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ASYMPTOTIC ANALYSIS OF PLANE-STRAIN MODE I STEADY-STATE CRACK GROWTH IN TRANSFORMATION TOUGHENING CERAMICS(II)*

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Abstract

Based on a constitutive law which includes the shear components of transformation plasticity, the asymptotic solutions to near-tip fields of plane-strain mode I steadity propagating cracks in transformed ceramics are obtained for the case of linear isotropic hardening. The stress singularity, the distributions of stresses and velocities at the crack tip are determined for various material parameters. The factors influencing the near-tip fields are discussed in detail.

Key words transformation toughening ceramics, shear effect, crack growth, asymptotic method

I. Introduction

The mechanism of fracture toughness enhancement in ceramics has been widely studied since the early 1980s. The pioneering constitutive model developed by Budiansky et al.[1] includes the effect of plastic dilation, but neglects the transformation-induced shear strain. And their computational results of toughness increment are much less than experimental observation. There arises doubt from researchers about the validity of the model of Budiansky et al[1]. Recently, increasing experimental evidence found by Chen and Reyes-Morel[2,3] indicated that plastic shear and dilatant effects are of comparable magnitude and both can not been ignored in the assessment of toughness increment. Based on these and the related observations, Huang et al[4] presented a new micromechanics-based continuum model to account for both dilatant and shear effects. by means of Hill-Rice's internal variable constitutive theory. Ye et al[5] also developed a new constitutive law including dilatant and shear effects, and used it in their FEM calculation of stationary cracks. Stam et al[6] carried out FEM analysis to study the crack growth behavior by using the constitutive law in reference [4]. It is found that the shear components of transformation deformation which has been neglected in previous investigations have played an important role in the estimate of toughness behavior in ceramics.

Asymptotic analyses of stress and strain fields near the crack tip in those pressure-

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sensitive materials have been paid much attention during the past years. Li and Pan[7] studied the asymptotic fields of a stationary crack for deformation-plasticity theory adopting Drucker-Prager yield surface with associative flow-rule. For incremental small-strain elastoplasticity obeying the Drucker-Prager yielding condition, an asymptotic determination of near-tip fields at the growing crack tip has been presented by Bigoni and Radi[8]. But all those investigations not taking the full transformation zone into account, resulted in a plastic reloading zone near the crack flanks. Furthermore, because a limit value of the pressure-sensitive factors exists in their asymptotic analyses, the constitutive model in references [7] and [8] can not be used for transformation toughening ceramics.

In our previous paper [9], the asymptotic analysis of stress and strain field near the model I steadily propagating crack has been carried out, based on the constitutive model of Budiansky et al[1]. It is seen that with the decrease of plastic volumerical tangential modules, the singularity of stress fields and the level of mean stress ahead of the crack tip will decrease, and obviously this will lead to an increase in the fracture toughness. In this paper, a brief outline of the constitutive model developed by Ye et al[5] is firstly given. Then the paper is addressed to the asymptotic study of plane-strain mode I steady-state growing cracks by using a variable-separable expression similar to HRR-type fields. The results of detailed computations for various material parameters are presented and the effects of shear effect on the near-tip fields are discussed.

II. Basic Equations

2.1 The constitutive relations

A homogeneous isotropic hardening material characterised by a nonlinear constitutive relation is adopted in reference [5]. It is assumed that the yield criterion contains two stress invariants and the plastic deformation obeys the associative flow rule. These assumptions are consistent with the experimental observation in Chen and Reyes-Morel[2,3] and the theoretical analysis of Huang et al[4].

The transformation plastic loading surface, f, can be written as:

$$f=\sigma_m+\mu\sigma_e-\sigma_m^*-H\left(\int\overline{d\varepsilon^p}\right)=0 \tag{2.1}$$

which σ_e is the effective stress, $\sigma_e=\sqrt{\frac{3}{2}s_{ij}s_{ij}}$, s_{ij} is the stress deviator, $s_{ij}=\sigma_{ij}-\sigma_m\delta_{ij}$, σ_m is the mean stress, $\sigma_m=\frac{1}{3}\delta_{ij}\sigma_{ij}$ and σ_{ij} is Cauchy stress, the material constant μ measures the effective stress sensitivity of yielding. $\overline{d\varepsilon^p}=\sqrt{\frac{2}{3}d\varepsilon_{ij}^p d\varepsilon_{ij}^p}$, $d\varepsilon_{ij}^p$ is the increment in total plastic strain and H is a function of the accumulated effective plastic strain $\int\overline{d\varepsilon^p}$, σ^* is the characteristic mean stress at which transformation plasticity occurs.

