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A contact crack problem in an infinite plate

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Abstract

The problem of an infinite plate with crack of length $2a$ loaded by the remote tensile stress P and a pair of concentrated forces Q is discussed. The value of the force Q for the initial contact of crack face is investigated and the contact length elevated, while the Q force increases. The problem is solved assuming that the stress intensity factor vanishes at the end point of the contact portion. By the Fredholm integral equation for the multiple cracks, the reduction of stress intensity factor due to Q is found.

Keywords: Plane elasticity; Contact crack problem

1. Introduction

In many situations a crack may experience contact phenomena as a consequence of particular loading conditions, for example, a cracked strip in bending. In this case some portion of crack face is closed and the related crack problem is called the contact crack problem in this paper. Also within the range of linear elasticity, the contact problem is a typical nonlinear problem as the boundary conditions are unknowns and depend on loading. Recently, the problem has attracted much attention by several investigators [1–3].

In plane elasticity, the well known rules for the contact crack problem are as follows: (a) the crack opening displacement function is nonnegative in any point of the open portion, (b) the contact stress is nontensile in any point of the closed portion [2]. In fact, by applying these rules, the solution of the contact crack problem has inherent difficulty. Alternative rules were

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suggested in Ref. [1]: (a) if some portion of crack is closed the closed portion becomes a continuous medium; (b) after the crack is partly closed, the stress intensity factor vanishes at the end point of the closed portion. The mentioned hypothesis coincides with the physical situation of the contact problem. These rules will be used in the present study. Clearly, the suggested procedure requires the symmetry of both geometry and loading and it cannot be applied for cracks under mixed mode case or if the contact stress between crack faces has to be evaluated.

In the following analysis, an infinite plate with crack of length $2a$ loaded by the remote tensile stress P and a pair of compressive concentrated forces Q symmetrically applied on the mid-plane is discussed (Fig. 1). The value of the Q force for the initial contact of crack face is investigated. After the initial contact, the contact length $2b$ will be increased, while the Q force increases (Fig. 2). Assuming that the stress intensity factor vanishes at the end point of the contact portion, the relation between Q force and the contact length $2b$ is obtained by the Fredholm integral equation approach for the multiple crack problem, and the reduction of stress intensity factor at the crack tip A is found. Numerical examples with calculated results are given.

2. Analysis

Depending on the loading tension P and the applied Q force (Fig. 1a), two cases will be encountered: (a) for a relative small value of $Q(Q < Q_c)$, no contact of crack face occurs; (b) for larger values of $Q(Q > Q_c)$, the contact of crack face happened, where Q_c is a critical value. In the second case, the boundary value condition is not given before hand.

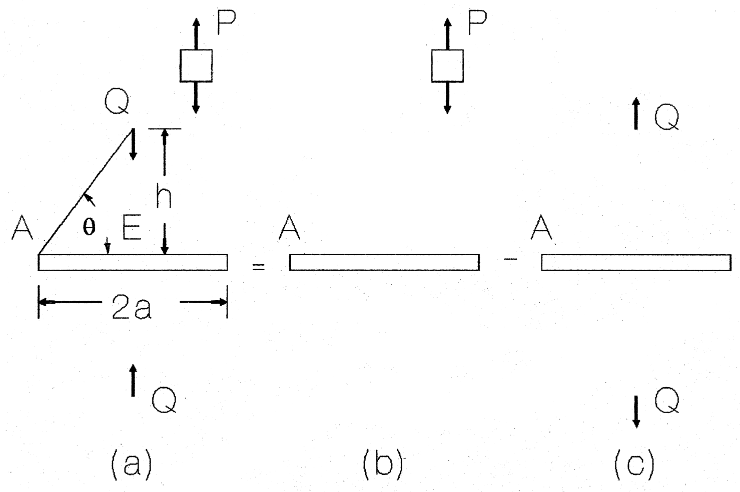

Fig. 1. The relative small concentrated force Q case ($Q < Q_c$): (a) the original problem; (b) the problem for loading P; (c) the problem of loading Q.

