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ABSTRACT: A previously published refined shear deformation theory is used to analyse free vibration of laminated shells. The theory includes the assumption that the transverse shear strains across any two layers are linearly dependent on each other. The theory has the same dependent variables as first-order shear deformation theory, but the set of governing differential equations is of twelfth order. No shear correction factors are required. Free vibration of symmetric cross-ply laminated cylindrical shells, symmetric and antisymmetric cross-ply cylindrical panels is calculated. The numerical results are in good agreement with those from three-dimensional elasticity theory.

KEY WORDS: vibration analysis, laminated shells, shear deformation, governing equations, cylindrical shells and panels.

INTRODUCTION

CLASSICAL LAMINATION THEORY (CLT) based on the Kirchhoff-Love hypothesis [1] is the earliest theory of thin shells laminated of anisotropic material. However, transverse shear deformation effects are important in vibration analysis of laminated shells. The first-order shear deformation theory (FSDT) for

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laminated shells [2–4] is much more accurate than classical lamination theory for the prediction of natural frequencies. However, since the tangential displacements are still assumed to be linear across the shell thickness, there is less improvement in the accuracy of the modal tangential displacements and modal stresses. Furthermore, shear correction factors have to be used in order to adjust the transverse shear stiffnesses. To overcome these drawbacks, several refined shear deformation theories (displacement based) have been developed. These include the higher-order shear deformation theories based on nonlinear through thickness tangential displacement assumptions [5–7] and the discrete layer theories based on piecewise linear tangential displacement assumptions [8]. The latter are more capable of modelling the warpage of the cross section of a laminated shell with many layers. However, they are computationally expensive. The imposition of the continuity of transverse stresses at layer interfaces can be used to reduce the total number of dependent variables and save effort. Such a simplified discrete-layer theory (SDLT) has been presented in Reference [9]. However, since in that theory the transverse shear stresses are assumed implicitly to be nearly constant, that is not entirely satisfactory. Recently, a refined shear deformation theory with a modified model was presented. The tangential displacements are also assumed to be piecewise linear and the normal displacement constant across the thickness. In addition, the transverse shear strains across any two different layers are assumed to be linearly dependent on each other. The theory has been used to analyse bending and free vibration of laminated plates in References [10] and [11], respectively. The solutions of the theory were found to be in good agreement with the exact solutions of the three-dimensional theory of elasticity. Static analysis of laminated shells was successively given in Reference [12]. The present paper deals with the free vibration analysis of laminated shells. Symmetric cross-ply laminated cylindrical shells, symmetric and antisymmetric cross-ply cylindrical panels are calculated. The numerical results are compared with CLT, FSDT, SDLT and three-dimensional elasticity theory solutions.

FORMULATION

Consider a shell of constant thickness h composed of $N_1 + N_2 + 1$ thin layers of anisotropic material bonded together. A system of orthogonal curvilinear coordinates is defined by the coordinates α_1 and α_2 corresponding to the lines of curvature on the middle surface of the shell and the coordinate z along the normal to the middle surface. The Lame's coefficients and the normal curvatures in the directions of α_1 and α_2 are denoted by A_1, A_2 and k_1 $(=1/R_1)$, k_2 $(=1/R_2)$ respectively. We assume that the thickness of the shell h is very small compared to the principal radii of curvature, i.e., k_1h, $k_2h \ll 1$. However, the value of span-to-thickness ratio may not be very large for a moderately thick shell. The thickness of the ith layer is t_i $(i = -N_2, \ldots 0, \ldots N_1)$. The coordinate in the z-direction of the mid-surface of the ith layer is z_i. The layer corresponding to $i=0$ is determined from the condition: $-t_0/2 < z_0 \le t_0/2$, i.e., it includes the middle surface of the shell. We begin with the displacement field of the ith layer:

$$
\begin{aligned}
u_1^{(i)}(\alpha_1,\alpha_2,z) &= u_{1m}^{(i)}(\alpha_1,\alpha_2)+(z-z_i)\psi_1^{(i)}(\alpha_1,\alpha_2)\\
u_2^{(i)}(\alpha_1,\alpha_2,z) &= u_{2m}^{(i)}(\alpha_1,\alpha_2)+(z-z_i)\psi_2^{(i)}(\alpha_1,\alpha_2) \qquad (1)\\
w^{(i)}(\alpha_1,\alpha_2,z) &= W(\alpha_1,\alpha_2)
\end{aligned}
$$

where $u_{1m}^{(i)}$, $u_{2m}^{(i)}$ and W denote the displacements of a point (α_1, α_2) in the mid-surface of the ith layer, and $\psi_1^{(i)}$ and $\psi_2^{(i)}$ are the rotations of normals to mid-surface. In fact, we assume that the normal displacement W is constant through the thickness of the shell. Considering $k_1h, k_2h \ll 1$, we can obtain the transverse shear strain components $\gamma_{13}^{(i)}$ and $\gamma_{23}^{(i)}$ within the ith layer as

$$
\begin{aligned}
\gamma_{13}^{(i)} &= \psi_1^{(i)} + \frac{1}{A_1}\frac{\partial W}{\partial \alpha_1} - k_1 u_{1m}^{(i)} = \psi_1^{(i)} - \varphi_1^{(i)}\\
\gamma_{23}^{(i)} &= \psi_2^{(i)} + \frac{1}{A_2}\frac{\partial W}{\partial \alpha_2} - k_2 u_{2m}^{(i)} = \psi_2^{(i)} - \varphi_2^{(i)}
\end{aligned} \qquad (2)
$$

