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USING WAVELET TRANSFORM TO STUDY THE LIPSCHITZ LOCAL SINGULAR EXPONENT IN WALL TURBULENCE*

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Abstract

In this paper, wavelet transform is introduced to study the Lipschitz local singular exponent for characterizing the local singularity behavior of fluctuating velocity in wall turbulence. It is found that the local singular exponent is negative when the ejections and sweeps of coherent structures occur in a turbulent boundary layer.

Key words wavelet transform, coherent structure, Lipschitz local singular exponent

I. Introduction

The near wall coherent structure in a turbulent boundary layer has long been the subject for a large number of investigations. There is no longer any doubt that coherent structures are a major component in wall-bounded turbulent shear flows. They play an important role in the production, dissipation and transportation of turbulent energy. One of the numerous difficulties of these studies is how to quantitatively measure coherent structure from physical experiments or from direct numerical simulations. The goal is to isolate or characterize the coherent structures which drive the near wall turbulence dynamics.

Wavelet transforms[1] are recently developed mathematical techniques based on group theory and square integrable representations. These techniques allow one to unfold a signal into both physical space and scale space at the same time by convoluting the signal with a given analyzing function called wavelet. The wavelet is obtained by translating and dilating the chosen mother wavelet function. The limited spatial support of the wavelet localized in time space is important because the behavior of the signal at infinity does not play any role. Therefore the wavelet analysis can be performed locally for the signal in time space. This is opposed to the Fourier transform which is inherently nonlocal due to the space filling nature of the trigonometric functions. On the other hand, wavelet transform provides a frequency analysis of the signal with a filter of constant relative resolution by means of dilating or contracting the window width of the wavelet function. So wavelet transform can be regarded as a time-frequency adjoining analysis.

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As a new tool, wavelet transform can be devoted to the use for identifying coherent structure arising in wall trubulence instead of the conditional sampling methods usually used. Jiang Nan (1997)[2] has performed the time-frequency adjoining decompositions of the hot-film probe output signal of the longitudinal velocity fluctuation in a turbulent boundary layer with two mother wavelet. The energy evolution across scales are obtained from the wavelet coefficient modulus. The energy maxima criterion is established to determine the scale that corresponds to coherent structure. The coherent structure velocity is extracted from the turbulent fluctuating velocity with wavelet inverse transform. It is found that the coherent structure average period made normalized with inner scale is $t^+ = 1138$ where $t^+ = tu_\tau^2/\nu$ (u_τ is the wall friction velocity and ν the kinematic viscosity). The results are in good agreement with the experimental measurements.

This paper is devoted to the study of Lipschitz local singular exponent[3] of wall turbulence signal as a new defined quantity for coherent structure intensity using wavelet transform. It is found that when the ejections and sweeps of coherent structures take place, the Lipschitz local exponent of fluctuating velocity within the near wall region of a turbulent boundary layer is negative. This conflicts with Kolmogorov theory[4] that the velocity field has the same scaling behavior $\delta v(t,\Delta t) = v(t+\Delta t) - v(t) = \Delta t^{1/3}$ which yields the well-known $E(k) \infty k^{-5/3}$ power law behavior of the energy spectrum. Negative exponents which do not seem to have been previously reported correspond to dynamically significant events have strong localized gradients. The interpretation of these events in the near wall region of a turbulent boundary layer is the occasional passage or the intermittency phenomenon of the coherent structures which is localized in space near the probe. The ejections and sweeps of the coherent structures can lead to the strong spike of the fluctuating velocity and they are involved in a large portion of the momentum transport in wall turbulence.

II. Wavelet Transform and Lipschitz Local Singular Exponent

2.1 Wavelet transform and eddies in turbulence

Wavelet transform is a new mathematical technique developed recently which consists in decomposing an signal into elementary contributions called wavelets. By convoluting the signal with the wavelets, this decomposition comprises an expansion of the signal over the wavelets. In the one dimensional case, it provides a two dimensional unfolding of the signal resolving both in the position and scale as indenpendent variables. Wavelets are constructed from an analysing function called mother wavelet by means of translations and dilatations.