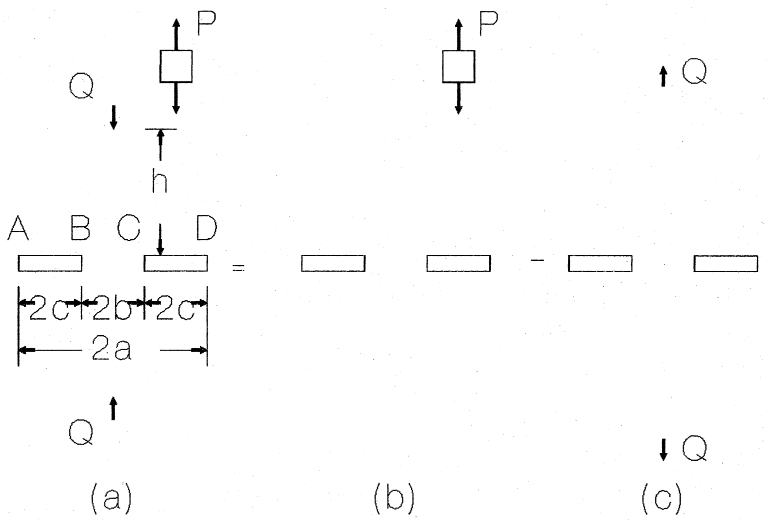

Fig. 2. The relative large concentrated force Q case ($Q > Q_c$): (a) the original problem; (b) the problem for loading P; (c) the problem of loading Q.

2.1. The relative small concentrated force Q case ($Q < Q_c$) (Fig. 1a)

For the relative small concentrated force, no contact actually occurs along the crack faces. In this case, the original problem shown in Fig. 1a can be considered as a superposition of two problems as shown in Fig. 1b,c.

For the problem shown in Fig. 1b, the following equations hold:

$$K_{A(P)} = P\sqrt{\pi a} \tag{1}$$

$$2Gv_{E(P)} = \frac{(\kappa + 1)Pa}{2}, \tag{2}$$

where a is the half length of crack, G the shear modulus of elasticity, $\kappa = (3-\nu)/(1 + \nu)$ for the plane stress case, ν the Poisson's ratio, $K_{A(P)}$ the stress intensity factor at the tip A caused by the tension P, $v_{E(P)}$ the vertical displacement at the point E caused by the tension P (Fig. 1).

For the problem shown in Fig. 1c, the equivalent equations are:

$$K_{A(Q)} = \frac{Q}{(\kappa + 1)\sqrt{\pi a}}(\kappa + 1 + 2\sin^2\ \theta)\cos\theta \tag{3}$$

$$2Gv_{E(Q)} = \frac{Q}{2\pi}\left\{\frac{4H}{H^2 + 1} + (\kappa + 1)(\mathrm{Log}(H + 1) - \mathrm{Log}(H - 1))\right\}, \tag{4}$$

where $K_{A(Q)}$ denotes the stress intensity factor at the crack tip A, $v_{E(Q)}$ the vertical displacement at the point E caused by the force Q, and

$$H = \frac{h}{a} + \sqrt{1 + (h/a)^2}, \quad \theta = \text{Arc}\tan(h/a), \tag{5}$$

where h is shown in Fig. 1. The solution approach for the problem shown by Fig. 1c will be referred to the Appendix A.

It is assumed that the initial contact process is governed by the crack opening displacement at the point E caused by the tension P and the concentrated force Q. In addition, it can be proved numerically that the first contact point is the middle point E of the crack face when Q value increases. Therefore, the initial contact condition takes the form

$$v_{\mathrm{E}} = v_{\mathrm{E(P)}} - v_{\mathrm{E(Q)}} = 0. \tag{6}$$

Substituting (2) and (4) into (6) the critical value for Q which initiates the contact is obtained by

$$Q_{\mathrm{c}} = C(h/a)(2Pa), \tag{7}$$

where the function $C(h/a)$ is plotted in Fig. 3.