As real transverse shear stresses are continuous between layers, we assume that the transverse shear strain components across any two different layers are linearly dependent on each other. So we have

$$
\gamma_{13}^{(i)} = \lambda_{11}^{(i)}\gamma_{13}^{(0)} + \lambda_{12}^{(i)}\gamma_{23}^{(0)}, \quad \gamma_{23}^{(i)} = \lambda_{21}^{(i)}\gamma_{13}^{(0)} + \lambda_{22}^{(i)}\gamma_{23}^{(0)} \qquad (3)
$$

where $\gamma_{13}^{(0)}$ and $\gamma_{23}^{(0)}$ represent the transverse shear strain components within layer zero. $\lambda_{rs}^{(i)}$ $(rs = 11, 22, 12, 21)$ are undetermined constants. Furthermore, the continuity of interlaminar tangential displacements has to be preserved. Hence, the tangential displacements of points in every layer can be expressed in terms of five unknown functions: the displacements of points on the middle surface of the shell, $U_1(\alpha_1, \alpha_2)$, $U_2(\alpha_1, \alpha_2)$ and $W(\alpha_1, \alpha_2)$, and rotations of normals to mid-surface of layer zero, $\psi_1^{(0)}(\alpha_1, \alpha_2)$ and $\psi_2^{(0)}(\alpha_1, \alpha_2)$. The expressions are

$$
u_1^{(i)}(\alpha_1,\alpha_2,z) = U_1(\alpha_1,\alpha_2) + \left[z_0 + S(i)t_{11}^{(i)}\right]\psi_1^{(0)}(\alpha_1,\alpha_2) \qquad (4)
$$

$$+ S(i)t_{12}^{(i)}\psi_2^{(0)}(\alpha_1,\alpha_2)+\left[z_i - z_0 - S(i)t_{11}^{(i)}\right]\varphi_1^{(0)}(\alpha_1,\alpha_2)$$

$$- S(i)t_{12}^{(i)}\varphi_2^{(0)}(\alpha_1,\alpha_2)+(z-z_i)\Big[\lambda_{11}^{(i)}\psi_1^{(0)}(\alpha_1,\alpha_2)$$

$$+ \lambda_{12}^{(i)}\psi_2^{(0)}(\alpha_1,\alpha_2)+\left(1-\lambda_{11}^{(i)}\right)\varphi_1^{(0)}(\alpha_1,\alpha_2)-\lambda_{12}^{(i)}\varphi_2^{(0)}(\alpha_1,\alpha_2)\Big]$$

(4 con't)

$$u_2^{(i)}(\alpha_1,\alpha_2,z)= U_2(\alpha_1,\alpha_2)+\left[z_0 + S(i)t_{22}^{(i)}\right]\psi_2^{(0)}(\alpha_1,\alpha_2)$$

$$+ S(i)t_{21}^{(i)}\psi_1^{(0)}(\alpha_1,\alpha_2)+\left[z_i - z_0 - S(i)t_{22}^{(i)}\right]\varphi_2^{(0)}(\alpha_1,\alpha_2)$$

$$- S(i)t_{21}^{(i)}\varphi_1^{(0)}(\alpha_1,\alpha_2)+(z-z_i)\Big[\lambda_{22}^{(i)}\psi_2^{(0)}(\alpha_1,\alpha_2)$$

$$+ \lambda_{21}^{(i)}\psi_1^{(0)}(\alpha_1,\alpha_2)+\left(1-\lambda_{22}^{(i)}\right)\varphi_2^{(0)}(\alpha_1,\alpha_2)-\lambda_{21}^{(i)}\varphi_1^{(0)}(\alpha_1,\alpha_2)\Big]$$

$$\varphi_1^{(0)}(\alpha_1,\alpha_2)\approx k_1U_1-\frac{1}{A_1}\frac{\partial W}{\partial\alpha_1},\quad \varphi_2^{(0)}(\alpha_1,\alpha_2)\approx k_2U_2-\frac{1}{A_2}\frac{\partial W}{\partial\alpha_2}$$

where the definitions of $t_{rs}^{(i)}$ and $S(i)$ are the same as in Reference [11]. Define

$$\varepsilon_1=\frac{1}{A_1}\frac{\partial U_1}{\partial\alpha_1}+\frac{U_2}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}+k_1W,\quad \varepsilon_2=\frac{1}{A_2}\frac{\partial U_2}{\partial\alpha_2}+\frac{U_1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}+k_2W$$

$$\omega_1=\frac{1}{A_1}\frac{\partial U_2}{\partial\alpha_1}-\frac{U_1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2},\qquad \omega_2=\frac{1}{A_2}\frac{\partial U_1}{\partial\alpha_2}-\frac{U_2}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}$$

$$\gamma_{12}=\omega_1+\omega_2=\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{U_1}{A_1}\right)+\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{U_2}{A_2}\right) \tag{5}$$

$$\kappa_1'=\frac{1}{A_1}\frac{\partial\psi_1^{(0)}}{\partial\alpha_1}+\frac{\psi_2^{(0)}}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2},\qquad \kappa_2'=\frac{1}{A_2}\frac{\partial\psi_2^{(0)}}{\partial\alpha_2}+\frac{\psi_1^{(0)}}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}$$

$$\kappa'_{12} = \frac{1}{A_1}\frac{\partial \psi_2^{(0)}}{\partial \alpha_1} - \frac{\psi_1^{(0)}}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + k_1\omega_2, \quad \kappa'_{21} = \frac{1}{A_2}\frac{\partial \psi_1^{(0)}}{\partial \alpha_2} - \frac{\psi_2^{(0)}}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + k_2\omega_1$$

$$\kappa''_1 = \frac{1}{A_1}\frac{\partial \varphi_1^{(0)}}{\partial \alpha_1} + \frac{\varphi_2^{(0)}}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2}, \quad \kappa''_2 = \frac{1}{A_2}\frac{\partial \varphi_2^{(0)}}{\partial \alpha_2} + \frac{\varphi_1^{(0)}}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} \qquad \text{(5 con't)}$$

$$\kappa''_{12} = \frac{1}{A_1}\frac{\partial \varphi_2^{(0)}}{\partial \alpha_1} - \frac{\varphi_1^{(0)}}{A_1 A_2}\frac{\partial A_1}{\partial \alpha_2} + k_1\omega_2 = \frac{1}{A_2}\frac{\partial \varphi_1^{(0)}}{\partial \alpha_2} - \frac{\varphi_2^{(0)}}{A_1 A_2}\frac{\partial A_2}{\partial \alpha_1} + k_2\omega_1 = \kappa''_{21}$$

We neglect the transverse normal stress and obtain the expression for the strain energy density per unit area of the middle surface of the laminated shell as

$$E = \frac{1}{2}\Big[N_1\varepsilon_1 + N_2\varepsilon_2 + S\gamma_{12} + M'_1\kappa'_1 + M'_2\kappa'_2 + M'_{12}\kappa'_{12} + M'_{21}\kappa'_{21} + M''_1\kappa''_1$$

$$+ M''_2\kappa''_2 + M''_{12} \times (2\kappa''_{12}) + Q'_1\gamma_{13}^{(0)} + Q'_2\gamma_{23}^{(0)}\Big] \qquad (6)$$