Let the plastic volumetric tangential modules be $\bar{B}=d\sigma_m/d\varepsilon_m^p$ and the plastic effective tangential modules be $\bar{G}=d\sigma_e/d\bar{e}^p$ in which $d\bar{e}^p=\sqrt{(2/3)de_{ij}^p de_{ij}^p}$, $de_{ij}^p=d\varepsilon_{ij}^p-\frac{1}{3}d\varepsilon_m^p\delta_{ij}$.

With the developing transformation plastic deformation, we have

$$\dot{f}=d\sigma_m+\mu d\sigma_e-H'\overline{d\varepsilon^p}=0 \tag{2.2}$$

where $H'=dQ/\overline{d\varepsilon^p}$, $Q=\sigma_m+\mu\sigma_e$

According to Drucker's postulate, we have:

$$d\varepsilon_{ij}^p=d\lambda\frac{\partial f}{\partial\sigma_{ij}}=d\lambda Q_{ij} \tag{2.3}$$

where $Q_{ij}=\frac{1}{3}\delta_{ij}+\frac{3}{2}\mu\frac{s_{ij}}{\sigma_e}$, $d\lambda$ is a scalar multiplier. It is easy to be found that $d\varepsilon_{kk}=d\lambda$ and $\overline{d\varepsilon^p}=\mu d\lambda$, so we can get the hardening modulus H': $H'=3(\bar{B}+\mu\bar{G})/\sqrt{2+9\mu^2}$.

The constitutive relations in rate form can be expressed as follows:

$$\dot{\varepsilon}_{ij}=\frac{1}{E}[(1+\nu)\dot{\sigma}_{ij}-\nu\dot{\sigma}_m\delta_{ij}]+\frac{\alpha}{H'}(Q_{kl}\dot{\sigma}_{kl})Q_{ij} \tag{2.4}$$

where

$$\alpha=\begin{cases}0 & \text{before the onset and after the end of transformation}\\ 1 & \text{at transformation loading stage}\end{cases}$$

In the case of $\mu=0$, the above equations are reduced to the constitutive equations of Budiansky et al[1]. From the experimental results mentioned above, the tangential modules $H'>0$ holds for Mg-PSZ ceramics, because $\bar{B}>0$ and $\bar{G}>0$. Thus loading and unloading criterion can be given as follows:

$$Q_{kl}\dot{\sigma}_{kl}\begin{cases}>0 & \text{loading stage}\\ =0 & \text{netural loading}\\ <0 & \text{unloading stage}\end{cases} \tag{2.5}$$

The yielding criteria can be formulated as:

$$\sigma_m+\mu\sigma_e=\sigma_m^* \tag{2.6}$$

The condition for the completion of transformation is:

$$\Theta=\Theta^T \tag{2.7}$$

in which Θ is the plastic dilatation and Θ^T is a constant. and the plastic effective strain is not constrained by any conditions.