The above-mentioned analysis demonstrates that no contact is encountered in the range of $0 < Q < Q_{\mathrm{c}}$. From Fig. 3 it can be observed that the $C(h/a)$ is an increasing function of the ratio h/a. In this case, the stress intensity factor can be expressed by

$$K_{\mathrm{A}} = K_{\mathrm{A(P)}} - K_{\mathrm{A(Q)}} \quad (0 < Q < Q_{\mathrm{c}}). \tag{8}$$

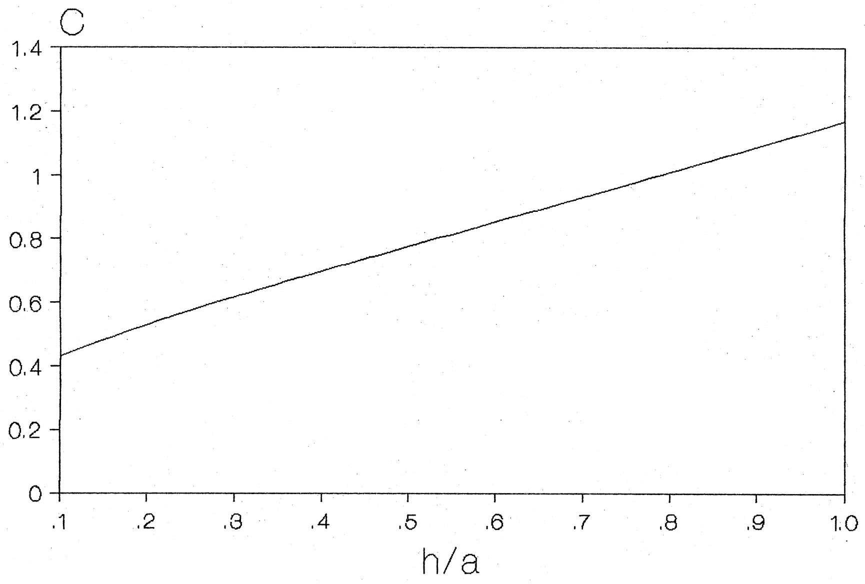

Fig. 3. The critical concentrated force $Q_{\mathrm{c}} = C(h/a)\ (2Pa)$ for the initial contact [see Fig. 1 and Eq. (7)].

2.2. The relative large concentrated force Q case ($Q > Q_c$) (Fig. 2a)

In this case, a central portion of crack, the zone BC in Fig. 2, is closed. By assuming that: (a) the closed portion BC becomes a continuous medium; (b) after the crack is partly closed, for the crack AB the stress intensity factor at the tip B vanishes, and the following equations hold [1]

$$K_B = K_C = 0 \qquad \text{(at the crack tips B and C).} \tag{9}$$

In order to avoid iterative technique in the solution, it is convenient to determine the value of Q producing a given closed zone equal to $2b$. In the solution, the original problem shown in Fig. 2a is considered as a superposition of two problems shown by Fig. 2b,c. The solution for two collinear cracks was obtained by the Fredholm integral equation (Appendix B).

For the remote stress P case (Fig. 2b), the stress intensity factors at the crack tips A and B can be expressed as

$$K_{A(P)} = g_1\left(\frac{b}{a}\right)P\sqrt{\pi a} \tag{10}$$

$$K_{B(P)} = g_2\left(\frac{b}{a}\right)P\sqrt{\pi a}. \tag{11}$$

Similarly, for the concentrated force Q case (Fig. 2c), the stress intensity factors at the crack tips A and B can be expressed by

$$K_{A(Q)} = h_1\left(\frac{b}{a},\frac{h}{a}\right)\frac{Q}{\sqrt{\pi a}} \tag{12}$$

$$K_{B(Q)} = h_2\left(\frac{b}{a},\frac{h}{a}\right)\frac{Q}{\sqrt{\pi a}}, \tag{13}$$

where the functions h_1, h_2, g_1 and g_2 were obtained by the integral equations reported in Appendix B.

In this case, the condition (9) can be rewritten as

$$K_B = K_{B(P)} - K_{B(Q)} = 0. \tag{14}$$

Substituting (11) and (13) into (14) the value of Q required to produce a closure zone equal to $2b$ is given by

$$Q_d = D(b/a, h/a)(2Pa), \tag{15}$$

where the function $D(b/a, h/a)$ is plotted in Fig. 4.