Here $N_1, N_2, \ldots Q'_2$ are all generalised internal forces. Defining

$$\{N\} = [N_1\ N_2\ S]^T, \quad \{M'\} = [M'_1\ M'_2\ M'_{12}\ M'_{21}]^T$$

$$\{M''\} = [M''_1\ M''_2\ M''_{12}]^T, \quad \{Q'\} = [Q'_1\ Q'_2]^T$$

$$\{\varepsilon\} = [\varepsilon_1\ \varepsilon_2\ \gamma_{12}]^T, \quad \{\kappa'\} = [\kappa'_1\ \kappa'_2\ \kappa'_{12}\ \kappa'_{21}]^T \qquad (7)$$

$$\{\kappa''\} = [\kappa''_1\ \kappa''_2\ 2\kappa''_{12}]^T, \quad \{\gamma\} = [\gamma_{13}^{(0)}\ \gamma_{23}^{(0)}]^T$$

the expression for the overall generalised force-strain relations is of the form

$$\begin{bmatrix} \{N\} \\ \{M'\} \\ \{M''\} \\ \{Q'\} \end{bmatrix} = \begin{bmatrix} [A] & [B'] & [B''] & [F] \\ [B']^T & [D'] & [D_c] & [H'] \\ [B'']^T & [D_c]^T & [D''] & [H''] \\ [F]^T & [H']^T & [H'']^T & [G] \end{bmatrix} \begin{bmatrix} \{\varepsilon\} \\ \{\kappa'\} \\ \{\kappa''\} \\ \{\gamma\} \end{bmatrix} \qquad (8)$$

The expressions of the sub-matrices are the same as in Reference [12].

We use the principle of stationary potential energy presented in Reference [13] to derive the differential equations of free vibration and the equations that $\gamma_{rs}^{(i)}$ must satisfy. The former can be given as follows:

$$\begin{aligned}
&\frac{\partial}{\partial\alpha_1}(N_1A_2)+\frac{\partial}{\partial\alpha_2}(N_{21}A_1)+N_{12}\frac{\partial A_1}{\partial\alpha_2}-N_2\frac{\partial A_2}{\partial\alpha_1}+k_1A_1A_2(Q_1'+Q_1'')\\
&\qquad+\omega^2(M_{11}U_1+M_{13}W+M_{14}\psi_1^{(0)}+M_{15}\psi_2^{(0)})=0\\
&\frac{\partial}{\partial\alpha_1}(N_{12}A_2)+\frac{\partial}{\partial\alpha_2}(N_2A_1)+N_{21}\frac{\partial A_2}{\partial\alpha_1}-N_1\frac{\partial A_1}{\partial\alpha_2}+k_2A_1A_2(Q_2'+Q_2'')\\
&\qquad+\omega^2(M_{22}U_2+M_{23}W+M_{24}\psi_1^{(0)}+M_{25}\psi_2^{(0)})=0\\
&\frac{\partial}{\partial\alpha_1}[(Q_1'+Q_1'')A_2]+\frac{\partial}{\partial\alpha_2}[(Q_2'+Q_2'')A_1]-A_1A_2(k_1N_1+k_2N_2)\\
&\qquad-\omega^2(M_{31}U_1+M_{32}U_2+M_{33}W+M_{34}\psi_1^{(0)}+M_{35}\psi_2^{(0)})=0\\
&\frac{\partial}{\partial\alpha_1}(M_1'A_2)+\frac{\partial}{\partial\alpha_2}(M_{21}'A_1)+M_{12}'\frac{\partial A_1}{\partial\alpha_2}-M_2'\frac{\partial A_2}{\partial\alpha_1}-A_1A_2Q_1'\\
&\qquad+\omega^2(M_{41}U_1+M_{42}U_2+M_{43}W+M_{44}\psi_1^{(0)}+M_{45}\psi_2^{(0)})=0\\
&\frac{\partial}{\partial\alpha_1}(M_{12}'A_2)+\frac{\partial}{\partial\alpha_2}(M_2'A_1)+M_{21}'\frac{\partial A_2}{\partial\alpha_1}-M_1'\frac{\partial A_1}{\partial\alpha_2}-A_1A_2Q_2'\\
&\qquad+\omega^2(M_{51}U_1+M_{52}U_2+M_{53}W+M_{54}\psi_1^{(0)}+M_{55}\psi_2^{(0)})=0
\end{aligned}\tag{9}$$

where

$$\begin{aligned}
&N_{12}=S+k_2(M_{21}'+M_{12}''),\qquad N_{21}=S+k_1(M_{12}'+M_{12}'')\\
&Q_1''=\frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}(M_1''A_2)+\frac{\partial}{\partial\alpha_2}(M_{12}''A_1)+M_{12}''\frac{\partial A_1}{\partial\alpha_2}-M_2''\frac{\partial A_2}{\partial\alpha_1}\right]
\end{aligned}\tag{10}$$

$$Q_2'' = \frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}(M_{12}''A_2) + \frac{\partial}{\partial\alpha_2}(M_2''A_1) + M_{12}''\frac{\partial A_2}{\partial\alpha_1} - M_1''\frac{\partial A_1}{\partial\alpha_2}\right] \quad \text{(10 con't)}$$