2.2 Basic equations

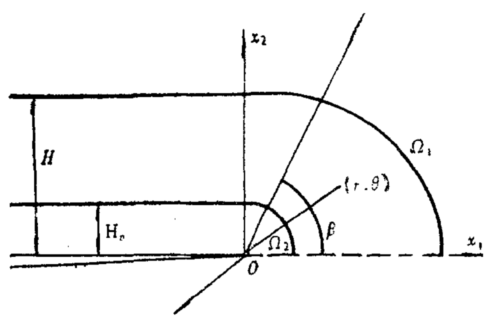

Fig. 1 Typical configuration of steadily- growing plane cracks

Under the small transformation condition. there are two distinct regions, actively loading transformation and elastic, in the neighborhood of the crack tip, which are separated by the boundary Γ on which elastic unloading begins. Those are shown in Fig. 1. Transformation loading occurs in domain A_1 and region A_2 corresponds to elastic unloading and full transformation stage. The height of transformation zone is H and the height of the fully transformed zone is H_r. The transformation zone is separated from elastic zone by the boundary Ω_1 and there exists a fully transformed zone circled by Ω_2 in the immediate vicinity of the crack tip. In this paper, we require that the material parameters B and the strength of transformation $\omega=E\Theta^T(1+\nu)/\sigma_m^*/(1-\nu)$ should satisfy the conditions $\bar{B}>0$ and $\omega>10$, which are always true for some materials, such as Mg-PSZ and TZP. The FEM results in references [3, 5, 7] indicated that H_r is much smaller, compared with H, in the case of above conditions. And the experimental observation of Marshall et al[10] also showed that in toughened Mg-PSZ ceramics, fully transformed zone can not be found by Raman Spectroscopic in the regions adjacent to crack tip, and the maximum amount of transformed particles was approximately 80% of the original tetragonal phase particles. So, the ring region circled by Ω_1 and Ω_2 is big enough for the existence of asymptotic solutions, because the deformation in fully-transformed zone has very little influence on the stress fields in it.

A Cartesian reference system is employed, as shown in Fig. 1, with the origin attached to the moving crack tip. It moves together with the crack at a constant velocity V. Therefore, the steady-state propagation condition implies that any material derivative can be identified with a spatial derivative in the direction x_1

$$d(\ )/dt=-V\partial(\ )/\partial x_1=V\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}-\cos\frac{\partial}{\partial r}\right] \tag{2.8}$$

Referring to the co-ordinates (r, θ), the equations of equilibrium are:

$$\begin{aligned}&(r\sigma_{rr})_{,r}+\sigma_{r\theta,\theta}-\sigma_{\theta\theta}=0\\&(r\sigma_{r\theta})_{,r}+\sigma_{\theta\theta,\theta}+\sigma_{r\theta}=0\end{aligned} \tag{2.9}$$

The strain components are related to the two in-plane velocities v_θ and v_r, v_θ and v_r by

$$\begin{aligned}&\dot{\varepsilon}_{rr}=v_{r,r},\quad \dot{\varepsilon}_{\theta\theta}=(v_{\theta,\theta}+v_r)/r\\&\dot{\varepsilon}_{r\theta}=\frac{1}{2}[v_{\theta,r}+(v_{r,\theta}-v_\theta)/r]\\&\dot{\varepsilon}_{33}=0\end{aligned} \tag{2.10}$$

Let $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ be the unit vectors in the directions of r and θ respectively, and the material derivative of stress components can be obtained by the following equations:

$$\begin{aligned}&(\boldsymbol{e}_i\cdot\sigma\cdot\boldsymbol{e}_j)^{\cdot}=\dot{\boldsymbol{e}}_i\cdot\sigma\cdot\boldsymbol{e}_j+\boldsymbol{e}_i\cdot\dot{\sigma}\cdot\boldsymbol{e}_j+\boldsymbol{e}_i\cdot\sigma\cdot\dot{\boldsymbol{e}}_j\\&\dot{\boldsymbol{e}}_r=V\frac{\sin\theta}{r}\boldsymbol{e}_\theta\\&\dot{\boldsymbol{e}}_\theta=-V\frac{\sin\theta}{r}\boldsymbol{e}_r\end{aligned} \tag{2.11}$$

Using Eq. (2.11), the following expressions of the components of stress rate tensors are obtained:

$$\begin{aligned}&\dot{\sigma}_{r\theta}=V\left[\frac{\sin\theta}{r}(\sigma_{r\theta,\theta}-\sigma_{rr}+\sigma_{\theta\theta})-\cos\theta\sigma_{r\theta,r}\right]\\&\dot{\sigma}_{rr}=V\left[\frac{\sin\theta}{r}(\sigma_{rr,\theta}-2\sigma_{r\theta})-\cos\theta\sigma_{rr,r}\right]\\&\dot{\sigma}_{\theta\theta}=V\left[\frac{\sin\theta}{r}(\sigma_{\theta\theta,\theta}+2\sigma_{r\theta})-\cos\theta\sigma_{\theta\theta,r}\right]\\&\dot{\sigma}_{33}=V\left[\frac{\sin\theta}{r}\sigma_{33,\theta}-\cos\theta\sigma_{33,r}\right]\end{aligned} \tag{2.12}$$

Across the loading-unloading boundary Γ, all the stress components must be continuous in this paper. As a mater of fact, it has been proved by Narasimhan and Rosakis[11] that the requirement is correct if the material is stable in the Drucker sense.