Definition 1 if $W(t) \in L^2(R)$ satisfies the "admissibility" condition:

$$C_W = \int_{-\infty}^{+\infty} \frac{|\hat{W}(\omega)|^2}{|\omega|} d\omega < \infty \tag{2.1}$$

where $\hat{W}(\omega)$ is the Fourier transform of $W(t)$, then $W(t)$ is called a "basic wavelet".

Relative to every basic wavelet $W(t)$, $W_{ab}(t)$ is the translation (factor b) and dilatation (factor a) of $W(t)$:

$$W_{ab}(t) = \frac{1}{a} W\left(\frac{t-b}{a}\right) \tag{2.2}$$

where $a, b \in R$ with $a \neq 0$.

The wavelet transform $W_s(a,b)$ of signal $s(t) \in L^2(R)$ with respect to $W_{ab}(t)$ is defined as their scalar product by

$$W_s(a,b) = \int_{-\infty}^{+\infty} s(t)\,\overline{W_{ab}(t)}\,dt \tag{2.3}$$

Remark If, in addition, both $W(t)$ and $\hat{W}(\omega)$ satisfy

$$tW(t) \in L^2(R), \omega\hat{W}(\omega) \in L^2(R)$$

then the basic wavelet $W(t)$ provides a time-frequency window with finite area and $\hat{W}(\omega)$ is a continuous function, so that the finiteness of C_W implies

$$\hat{W}(0) = 0 \tag{2.4}$$

or equivalently,

$$\int_{-\infty}^{+\infty} W(t)\,dt = 0 \tag{2.5}$$

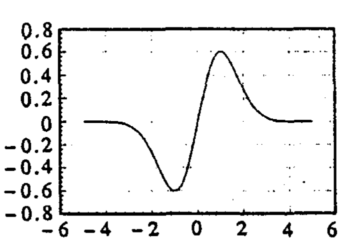
This means that its average is zero or the wavelet is said to have zero vanishing moment. A wavelet must have at least zero vanishing moment. A wavelet is said to have n vanishing moments, if and only if for all non-negative integer $0 \leqslant k \leqslant n$, it satisfies

$$\int_{-\infty}^{+\infty} t^k W(t)\,dt = 0 \tag{2.6}$$

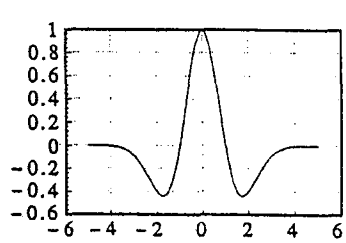
A wavelet has n vanishing moments is also called to have cancellations up to order n. $W_{ab}(t)$ also has zero mean. It follows that if $s(t) \in L^2(R)$ has no variation at the scale of a in the vicinity of a given position b (that is to say that, within the support of $W_{ab}(t)$, $s(t)$) is approximately constant) the wavelet coefficient $W_s(a,b)$ is zero. Conversely, if $s(t)$ evolves around a given position b at a given scale a obviously (that is to say $s(t)$ has significant varietions within the support of $W_{ab}(t)$, the corresponding coefficient $W_s(a,b)$ is large. So the wavelet transform can be regarded as a mathematical microscope, for which position and magnification correspond to b and a respectively, and the performance of the optics is determined by the choice of the analysing basic wavelet $W(t)$. Fig. 1 shows the shape of Gauss basic wavelet and Marr basic wavelet which are defined respectively by

$$W(t) = te^{-t^2/2} \tag{2.7a}$$

$$W(t) = (1 - t^2)e^{-t^2/2} \tag{2.7b}$$



(a) Gauss basic wavelet



(b) Marr basic wavelet

Fig. 1

The expression of $W_s(a,b)$ in equation (2.3) shows that the wavelet coefficient is proportional to the intercorrelation of $s(t)$ with $W_{ab}(t)$. As far as turbulence is concerned, wavelet function has special physical meaning. "Eddy" provides the most suitable elementary decomposition of turbulence. Wavelet representation provides the decompositions of turbulence into eddies modes. "Eddies" are to turbulence study, what wavelets are, more generally, to signal decomposition. They are fairly broad contributions in the spectral domain and localized contributions in the physical space. Large $W_s(a,b)$ means that $s(t)$ and $W_{ab}(t)$ have strong intercorrelation and that there is an eddy of scale a at the time $t=b$ passing through the probe. From the standard (a, b) plane representation of the wavelet coefficients, turbulence can be unfolded into one-to-one eddies at different positions $t=b$ and different scales a.