In Equation (9) ω is the natural circular frequency, and M_{11}, etc., are coefficients and differential operators with respect to α_1 and α_2. The expressions for them are given in Appendix A. The set of Equations (9) can be expressed in terms of the amplitudes of the displacements U_1, U_2, W and the rotations $\psi_1^{(0)}$ and $\psi_2^{(0)}$. It is of twelfth order and no shear correction factors are introduced. The consistent homogeneous boundary conditions are of the form

$$N_n + k_{ns}M_{ns}'' = 0 \quad \text{or} \quad U_n = 0$$

$$N_{ns} + k_sM_{ns}'' = 0 \quad \text{or} \quad U_s = 0$$

$$Q_n' + Q_n'' + \frac{\partial M_{ns}''}{\partial s} - \omega^2\Big[(R_{11} - R)U_1 + R_{21}U_2 - (I + I_{1111} + I_{2121} - 2I_{11})\varphi_1^{(0)}$$

$$- (I_{1112} + I_{2221} - I_{12} - I_{21})\varphi_2^{(0)} + (I_{1111} + I_{2121} - I_{11})\psi_1^{(0)}$$

$$+ (I_{1112} + I_{2221} - I_{12})\psi_2^{(0)}\Big]\cos(n,\alpha_1) - \omega^2\Big[R_{12}U_1 + (R_{22} - R)U_2$$

$$- (I_{1112} + I_{2221} - I_{12} - I_{21})\varphi_1^{(0)} - (I + I_{2222}I_{1212} - 2I_{22})\varphi_2^{(0)} \quad (11)$$

$$+ (I_{1112} + I_{2221} - I_{21})\psi_1^{(0)} + (I_{2222} + I_{1212} - I_{22})\psi_2^{(0)}\Big]\cos(n,\alpha_2) = 0$$

$$\text{or} \quad W = 0$$

$$M_n' = 0 \quad \text{or} \quad \psi_n^{(0)} = 0$$

$$M_{ns}' = 0 \quad \text{or} \quad \psi_s^{(0)} = 0$$

$$M_n'' = 0 \quad \text{or} \quad \varphi_n^{(0)} = 0$$

The expressions for the coefficients R_{11}, etc., are also given in Appendix A. At each corner of the shell there is the additional requirement that

$$M''_{ns}(s+0) - M''_{ns}(s-0) = 0 \quad \text{or} \quad W = 0 \tag{12}$$

Furthermore, two independent sets of simultaneous linear algebraic equations which $\lambda_{rs}^{(i)}$ ($i > 0$ or $i < 0$) must satisfy, respectively, can also be obtained. The coefficients of the algebraic equations consist of ω^2 and area integrals defined with respect to the region of the middle surface of the shell. The integrated functions are expressed in terms of the amplitudes of the displacements and rotations and the strain components in Equation (5). The details are given in Appendix B. Coupling between the set of differential Equations (9) and two sets of algebraic equations arise through the coefficients in those equations. We have to solve them together using iteration methods. A procedure has been suggested in Reference [11]. In general the iteration process converges quickly.

EXAMPLES

The exact analytical solution of Equation (9) for a general laminated shell under arbitrary boundary conditions is a difficult task. Here only the free vibrations of simply supported cross-ply circular cylindrical shells and panels are to be considered. The following three problems are solved:

1. Simply supported complete cylindrical shells with $[0°/90°]_s$ lay-up. The radius and axial length of their middle surfaces are denoted by R and L, respectively. The lamina properties are assumed to be as follows:

$$E_L/E_T = 25, \; G_{LT}/E_T = 0.5, \; G_{TT}/E_T = 0.2, \; \nu_{LT} = \nu_{TT} = 0.25 \tag{13}$$

   Fundamental frequency parameters $\bar{\omega} = \omega L^2(\rho/E_T)^{1/2}/h$ of the shells with common $L/h = 10$ but $R/L = 5$, 10, 20, 50 and 100, respectively, are calculated, where ρ is the mass density
2. Simply supported symmetric cylindrical panels with $[0°/90°]_s$ or $[90°/0°]_s$ lay-up. The middle surfaces of the panels have the same square plan form ($L \times L$). Hence, for given values of L and ϕ (shallowness angle), the circumferential length of the panel middle surface is given as $L_s = L\phi/[2\sin(\phi/2)]$. The lamina properties are given as follows:

$$E_L/E_T = 40, \; G_{LT}/E_T = 0.6, \; G_{TT}/E_T = 0.5, \; \nu_{LT} = \nu_{TT} = 0.25 \tag{14}$$

   Fundamental frequency parameters $\omega* = \omega h(\rho/\pi^2 G_{LT})^{1/2}$ of the panels with $h/L = 0.1$, 0.2, 0.3 and $\phi = 30°$, 60°, 90° are calculated.
3. Simply supported antisymmetric cylindrical panels with $[0°/90°]$ or $[0°/90°]_4$

Table 1. Fundamental frequency parameters $\bar{\omega}$ of simply supported cylindrical shells with $[0°/90°]_s$ lay-up (n in superscripts).

R/L	Exact	Present	SDLT	FSDT	CLT
5	10.305^{7}	10.363	10.462	10.958	13.704
10	10.027^{11}	10.087	10.187	10.698	13.499^{9}
20	9.902^{15}	9.964	10.063^{14}	10.579^{14}	13.401^{12}
50	9.834^{12}	9.896^{13}	9.996^{8}	10.496^{8}	13.345^{5}
100	9.815^{1}	9.878	9.971^{2}	10.496^{2}	13.336^{2}

Exact: given in Reference [14].
SDLT, FSDT, CLT: tabulated in Reference [15].

lay-up. The geometrical form and the lamina properties of the panels are the same as given in Problem 2. Fundamental frequency parameters $\omega^* = \omega h(\rho/\pi^2 G_{LT})^{1/2}$ of the panels with $h/L = 0.2$, 0.3 and $\phi = 30°, 60°, 90°$ are calculated.

Let the axial and circumferential coordinates be denoted by x and θ, respectively, and let the axial ends of the cylindrical shells and panels be $x = 0$ and $x = L$. In all problems, the following solution form is assumed:

$$\begin{aligned} U_1 &= A\cos(\pi x/L)\sin(n\theta), \quad U_2 = B\sin(\pi x/L)\cos(n\theta) \\ W &= C\sin(\pi x/L)\sin(n\theta), \\ \psi_1^{(0)} &= D\cos(\pi x/L)\sin(n\theta), \quad \psi_2^{(0)} = E\sin(\pi x/L)\cos(n\theta) \end{aligned} \tag{15}$$

It can be verified that in this case $\lambda_{12}^{(i)} = \lambda_{21}^{(i)} = 0$ and the solution given in Equation (15) satisfies all the simply supported boundary conditions. Substituting Equation (15) into Equation (9) for cross-ply circular cylindrical shells, we obtain the solution for A, B, C, D, E and $\lambda_{11}^{(i)}$, $\lambda_{22}^{(i)}$ with iteration procedure. For Problem 1 the circumferential wave number n in Equation (15) is to be selected to associate with the fundamental frequency. For Problems 2 and 3 there are $n = 6$, 3 and 2 in accordance with $\phi = 30°$, $60°$ and $90°$, respectively.