In addition, the stress intensity factor at the crack tip A can be expressed by

$$K_A = K_{A(P)} - K_{A(Q)}. \tag{16}$$

Substituting (10) and (12) into (16) and using the obtained Q_d value, K_A can be rewritten as

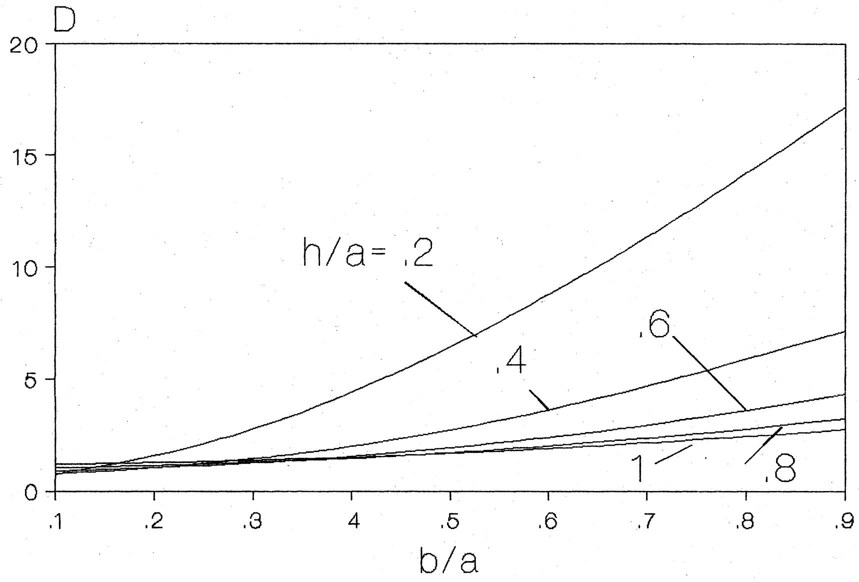

Fig. 4. The calculated $D(b/a, h/a)$ values [see Fig. 2 and Eq. (15)].

$$K_A = E(b/a, h/a)P\sqrt{\pi a}. \tag{17}$$

Reports and plot of function $E(b/a, h/a)$ in Fig. 5 show that, the stress intensity factor at the tip A decreases as the closed portion of crack increases.

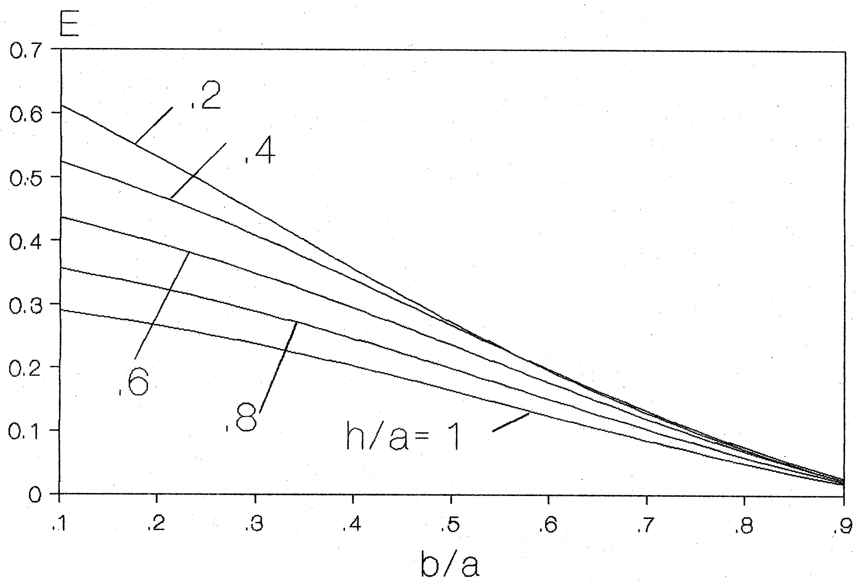

Fig. 5. The calculated normalized stress intensity factors $E(b/a, h/a)$ at the crack tip A [see Fig. 2 and Eq. (17)].