III. Asymptotic Solution

It is noted in reference [6] that the system of partial differential equations consisting of equations (2.4), (2.10), (2.11), which, are homogeneous in r, is strongly elliptic if $H'>0$.

Therefore asymptotic solutions can be sought in a variable-separable form, similar to the HRR-type fields[8, 12]

$$
\begin{aligned}
v_r &= K(V/E)y_1(\theta)(2\pi r)^s/s \\
v_\theta &= K(V/E)y_2(\theta)(2\pi r)^s/s \\
\sigma_{r\theta} &= Ky_3(\theta)(2\pi r)^s/s \\
\sigma_{rr} &= Ky_4(\theta)(2\pi r)^s \\
\sigma_{\theta\theta} &= Ky_5(\theta)(2\pi r)^s \\
\sigma_{33} &= Ky_6(\theta)(2\pi r)^s
\end{aligned}
\tag{3.1}
$$

where negative s is the stress singularity coefficient, Γ is called as transformation stress intensify factor (SIF), and $y_i(\theta)$ $(i=1, 6)$ are unknown functions.

The substitution of (3.1) into (3.12), (2.4), (2.9), (2.10) yields a system of six first-order ODEs in the forms

$$
\begin{aligned}
y_3' &= y_6-(1+s)y_4 \\
y_5' &= -(1+s)y_3 \\
y_4' &= (sy_4\cos\theta+\dot{\underline{\sigma}}_{rr})/\sin\theta+2y_3 \\
y_6' &= (sy_6\cos\theta+\dot{\underline{\sigma}}_{33})/\sin\theta \\
y_2' &= -y_1+s[\dot{\underline{\sigma}}_{\theta\theta}-\nu(\dot{\underline{\sigma}}_{rr}+\dot{\underline{\sigma}}_{33})+\underline{\Delta}Q_{\theta\theta}] \\
y_1' &= (1-s)y_2+2s[(1+\nu)\dot{\underline{\sigma}}_{r\theta}+\underline{\Delta}Q_{r\theta}]
\end{aligned}
\tag{3.2}
$$

where

$$\dot{\underline{\sigma}}_{ij}=\dot{\sigma}_{ij}/[vK(2\pi)^s r^{s-1}],\quad y_i'=dy_i/d\theta,$$

$$\dot{\underline{\sigma}}_{\theta\theta}=-s(y_3\sin\theta+y_5\cos\theta)$$

$$\dot{\underline{\sigma}}_{r\theta}=-s(y_4\sin\theta+y_3\cos\theta),$$

$$Q_{rr}=\frac{1}{3}+\frac{3}{2}\mu\frac{s_{rr}}{\sigma_e},\quad Q_{\theta\theta}=\frac{1}{3}+\frac{3}{2}\mu\frac{s_{\theta\theta}}{\sigma_e}$$

$$Q_{33}=\frac{1}{3}+\frac{3}{2}\mu\frac{s_{33}}{\sigma_e},\quad Q_{r\theta}=\frac{3}{2}\mu\frac{s_{r\theta}}{\sigma_e}$$

$$\dot{\underline{\sigma}}_{rr}=\frac{1}{\Delta}\Big\{\dot{\underline{\sigma}}_{\theta\theta}\Big[\nu(1+\nu)\frac{H'}{E}+\nu Q_{33}(Q_{33}-Q_{\theta\theta})-Q_{rr}(Q_{\theta\theta}+\nu Q_{33})\Big]$$

$$-2\dot{\underline{\sigma}}_{r\theta}Q_{r\theta}(Q_{rr}+\nu Q_{33})+y_1\Big(\frac{H'}{E}+Q_{33}^2\Big)\Big\}$$

$$\dot{\underline{\sigma}}_{33}=\frac{1}{H'/E+Q_{33}^2}\Big[\nu\frac{H'}{E}(\dot{\underline{\sigma}}_{rr}+\dot{\underline{\sigma}}_{\theta\theta})-Q_{33}(Q_{rr}\dot{\underline{\sigma}}_{rr}+Q_{\theta\theta}\dot{\underline{\sigma}}_{\theta\theta}+2Q_{r\theta}\dot{\underline{\sigma}}_{r\theta})\Big]$$

$$\Delta=(1-\nu^2)\frac{H'}{E}+Q_{33}^2+Q_{rr}^2+2\nu Q_{rr}Q_{33}$$

$$\underline{\Delta}=\frac{E}{H'}(Q_{rr}\dot{\underline{\sigma}}_{rr}+Q_{\theta\theta}\dot{\underline{\sigma}}_{\theta\theta}+Q_{33}\dot{\underline{\sigma}}_{33}+2Q_{r\theta}\dot{\underline{\sigma}}_{r\theta})$$

in which $\underline{\Delta}<0$ if the transformation unloading starts and then let $Q_{rr}=Q_{33}=Q_{\theta\theta}=Q_{r\theta}=0$ in the above experssions.