Table 2. Corresponding values of $\lambda_{11}^{(\pm1)}$ and $\lambda_{22}^{(\pm1)}$ for cylindrical shells with $[0°/90°]_s$ lay-up.

R/L	n	$\lambda_{11}^{(1)}$	$\lambda_{11}^{(-1)}$	$\lambda_{22}^{(1)}$	$\lambda_{22}^{(-1)}$
5	7	0.2364	0.2307	0.1417	1.127
10	11	0.2350	0.2325	0.2825	1.224
20	15	0.2343	0.2335	0.3834	1.470
50	13	0.2340	0.2341	0.4876	1.829
100	1	0.2340	0.2342	0.7741	1.631

Table 3. Fundamental frequency parameters ω^* of cylindrical panels with symmetric cross-ply lay-ups

		0°/90°/90°/0°			90°/0°/0°/90°		
h/L	ϕ	Exact	Present	CLT	Exact	Present	CLT
0.1	30	0.062592	0.063082	0.076479	0.060935	0.061538	0.073960
	60	0.064600	0.064899	0.076832	0.059130	0.059735	0.068307
	90	0.067844	0.067940	0.078016	0.058942	0.059383	0.063728
0.2	30	0.174098	0.177001	0.300777	0.170682	0.173862	0.290140
	60	0.170868	0.172980	0.287680	0.158886	0.161847	0.250230
	90	0.168514	0.169729	0.271978	0.147231	0.149505	0.203953
0.3	30	0.293392	0.299525	0.673544	0.289056	0.295427	0.647908
	60	0.283798	0.288193	0.635520	0.268258	0.273476	0.546926
	90	0.274143	0.276674	0.588725	0.245519	0.249260	0.428307

Exact: given in Reference [16].

Table 4. Corresponding values of $\lambda_{11}^{(\pm1)}$ and $\lambda_{22}^{(\pm1)}$ for cylindrical panels with symmetric cross-ply lay-ups.

	h/L	ϕ	$\lambda_{11}^{(1)}$	$\lambda_{11}^{(-1)}$	$\lambda_{22}^{(1)}$	$\lambda_{22}^{(-1)}$
0°/90°/90°/0°	0.1	30	0.4818	0.4577	0.0946	0.1547
		60	0.4938	0.4469	0.0575	0.1959
		90	0.5047	0.4377	−0.0034	0.2637
	0.2	30	0.5160	0.4914	0.1055	0.1205
		60	0.5283	0.4807	0.0916	0.1336
		90	0.5392	0.4717	0.0611	0.1578
	0.3	30	0.5649	0.5371	0.1164	0.1154
		60	0.5786	0.5249	0.1077	0.1185
		90	0.5907	0.5149	0.0793	0.1270
90°/0°/0°/90°	0.1	30	0.0727	0.1704	0.4687	0.4698
		60	0.0185	0.2096	0.4711	0.4654
		90	−0.0362	0.2395	0.4826	0.4511
	0.2	30	0.0747	0.1483	0.4898	0.5154
		60	0.0356	0.1774	0.4795	0.5210
		90	−0.0033	0.1976	0.4779	0.5150
	0.3	30	0.0785	0.1520	0.5289	0.5715
		60	0.0404	0.1814	0.5097	0.5830
		90	0.0026	0.2009	0.5001	0.5809

Table 5. Fundamental frequency parameters ω^* of cylindrical panels with antisymmetric cross-ply lay-ups.

		0°/90°			[0°/90°]$_4$		
h/L	ϕ	Exact	Present	CLT	Exact	Present	CLT
0.2	30	0.134307	0.137924	0.164664	0.182559	0.184182	0.288075
	60	0.128319	0.131263	0.150741	0.172616	0.174399	0.262097
	90	0.125689	0.127278	0.140344	0.163481	0.164832	0.234160
0.3	30	0.243670	0.252566	0.352954	0.308913	0.312650	0.639104
	60	0.226519	0.233808	0.312875	0.289353	0.293315	0.570330
	90	0.213085	0.217317	0.277655	0.269429	0.272700	0.496384

Exact: given in Reference [16].

Table 6. Corresponding values of $\lambda_{11}^{(i)}$ and $\lambda_{22}^{(i)}$ for cylindrical panels with antisymmetric cross-ply lay-ups.

		h/L	0.2			0.3		
		ϕ / i	30	60	90	30	60	90
0°/90°	$\lambda_{11}^{(i)}$	−1	4.2768	5.9778	10.7030	3.8041	4.8777	7.0858
	$\lambda_{22}^{(i)}$	−1	0.3994	0.4369	0.5047	0.4343	0.4599	0.5162
[0°/90°]$_4$	$\lambda_{11}^{(i)}$	3	0.0222	0.0087	−0.0051	0.0239	0.0079	−0.0086
		2	0.4916	0.4854	0.4787	0.5274	0.5209	0.5136
		1	1.0443	1.0427	1.0413	1.0771	1.0761	1.0750
		−1	1.2940	1.2952	1.2980	1.2691	1.2711	1.2736
		−2	0.9934	0.9933	0.9939	0.9976	0.9969	0.9968
		−3	1.0256	1.0247	1.0252	1.0674	1.0662	1.0658
		−4	0.4561	0.4523	0.4500	0.4951	0.4907	0.4879
	$\lambda_{22}^{(i)}$	3	0.3632	0.3690	0.3757	0.4069	0.4151	0.4239
		2	0.8075	0.8184	0.8274	0.8695	0.8872	0.9022
		1	0.7754	0.7809	0.7855	0.8029	0.8121	0.8192
		−1	0.8199	0.8624	0.8983	0.8571	0.9051	0.9515
		−2	0.8481	0.8862	0.9213	0.9043	0.9529	0.9968
		−3	0.4023	0.4200	0.4375	0.4443	0.4647	0.4829
		−4	0.0292	0.0341	0.0434	0.0307	0.0331	0.0390