3. Remarks

It was reported in some references, the obtained stress intensity factor is negative, for example in [7, Fig. 4]. This means that the solution violate the nature of deformation geometry and it is of no sense. In the present study, the contact effect has been considered and the obtained result is reasonable from view point of the deformation geometry. Meantime, from the calculated results we see that the applied Q force can considerably reduce the stress intensity factor at the crack tip. For example, if $h/a = 1.0$ and $Q = 3.796$ Pa, the closed portion is $b = 0.6a$ and $K_A = 0.124\ P\sqrt{\pi a}$.

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Appendix A

A.1. A crack in an infinite plate under the action of the concentrated force (Fig. A1)

The geometry and the loading condition of the problem are shown in Fig. A1. To solve the problem, the complex variable function method for plane elasticity is used [4]. According to this method, the stresses (σ_x, σ_y, σ_{xy}), the resultant force functions (X, Y) and the displacements (u, v) can be defined by two complex potentials $\phi_1(z)$, $\psi_1(z)$.

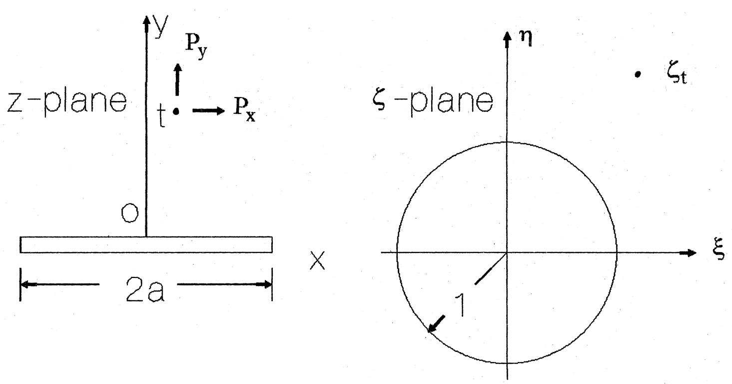

Fig. A1. A crack with the applied concentrated forces P_x and P_y.

In the solution, we use the following mapping function [4, 5]

$$z = \omega(\zeta) = \frac{a}{2}\left(\zeta + \frac{1}{\zeta}\right), \tag{A1}$$

which maps the region exterior to the unit circle (in ζ-plane) into the region exterior to the crack configuration (in z-plane) (Fig. A1). Also, we denote

$$t = \omega(\zeta_t) = \frac{a}{2}\left(\zeta_t + \frac{1}{\zeta_t}\right), \tag{A2}$$

where $z = t$ is the point of applied forces P_x and P_y. In the following analysis we express

$$\phi(\zeta) = \phi_1(z)|_{z=\omega(\zeta)}, \quad \psi(\zeta) = \psi_1(z)|_{z=\omega(\zeta)}. \tag{A3}$$

Using the continuation approach of complex variable function [5], we can obtain the following solution

$$\phi(\zeta) = F\,\mathrm{Log}(\zeta - \zeta_t) + \kappa F\,\mathrm{Log}\left(\frac{1}{\zeta} - \bar{\zeta}_t\right) - \frac{\bar{F}(\omega(1/\bar{\zeta}_t) - \omega(\zeta_t))}{\overline{\omega'(\zeta_t)}((1/\zeta) - \bar{\zeta}_t)} \tag{A4}$$

$$\psi(\zeta) = -w(\zeta) - \bar{\omega}\left(\frac{1}{\zeta}\right)\frac{\phi'(\zeta)}{\omega'(\zeta)} \tag{A5}$$

$$w(\zeta) = \kappa\bar{F}\,\mathrm{Log}(\zeta - \zeta_t) - \frac{F(\bar{\omega}(1/\zeta_t) - \overline{\omega(\zeta_t)})}{\omega'(\zeta)(\zeta - \zeta_t)} + \bar{F}\,\mathrm{Log}\left(\frac{1}{\zeta} - \bar{\zeta}_t\right), \tag{A6}$$

where

$$F = -\frac{P_x + iP_y}{2\pi(\kappa + 1)} \tag{A7}$$

and $\kappa = (3-\nu)/(1+\nu)$ for the plane stress problem, $\kappa = 3 - 4\nu$ for the plane strain problem, ν is the Poisson's ratio.