Coherent structures are particularly important large-scale "eddies" in wall-bounded turbulence and they are localized in spatial space and time space. The strong intermittency and inhomogeneity lead the well-known physical meaning of Fourier Transform is lost and the wavelet projection could be a very good alternative.

2.2 Lipschitz local singular exponent

Definition 2 Let n be a non-negative integer, $n \leqslant \alpha < n+1$, signal $s(t)$ is said to have Lipschitz local exponent α at point $t = t_0$, if and only if there exists two constants $\delta > 0$ and C, such that for $0 < | t - t_0 | < \delta$

$$s(t) = s(t_0) + (t - t_0)s'(t_0) + \cdots + \frac{(t-t_0)^n s^{(n)}(t_0)}{n!} + C\,| t - t_0 |^{\alpha(t_0)} + o(| t - t_0 |^{\alpha(t_0)}) \tag{2.8}$$

Signal $s(t)$ is said singular at point $t = t_0$, if its Lipschitz local exponent $\alpha(t_0) < 1$

$$s(t) = s(t_0) + C\,| t - t_0 |^{\alpha(t_0)} + o(| t - t_0 |^{\alpha(t_0)}) \tag{2.9}$$

Signal $s(t)$ that is continuously differentiable at point $t = t_0$ is Lipschitz 1 at this point. If the derivative of $s(t)$ is bounded but discontinuous at point $t = t_0$, $s(t)$ is still Lipschitz 1 at this point. The Lipschitz local exponent gives an indication of the differentiability of $s(t)$ but it is more precise. If the Lipschitz local exponent α of $s(t)$ satisfies $n < \alpha < n+1$, it can be known that $s(t)$ is n times differentiable at this point but its nth derivative is singular $\alpha - n < 1$ at this point and α characterizes this singularity.

A wavelet has vanishing moment (cancelation) of order zero as equation (2.5) can measure the Lipschitz local singular exponent $\alpha < 1$. It can be proven that if a signal $s(t)$ has Lipschitz local singular exponent $\alpha(t_0) < 1$ at point $t = t_0$, when $0 < \max(\lambda a, a) < \delta/(t^* + \sigma_t)$, the wavelet transform of $s(t)$ satisfies

$$W_s(\lambda a, t_0) = \lambda^{\alpha(t_0)} W_s(a, t_0) \tag{2.10}$$

As a result, the Lipschitz local singular exponent $\alpha(t_0)$ of $s(t)$ satisfies

$$\alpha(t_0) = \frac{\log | W_s(\lambda a, t_0)/W_s(a; t_0) |}{\log | \lambda |} \tag{2.11}$$

In fact,

$$W_s(\lambda a, t_0) = \frac{1}{\lambda a}\int_{-\infty}^{+\infty} s(t)\,\overline{W\left(\frac{t-t_0}{\lambda a}\right)}dt$$

$$t = \lambda a x + t_0$$

$$W_s(\lambda a, t_0) = \int_{-\infty}^{+\infty} s(\lambda a x + t_0)\, \overline{W(x)}\, dx$$

$$\int_{-\infty}^{+\infty} s(t_0)\, \overline{W(-x)}\, dx = 0$$

$$\begin{aligned} W_s(\lambda a, t_0) &= \int_{-\infty}^{+\infty} [s(\lambda a x + t_0) - s(t_0)]\, \overline{W(x)}\, dx \\ &= \int_{-\infty}^{+\infty} C \mid \lambda a x \mid^{\alpha(t_0)} \overline{W(x)}\, dx \\ &= \lambda^{\alpha(t_0)} \int_{-\infty}^{+\infty} C \mid a x \mid^{\alpha(t_0)} \overline{W(x)}\, dx \\ &= \lambda^{\alpha(t_0)} \int_{-\infty}^{+\infty} [s(a x + t_0) - s(t_0)]\, \overline{W(x)}\, dx \\ &= \lambda^{\alpha(t_0)} \int_{-\infty}^{+\infty} s(a x + t_0)\, \overline{W(x)}\, dx \\ &= \lambda^{\alpha(t_0)} \int_{-\infty}^{+\infty} s(t)\, \overline{W\left(\frac{t - t_0}{a}\right)}\, \frac{1}{a} dt \\ &= \lambda^{\alpha(t_0)} W_s(a, t_0) \end{aligned}$$