In Table 1, fundamental frequency parameters $\overline{\omega}$ for Problem 1 are presented for differential values of R/L and compared with corresponding exact frequencies given in Reference [14], as well as corresponding results obtained by SDLT, FSDT (for a 5/6 shear correction factor) and CLT which were tabulated in Reference [15]. The superscript accompanying some of the numerical results in Table 1 indicates the circumferential wave number, n, for which the fundamental frequency was detected. In any case that corresponding three-dimensional and shell theory predictions were obtained for the same value of n, such a superscript is omitted from the corresponding shell theory results. However, it has to be noticed that, for such short shells with $L/h = 10$, neighbouring values of n result in slight or even negligible changes of the lowest frequency predictions independently of the shell theory employed. For instance, according to the present theory, for the shell with $R/L = 50$, $\overline{\omega} = 9.89629$ for $n = 13$ and $\overline{\omega} = 9.89632$ for $n = 12$. The difference between these lowest two values of $\overline{\omega}$ is nearly imperceptible. It is shown in Table 1 that the present theory gives more accurate results than SDLT, FSDT and CLT. The corresponding values of $\lambda_{11}^{(\pm 1)}$ and $\lambda_{22}^{(\pm 1)}$ for the present model are given in Table 2. The middle two 90° layers of the shells are regarded as one layer, layer zero, in the present theory.

The fundamental frequency parameters ω^* of the cylindrical panels in

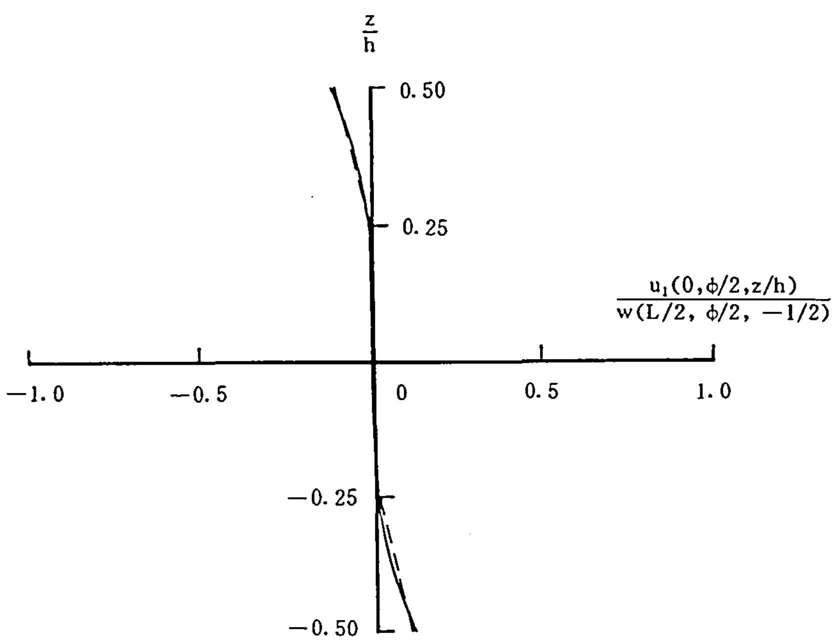

Figure 1. The fundamental mode shapes of axial displacement component for a $[0°/90°]_s$ cylindrical panel. (——) Exact one given in Reference [16]; (- - -) present.

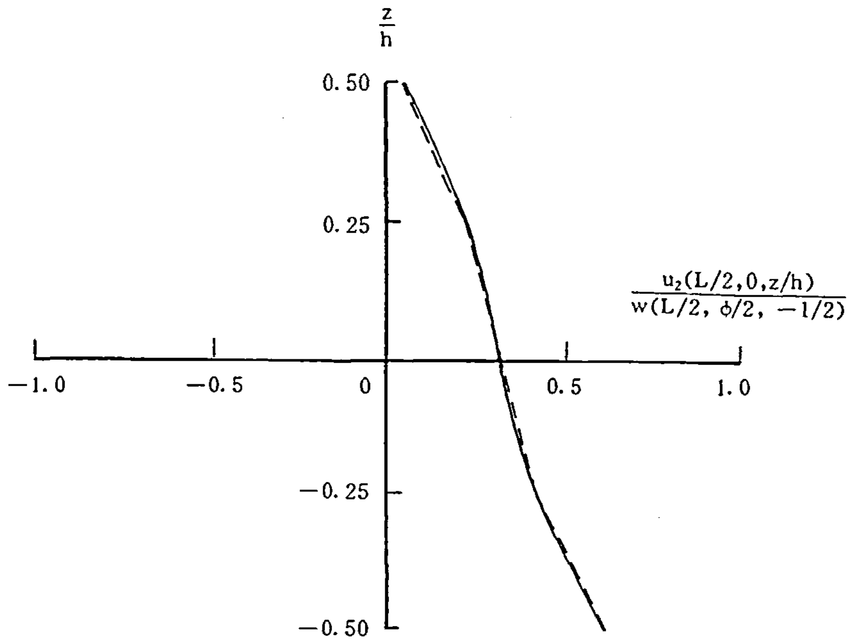

Figure 2. *The fundamental mode shapes of circumferential displacement component for a $[0°/90°]_s$ cylindrical panel. (——) Exact one given in Reference [16]; (- - -) present.*

Problem 2 obtained using the present theory are given in Table 3. They are compared with those given by CLT, as well as with those obtained by three-dimensional theory which were given in Reference [16]. For all theories the fundamental frequency parameters are always higher for $[0°/90°]_s$ laminates than for $[90°/0°]_s$ ones. It can be seen that the present theory gives much more accurate results than CLT. The corresponding values of $\lambda_{11}^{(i)}$ and $\lambda_{22}^{(i)}$ for the two kinds of laminates are given in Table 4. Table 5 presents fundamental frequency parameters ω^* of the antisymmetrical cylindrical panels in Problem 3. The comparisons made in Table 5 show that the present theory also gives much more accurate results than CLT. The corresponding values of $\lambda_{11}^{(i)}$ and $\lambda_{22}^{(i)}$ for those panels are given in Table 6.

