1]$) of the reduced domain and their forward mapping in the correspondent lifted space. The different types of lines display the relative changes between original positions and their forward mappings along the s coordinate axis.

Fig. 4. UPOs with different symmetry in the configuration space and periodic points with unstable invariant manifolds in rotation Poincaré sections: (a) the UPO (4); (b) the UPO (5); (c) the UPO (14); (d) the UPO (15).

Fig. 5. Periodic points encoded by (a) $R_0^2 R_1 R_2^2 R_1$; (b) $L_0 R_0^2 R_1 L_2 R_2^2 R_1$; (c) $L_0 R_1 R_2 R_1^2 L_2 R_1 R_0 R_1^2$; (d) $L_0 R_1^2 R_0^2$ in the 5-letter encoding and their stable and unstable invariant manifolds.

Fig. 6. Stable and unstable manifolds with partition lines in the reduced domain.

Fig. 7. Twisting parts of topological template describing the forward mapping on the regions (a) ($s \in [0, 1)$, $v \in [-1, 1]$) and (b) ($s \in [1, 2)$, $v \in [-1, 1]$). For crossing of two lines in suspension of the Poincaré mapping, the front (back) one is denoted by solid lines (short dashes or the combination of solid lines and short dashes). The top and bottom belts are denoted by solid lines or dashes depending on their positions. The connecting lines of two parts of the broken belt in the forward mapping are denoted by long dashes.

Fig. 8. A flow of topological template of the reduced domain. The notions for belts are the same as Fig. 7, except that projection of several parts in the bottom belt on one belt in terms of forward symbolic codings is connected by short dashes. The flow is denoted by short dashes.

UPOs	I.	Rotation numbers and symbolic sequences of	diamagnetic Kepler problem at $\epsilon = 0$.	3-lett. Seq. & its Period	5-lett. Seq. & its Period	Orb. Period	Rot. Num.	
1		R_0		1	R_0	1	4	-2
2		L_0R_1		2	$L_0R_1L_2R_1$	4	4	-1
3		R_0R_1		2	$R_0R_1R_2R_1$	4	4	-2
4		$R_0^2R_1$		3	$R_0^2R_1R_2^2R_1$	6	6	-3
5		$L_0R_0^2R_1$		4	$L_0R_0^2R_1L_2R_2^2R_1$	8	8	-3
6		$L_0R_1R_0^2$		4	$L_0R_1R_2^2L_2R_1R_0^2$	8	8	-3
7		$L_0R_1R_0R_1$		4	$L_0R_1R_2R_1$	4	4	-1.5
8		$R_0^3R_1$		4	$R_0^3R_1R_2^3R_1$	8	8	-4
9		$R_0^2R_1^2$		4	$R_0^2R_1^2$	4	8	-4
10		$L_0R_0L_0R_1^2$		5	$L_0R_0L_0R_1^2$	5	20	-6
11		$L_0R_0^2R_1^2$		5	$L_0R_0^2R_1^2$	5	20	-8
12		$L_0R_1L_0R_1^2$		5	$L_0R_1L_2R_1^2L_2R_1L_0R_1^2$	10	10	-3
13		$L_0R_1R_0^2R_1$		5	$L_0R_1R_2^2R_1$	5	20	-8
14		$L_0R_1R_0R_1^2$		5	$L_0R_1R_2R_1^2L_2R_1R_0R_1^2$	10	10	-4
15		$L_0R_1^2R_0^2$		5	$L_0R_1^2R_0^2$	5	20	-8
16		$L_0R_1^2R_0R_1$		5	$L_0R_1^2R_0R_1L_2R_1^2R_2R_1$	10	10	-4
17		$R_0^4R_1$		5	$R_0^4R_1R_2^4R_1$	10	10	-5
18		$R_0^3R_1^2$		5	$R_0^3R_1^2$	5	20	-10
19		$R_0^2R_1R_0R_1$		5	$R_0^2R_1R_2R_1$	5	20	-10
20		$R_0^2R_1^3$		5	$R_0^2R_1^3R_2^3R_1^3$	10	10	-5
21		$R_0R_1R_0R_1^2$		5	$R_0R_1R_2R_1^2R_2R_1R_0R_1^2$	10	10	-5
22		$L_0R_0L_0R_1R_0R_1$		6	$L_0R_0L_0R_1R_2R_1$	6	12	-4
23		$L_0R_0L_0R_1^3$		6	$L_0R_0L_0R_1^3L_2R_2L_2R_1^3$	12	12	-4
24		$L_0R_0^2L_0R_1^2$		6	$L_0R_0^2L_0R_1^2$	6	6	-2
25		$L_0R_0^4R_1$		6	$L_0R_0^4R_1L_2R_2^4R_1$	12	12	-5
26		$L_0R_0^3R_1^2$		6	$L_0R_0^3R_1^2$	6	6	-2.5
27		$L_0R_0^2R_1^3$		6	$L_0R_0^2R_1^3L_2R_2^3R_1^3$	12	12	-5
28		$L_0R_1L_0R_1R_0R_1$		6	$L_0R_1L_2R_1R_0R_1L_2R_1L_0R_1R_2R_1$	12	12	-4
29		$L_0R_1R_0^4$		6	$L_0R_1R_2^4L_2R_1R_0^4$	12	12	-5
30		$L_0R_1R_0^3R_1$		6	$L_0R_1R_2^3R_1$	6	12	-5
31		$L_0R_1R_0R_1R_0R_1$		6	$L_0R_1R_2R_1R_0R_1L_2R_1R_0R_1R_2R_1$	12	12	-5
32		$L_0R_1^2R_0^3$		6	$L_0R_1^2R_0^3$	6	6	-2.5
33		$L_0R_1^3R_0^2$		6	$L_0R_1^3R_2^2L_2R_1^3R_0^2$	12	12	-5
34		$R_0^5R_1$		6	$R_0^5R_1R_2^5R_1$	12	12	-6
35		$R_0^4R_1^2$		6	$R_0^4R_1^2$	6	12	-6
36		$R_0^3R_1R_0R_1$		6	$R_0^3R_1R_2R_1$	6	12	-6
37		$R_0^3R_1^3$		6	$R_0^3R_1^3R_2^3R_1^3$	12	12	-6
38		$R_0^2R_1^4$		6	$R_0^2R_1^4$	6	12	-6

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