If $s(t)$ has a Lipschitz local exponent α, its primitive has a Lipschitz local exponent $\alpha + 1$. So it is possible to make an extension of Lipschitz local exponent to negative value. If the primitive of $s(t)$ has a Lipschitz local singular exponent $\alpha + 1 < 1$, then $s(t)$ has a Lipschitz local singular exponent $\alpha < 0$. For example, for a piece-wise linear function in the neighborhood of point $t = t_0$, its the first derivative is a stair function and its the second derivative is a Dirac at point $t = t_0$. It can be easily proven that a stair function has a Lipschitz local singular exponent $\alpha = 0$ and a Dirac has a Lipschitz local singular exponent $\alpha = -1$.

In fact, suppose a stair function

$$s(t) = \begin{cases} 1, & t > t_0 \\ -1, & t \leqslant t_0 \end{cases}$$

then for the Gauss basic wavelet

$$\frac{1}{a}\left[\int_{-\infty}^{t_0} - \overline{W\left(\frac{t - t_0}{a}\right)} dt + \int_{t_0}^{+\infty} \overline{W\left(\frac{t - t_0}{a}\right)} dt\right] \neq 0$$

$$\int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$

$$\delta(t) = \begin{cases} 0, & t \neq t_0 \\ +\infty, & t = t_0 \end{cases}$$

then

$$W_\delta(\lambda a, t_0) = \frac{1}{\lambda a} \int_{-\infty}^{+\infty} \delta(t)\, \overline{W\left(\frac{t - t_0}{\lambda a}\right)} dt = \frac{1}{\lambda a} \overline{W\left(\frac{t_0 - t_0}{\lambda a}\right)}$$

$$= \frac{1}{\lambda a} W(0) = \frac{1}{\lambda a} = \frac{1}{\lambda}\frac{1}{a}\int_{-\infty}^{+\infty} \delta(t)\, \overline{W\left(\frac{t-t_0}{a}\right)} dt = \frac{1}{\lambda} W_\delta(a, t_0)$$

So $\alpha(t_0) = -1$

III. Using Wavelet Transform to Study the Lipschitz Local Singular Exponent in Wall Turbulence

Most important and interestintg information of a signal is often carried by irregular structures and transient phenomena such as peaks. In physics, it is especially important to study these irregular structures to infer properties about the underlined physical phenomena. A well-known example is the large-scale coherent structures in wall-bounded turbulence, for which there is still no comprehensive theory to understand the nature and repartition of them. This often motivates us want to detect and characterize the irregular coherent structures from turbulent fluctuating signals. The Fourier transform is global and provides an overall description of the regularity of signals, but it is not well adapted for finding the location and the spatial distribution of singularities. By decomposing signals into elementary building blocks that are well localized both in space and frequency, wavelet transform can characterize the local irregularity of coherent structures arise in wall turbulent flows. This enable us to study the intensity of coherent structures quantitatively by defining the Lipschitz local singular exponent of turbulent fluctuating signals using wavelet transformation.