Figures 1, 2, 3 and 4 give the fundamental mode shapes of axial and circumferential displacement components of the symmetric four-layered $[0°/90°]_s$ and the antisymmetric two-layered [0°/90°] laminated cylindrical panels both with the same values of $h/L = 0.3$ and $\phi = 60°$, respectively. The modal axial displacement u_1 and circumferential one u_2 were nomalized by dividing them by the radial displacement w at the surface $z = -h/2$. The results show that the present study is in good agreement with the exact elasticity solution given in Reference [16]. It

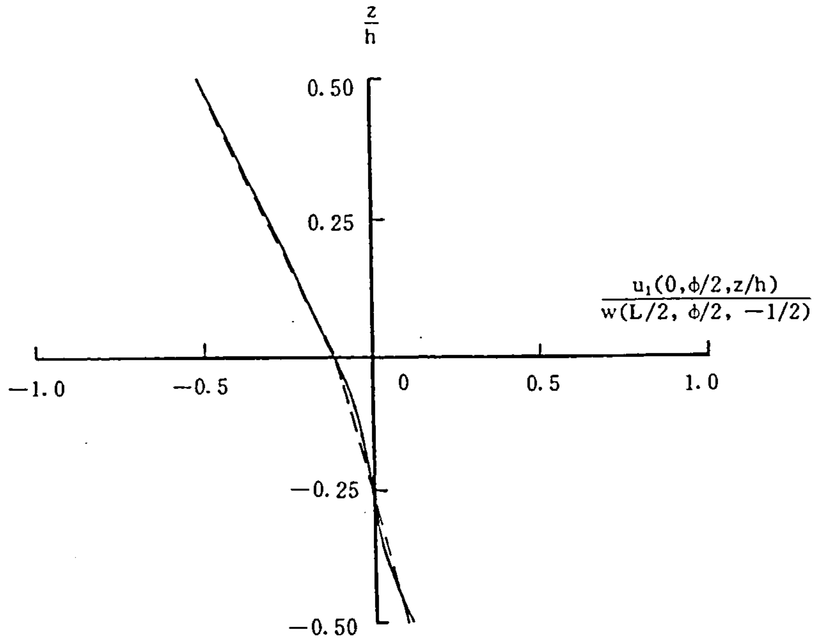

Figure 3. The fundamental mode shapes of axial displacement component for a [0°/90°] cylindrical panel. (——) Exact one given in Reference [16]; (- - -) present.

means that the present theory can give good prediction of the modal tangential stresses. The modal transverse shear stresses can be accurately obtained by integrating the motion equations of three-dimensional elasticity if need be.

CONCLUSIONS

A refined shear deformation laminated shell theory published in Reference [12] has been used here to analyse the free vibration of laminated shells. The theory contains the same number of dependent variables as first-order shear deformation theory, but the set of governing differential equations is of twelfth order. No shear correction factors are required. The theory can be used to analyse the free vibration of arbitrary laminated shells without limitation on the materials and the number of layers and the direction of the ply angle. The numerical results for simply supported symmetric cross-ply cylindrical shells, symmetric and antisymmetric cross-ply cylindrical panels have been compared with those given by three-dimensional elasticity theory. From the results it can be seen that the present theory gives accurate predictions of both the natural frequencies and the modal shapes even for fairly thick laminates with a thickness-to-span ratio equal

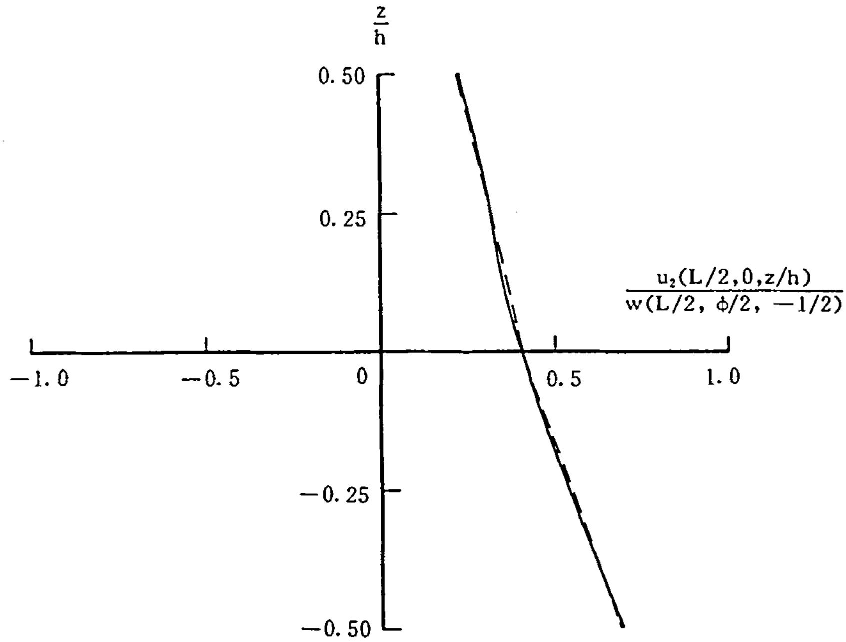

***Figure 4.** The fundamental mode shapes of circumferential displacement component for a [0°/90°] cylindrical panel. (——) Exact one given in Reference [16]; (- - -) present.*

to 0.3. The present theory can be regarded as a direct generalization and improvement of the first-order shear deformation theory. Although the analytical solution of the equations can be obtained only in a few cases, one can use approximate methods, e.g., finite element methods, to obtain numerical solutions in other cases.

In the present theory the undetermined constants $\lambda_{rs}^{(i)}$ may be regarded as average values for the whole laminated shell. The values of $\lambda_{rs}^{(i)}$ are different for each normal mode. As every normal mode of a laminated shell can be approximately obtained with the present theory, then the orthogonality relations between any two different normal modes also exist approximately. If the responses of the laminated shell to dynamic loads have to be calculated, they can be obtained using the modal expansion technique. Using the present theory, one can predict accurately not only the normal displacements but also the tangential stresses, which are also important for dynamic analysis of laminated shells.

ACKNOWLEDGEMENT

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APPENDIX A: THE EXPRESSIONS OF THE COEFFICIENTS AND DIFFERENTIAL OPERATORS M_{11}, ETC.