The mode I symmetry requires that

$$y_2(0)=y_3(0)=y'_4(0)=y'_5(0)=y'_6(0)=0 \tag{3.3}$$

Moreover, on the crack flank, tractions must vanish, hence:

$$y_3(\pi)=y_5(\pi)=0 \tag{3.4}$$

With the normalisation of stress field, thus:

$$y_5(0)=1 \tag{3.5}$$

By taking into account $y'_4(0)=y'_6(0)=0$ the following auxiliary boundary conditions can be established:

$$\begin{aligned} y_1(0)&=-s\{y_4(0)-\nu[y_5(0)+y_6(0)]+Q_{rr}(0)\Delta(0)\} \\ y_6(0)&=\nu[y_4(0)+y_5(0)]-Q_{33}(0)\Delta(0) \end{aligned} \tag{3.6}$$

The system of ODEs can be solved by using the standard Runge-Kutta procedure, except for the amplituder K in (3.1). With the values of s and $y_4(0)$ assigned tentatively, the integration is performed and the values of $y_3(\pi)$ and $y_5(\pi)$ are checked if the condition (3.4) is met. On the basis of the error, s and $y_4(0)$ are reassigned and the process is iterated. With the control of the error and the time step, a satisfactory result can be got.

IV. Numerical Results

The numerical analysis was carried out for Mg-PSZ containing 35 vol% t-ZrO. The material parameters used in this calculation are $E=208\text{GPa}$, $\nu=0.3$, $\bar{B}/E=0.063$, $\bar{G}/E=0.226$ and various values of $\mu=0.0$, 0.1, 0.2, 0.5, 0.6. The numerical results of many constants, such as the singularity s and the unloading angle β, are reported in Table 1. It can be seen that the value of s increases with the increase of the parameter, but the value of β changes very little.

Table 1 The constants for various values of μ

μ	0	0.1	0.2	0.5	0.6
s	−0.40825	−0.38109	−0.35505	−0.30110	−0.28767
β	64°	67°	68°	69°	72°

Plots of angular distributions of the mean stress and effective stress near the crack tip are shown in Figs. 2 and 3, respectively. It is demonstrated that the angular distributions of

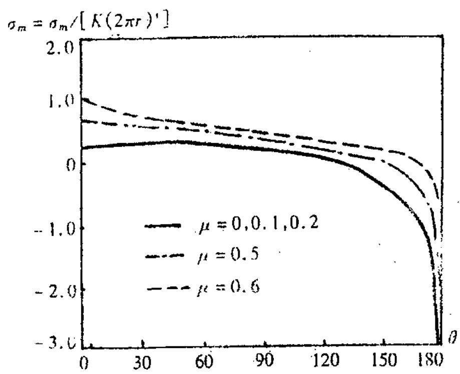

Fig. 2 The angular distribution of mean stress for various values of μ

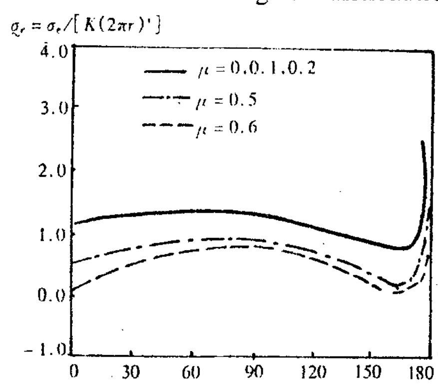

Fig. 3 The angular distribution of effective stress for various values of μ

stresses change very little if $\mu<0.2$ and the gradient of the stress curves ahead of the crack tip is much smaller compared with those in the case of $\mu=0.5$ and 0.6. When μ reaches a large value, the curve of stress ahead of the crack tip becomes steeper and steeper. It can be found that the values of effective stresses ahead of the crack tip decrease with the increasing value of μ, while the values of mean stresses increase a lot. Therefore the ratio of σ_m/σ_e increases rapidly with μ increasing. This indicates that the degree of the constraint ahead of the crack tip increases a lot.