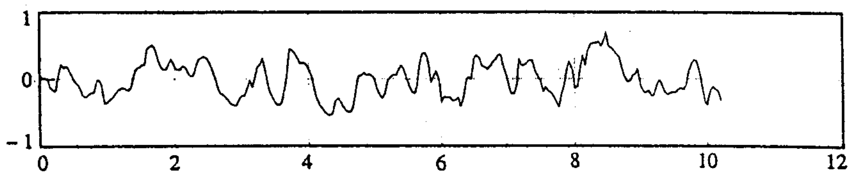


Fig. 2 Streamwise fluctuating velocity signal in wall turbulence at $y^+ = 32$

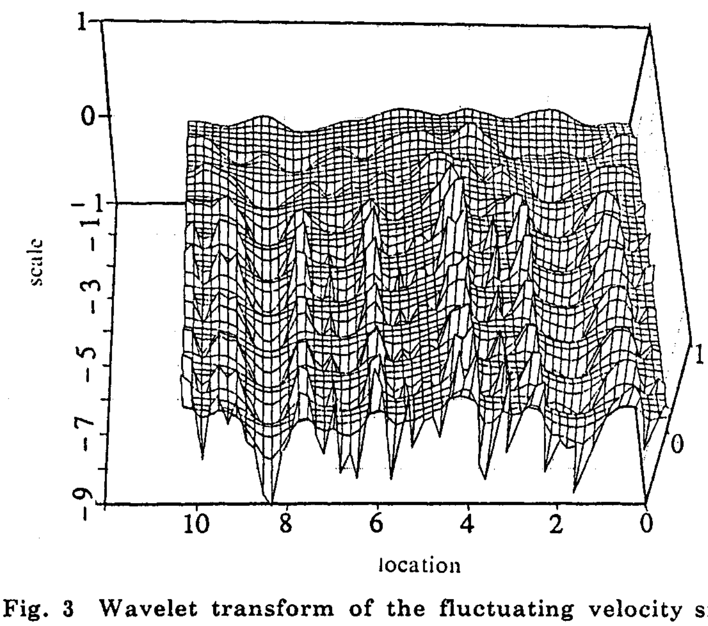


Fig. 3 Wavelet transform of the fluctuating velocity signal

Experiments are conducted in a fully developed turbulent flow of a free-surface recirculating water channel with a rectangular cross-section. A Plexiglass flat plate with a length of 1.3m and a width of 14cm is set on the bottom of the test section of the channel to trip a turbulent boundary layer. Simultaneous measurements of the streamwise velocity component are made using a TSI 1051-2D model anemometer with a 1210-20W model miniature hot-film probe. The original sreamwise velocity fuluctuating signal is shown in Fig. 2. The wavelet transform of the signal is shown in Fig. 3. The coherent structure velocity signal shown in Fig. 4 is extracted from the fluctuating signal using inverse wavelet transform[2]. The ejections and sweeps in Fig. 4 can be identified clearly. The Lipschitz local exponents at different time of the signal obtained by formula (2.10) and (2.11) are shown in Fig. 5. It can be seen from Fig. 5 that when the ejection and sweep take place, the Lipschitz local singular exponent is negative. This means that the signal has strong localized gradients when the ejections and sweeps take place. The ejections and sweeps of the coherent structures can lead to the strong spike of the fluctuating velocity and they are involved in a large portion of the momentum transport in wall turbulence.

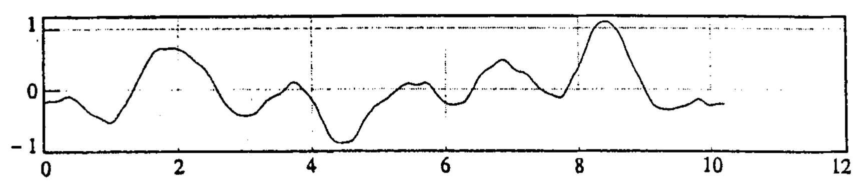


Fig. 4 Coherent structure velocity signal

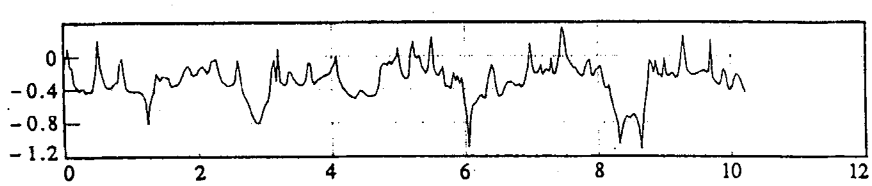


Fig. 5 Lipschitz local singular exponent for streamwise fluctuating velocity in wall turbulence

IV. Conclusion

Lipschitz local singular exponent of turbulent fluctuating signal can be a new defined quantity for coherent structure intensity using wavelet transform. It is found that when the ejections and sweeps take place, the Lipschitz local exponent of fluctuating velocity signal in the near wall region of a turbulent boundary layer is negative. Negative exponents correspond to dynamically significant events have strong localized gradients. The ejections and sweeps of the coherent structures can lead to the strong spike of the fluctuating velocity and they are involved in a large portion of the momentum transport in wall turbulence.

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