Meantime, the stress intensity factor at the right crack tip can be obtained by [5]

$$K = K_1 - iK_2 = 2\sqrt{\frac{\pi}{a}}\,\phi'(1). \tag{A8}$$

The above-mentioned result will give the required solution in the Section 2.1.

Appendix B

B.1. Fredholm integral equation for two cracks in series (Fig. B1)

The Fredholm integral equation for two cracks in series is introduced as follows (Fig. B1). In the problem, the boundary tractions along two crack faces are assumed as follows

$$\begin{aligned}&\sigma_{y(1)} = -pf_1(s_1), \quad |s_1| < c \qquad \text{(for the first crack in Fig. B1)}\\&\sigma_{y(2)} = -pf_2(s_2), \quad |s_2| < c \qquad \text{(for the second crack in Fig. B1)}\end{aligned} \tag{B1}$$

where p denotes a constant traction, $f_1(s_1)$ and $f_2(s_2)$ are two given functions.

The studied problem can be considered as a superposition of two single crack problems with the following undetermined boundary tractions

$$\begin{aligned}&\sigma_{y(1)} = -pF_1(s_1), \quad |s_1| < c\\&\sigma_{y(2)} = -pF_2(s_2), \quad |s_2| < c.\end{aligned} \tag{B2}$$

By using the the Fredholm integral equation approach [6], the following integral equations are obtainable

$$\begin{aligned}&\int_{-c}^{c} K_1(s_1, s_2)F_1(s_1)\,\mathrm{d}s_1 - F_2(s_2) = -f_2(s_2), \quad |s_2| < c\\&\int_{-c}^{c} K_2(s_2, s_1)F_2(s_2)\,\mathrm{d}s_2 - F_1(s_1) = -f_1(s_1), \quad |s_1| < c,\end{aligned} \tag{B3}$$

where

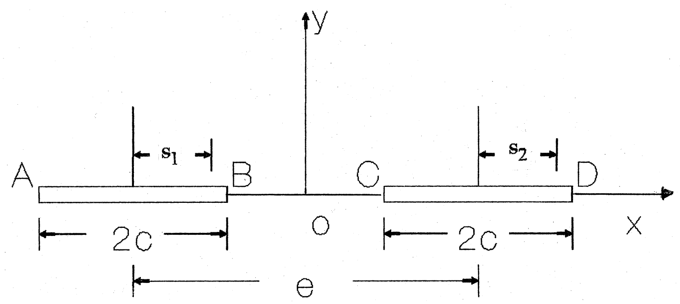

Fig. B1. Two cracks in series.

$$K_1(s_1, s_2) = \frac{1}{\pi}\left[\frac{c^2 - s_1^2}{(e + s_2)^2 - c^2}\right]^{1/2} \frac{1}{e + s_2 - s_1}$$

$$K_2(s_2, s_1) = \frac{1}{\pi}\left[\frac{c^2 - s_2^2}{(e - s_1)^2 - c^2}\right]^{1/2} \frac{1}{e + s_2 - s_1}. \tag{B4}$$

After the integral equation (B3) is solved, the stress intensity factors at four crack tips A, B, C and D can be evaluated from [6]

$$K_{\mathrm{A}} = \frac{\mathrm{P}}{\sqrt{\pi c}} \int_{-c}^{c} F_1(s)[(c - s)/(c + s)]^{1/2}\,\mathrm{d}s$$

$$K_{\mathrm{B}} = \frac{\mathrm{P}}{\sqrt{\pi c}} \int_{-c}^{c} F_1(s)[(c + s)/(c - s)]^{1/2}\,\mathrm{d}s$$

$$K_{\mathrm{C}} = \frac{\mathrm{P}}{\sqrt{\pi c}} \int_{-c}^{c} F_2(s)[c - s)/(c + s)]^{1/2}\,\mathrm{d}s$$

$$K_{\mathrm{D}} = \frac{\mathrm{P}}{\sqrt{\pi c}} \int_{-c}^{c} F_2(s)[(c + s)/(c - s)]^{1/2}\,\mathrm{d}s. \tag{B5}$$

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