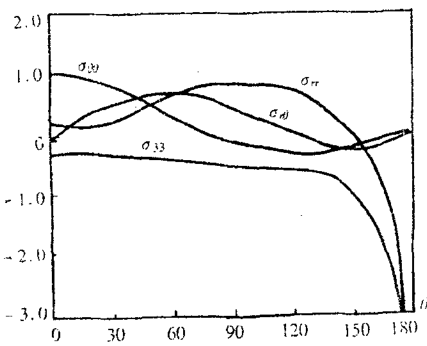

Fig. 4 **The angular distribution of stress components for** $\mu=0.1$

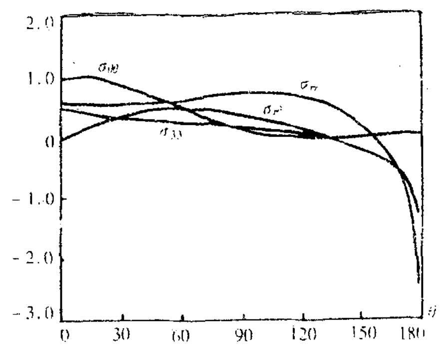

Fig. 5 **The angular distribution of stress components** $\mu=0.5$

Figs. 4 and 5 demonstrate the angular distributions of stress components at the case of $\mu=0.1$ and 0.5. It can be seen that by increasing the effective-stress sensitivity μ, the state of stress near $\theta=0$ approaching to the crack tip tends to be a hydrostatic state of tension. This is due to the singular behaviour at the vertex of the yield locus, similar to the Drucker-Prager yield surface used in Bigoni and Radi[8]. In the computation, when μ is greater than 0.613, we can not find any solution based on the present HRR-type formulation. Therefore, if we use the constitutive model including the plastic volumetric deformation to obtain the asymptotic crack-tip fields, it must be noticed that there is a limit value for its material constants, such as μ in this paper and the pressure-sensitivity in Drucker-Prager yield locus. For this reason, the asymptotic solution can not be obtained for the dilatation plasticity model in reference [1], by using the Drucker-Prager yield surface. In addition, from the result in Bigoni and Radi[8], the maximum value of the limit value of the pressure sensitivity is 0.5, much less than the value for the case of transformed ceramics, for example, 0.7~0.98 for Mg-PSZ and TZP ceramics. So, we can say that the constitutive law presented here is more suitable for the dilatation-dominated materials such as ceramics, rock and concrete, and the Drucker-Prager yield locus is suitable for steel and plastics in which the shear stress are much more important than volumetric deformations.

It can be seen that when θ approaches to π, the angular distribution of σ_{rr}, σ_{33} tends to be negative infinite. These results in the existence of constant compressive stresses in the region near the crack flank in the wake. It is consistent with the results of our previous work[9], Budiansy et al[1] and Stam et al[6]. But in the paper of Bigoni and Radi[8], we found that the same result occurs only at the case of large value of the pressure sensitivity, but at most cases, the stresses σ_{rr}, σ_{33} tend to be positive infinite as those in the classical elastic plastic materials.

V. Conclusions

In this paper, a constitutive model which includes the effect of shear part of transformation plasticity, is given and the asymptotic analysis of stress and velocity fields in the ring zone near the crack tip is carried out. The near-tip zone is comprised of the transformation loading zone and the elastic one in which the material is under unloading or at full-transformation stage. The near-tip asymptotic fields are dependent on the choice of the value of effective-stress sensitivity. It can be seen that an increase in μ will produce:

(1) a reduction in the singularity of the near-tip fields;

(2) a decrease in the effective stress ahead of the crack tip;

(3) an increase in the ratio of mean stress and effective stress, which means that the degree of constraint near the crack tip increases;

(4) a little change in the angular distribution of stress fields if the value of μ is less than 0.2.

In the meantime, we can find from the computational results that for volumetric-deformation-dominated materials, the constitutive model presented in this paper is more suitable for asymptotic analysis than the Drucker-Prager yield condition.

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