The coefficients and differential operators M_{11}, etc. are:

$$M_{11} = \rho_A A_1 A_2, \quad M_{13} = (R_{11} - R) A_2 \frac{\partial}{\partial \alpha_1} + R_{12} A_1 \frac{\partial}{\partial \alpha_2}$$

$$M_{14} = R_{11} A_1 A_2, \quad M_{15} = R_{12} A_1 A_2, \quad M_{22} = \rho_A A_1 A_2$$

$$M_{23} = R_{21} A_2 \frac{\partial}{\partial \alpha_1} + (R_{22} - R) A_1 \frac{\partial}{\partial \alpha_2}, \quad M_{24} = R_{21} A_1 A_2, \quad M_{25} = R_{22} A_1 A_2$$

$$M_{31} = (R_{11} - R)\left(A_2 \frac{\partial}{\partial \alpha_1} + \frac{\partial A_2}{\partial \alpha_1}\right) + R_{12}\left(A_1 \frac{\partial}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2}\right)$$

$$M_{32} = R_{21}\left(A_2 \frac{\partial}{\partial \alpha_1} + \frac{\partial A_2}{\partial \alpha_1}\right) + (R_{22} - R)\left(A_1 \frac{\partial}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2}\right) \tag{A1}$$

$$M_{33} = (I + I_{1111} + I_{2121} - 2I_{11})\left[\frac{A_2}{A_1}\frac{\partial^2}{\partial \alpha_1^2} + \frac{\partial}{\partial \alpha_1}\left(\frac{A_2}{A_1}\right) \cdot \frac{\partial}{\partial \alpha_1}\right]$$

$$+ 2(I_{1112} + I_{2221} - I_{12} - I_{21}) \cdot \frac{\partial^2}{\partial \alpha_1 \partial \alpha_2} + (I + I_{2222} + I_{1212} - 2I_{22})$$

$$\cdot \left[\frac{A_1}{A_2}\frac{\partial^2}{\partial \alpha_2^2} + \frac{\partial}{\partial \alpha_2}\left(\frac{A_1}{A_2}\right) \cdot \frac{\partial}{\partial \alpha_2}\right] - \rho_A A_1 A_2$$

$$M_{34} = (I_{1111} + I_{2121} - I_{11})\left(A_2 \frac{\partial}{\partial \alpha_1} + \frac{\partial A_2}{\partial \alpha_1}\right)$$

$$+ (I_{1112} + I_{2221} - I_{21})\left(A_1 \frac{\partial}{\partial \alpha_2} + \frac{\partial A_1}{\partial \alpha_2}\right)$$

$$M_{35} = (I_{1112} + I_{2221} - I_{12})\left(A_2\frac{\partial}{\partial\alpha_1} + \frac{\partial A_2}{\partial\alpha_1}\right)$$

$$+ (I_{2222} + I_{1212} - I_{22})\left(A_1\frac{\partial}{\partial\alpha_2} + \frac{\partial A_1}{\partial\alpha_2}\right)$$

$$M_{41} = R_{11}A_1A_2, \quad M_{42} = R_{21}A_1A_2$$

$$M_{43} = (I_{1111} + I_{2121} - I_{11})A_2\frac{\partial}{\partial\alpha_1} + (I_{1112} + I_{2221} - I_{21})A_1\frac{\partial}{\partial\alpha_2}$$

(A1 con't)

$$M_{44} = (I_{1111} + I_{2121})A_1A_2, \quad M_{45} = (I_{1112} + I_{2221})A_1A_2$$

$$M_{51} = R_{12}A_1A_2 \quad M_{52} = R_{22}A_1A_2$$

$$M_{53} = (I_{1112} + I_{2221} - I_{12})A_2\frac{\partial}{\partial\alpha_1} + (I_{2222} + I_{1212} - I_{22})A_1\frac{\partial}{\partial\alpha_2}$$

$$M_{54} = (I_{1112} + I_{2221})A_1A_2, \quad M_{55} = (I_{2222} + I_{1212})A_1A_2$$

The expressions of ρ_A, R, I, R_{11}, I_{11}, I_{1111}, etc. are the same as in Reference [11].

APPENDIX B: SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS WHICH $\lambda_{rs}^{(i)}$ MUST SATISFY

The forms of two sets of simultaneous linear algebraic equations that the unknown constants $\lambda_{rs}^{(i)}$ ($i > 0$ or $i < 0$) must satisfy respectively are the same as in Reference [11]. Define

$$r_1 = \kappa_1' - \kappa_1'', \quad r_2 = \kappa_2' - \kappa_2'', \quad r_{12} = \kappa_{12}' - \kappa_{12}'', \quad r_{21} = \kappa_{21}' - \kappa_{21}'' \qquad \text{(B1)}$$

$$d_{1,11} = r_1 - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\gamma_{23}^{(0)}, \quad d_{2,11} = \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\gamma_{13}^{(0)}$$

(B2)

$$d_{6,11} = r_{21} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\gamma_{13}^{(0)} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}\gamma_{23}^{(0)}, \quad d_{1,22} = \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}\gamma_{23}^{(0)}$$

$$d_{2,22} = \gamma_2 - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\gamma_{13}^{(0)}, \; d_{6,22} = \gamma_{12} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\gamma_{13}^{(0)} - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\gamma_{23}^{(0)}$$

$$d_{1,12} = r_{12} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\gamma_{13}^{(0)}, \quad d_{2,12} = \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\gamma_{23}^{(0)}$$

$$d_{6,12} = r_2 - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\gamma_{13}^{(0)} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\gamma_{23}^{(0)}, \quad d_{1,21} = \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\gamma_{13}^{(0)}$$

$$d_{2,21} = r_{21} + \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\gamma_{23}^{(0)} \tag{B2 con't}$$

$$d_{6,21} = r_1 - \frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}\gamma_{13}^{(0)} - \frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}\gamma_{23}^{(0)}$$

Then we have the similar expressions of the coefficients in the equations as in Reference [11]. However, we perform area integrations with respect to the region Ω of the shell. The subscripts, x, y and z of the quantities of Equations (B9), (B11), (B13) and (B14) in Reference [11] have to be changed into the subscripts 1, 2 and 3 here. Besides the variables U, V, $\partial W/\partial x$ and $\partial W/\partial y$ in Equation (B14) in Reference [11] have to be changed into U_1, U_2, $-\varphi_1^{(0)}$ and $-\varphi_2^{(0)}$ respectively here.

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