Dynamical symmetry of screened Coulomb potential and isotropic harmonic oscillator

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It is shown that for the screened Coulomb potential and isotropic harmonic oscillator, there exists an infinite number of closed orbits for suitable angular momentum values. At the aphelion (perihelion) points of classical orbits, an extended Runge-Lenz vector for the screened Coulomb potential and an extended quadrupole tensor for the screened isotropic harmonic oscillator are still conserved. For the screened two-dimensional (2D) Coulomb potential and isotropic harmonic oscillator, the dynamical symmetries SO_3 and SU(2) are still preserved at the aphelion (perihelion) points of classical orbits, respectively. For the screened 3D Coulomb potential, the dynamical symmetry SO_4 is also preserved at the aphelion (perihelion) points of classical orbits. But for the screened 3D isotropic harmonic oscillator, the dynamical symmetry SU(2) is only preserved at the aphelion (perihelion) points of classical orbits in the eigencoordinate system. For the screened Coulomb potential and isotropic harmonic oscillator, only the energy (but not angular momentum) raising and lowering operators can be constructed from a factorization of the radial Schrödinger equation.

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Bertrand's famous theorem in classical mechanics states that the only central forces that result in closed orbits for all bound particles are the inverse square law and Hooke's law [1,2]. In classical mechanics, the maximum number of functional independent conserved quantities of a closed system with N degrees of freedom is $2N-1$ [3]. For a system with independent conserved quantities, no fewer than N can be called integrable [4]. An integrable classical system with $N+\Lambda$ independent conserved quantities ($0\leqslant\Lambda\leqslant N-1$) is called Λ-fold degenerate, and there exist Λ linear relations with integer coefficients between the N frequencies ν_i ($i=1,2,\ldots,N$) of the system [5]. A classical system for $\Lambda=N-1$ is called a completely degenerate system, and there remains only one independent frequency, which implies the existence of closed orbits. The orbit of a particle in the attractive Coulomb potential ($V(r)=-k/r$) is always closed for any continuous energy ($E<0$) and angular momentum $\mathbf{L}$, i.e., an ellipse, of which the length of the semimajor axis is ($m=k=1$) $a=1/(2|E|)$, and the eccentricity is $e=\sqrt{1-2|E|L^2}$. The period of motion is $T=1/\nu=\pi|E|^{-3/2}/\sqrt{2}=2\pi a^{3/2}$ (Kepler's law). The closeness of the orbits is guaranteed by the existence of an additional conserved quantity—the Runge-Lenz vector $\mathbf{R}=\mathbf{p}\times\mathbf{L}-\mathbf{r}/r$ [6]. In fact, the direction of $\mathbf{R}$ is just that of the major axis of the elliptic orbit, and the magnitude of $\mathbf{R}$ is the eccentricity ($|\mathbf{R}|=e$). It is seen that $\mathbf{R}\cdot\mathbf{L}=0$ and $\mathbf{R}^2=2H\mathbf{L}^2+1$, so the number of independent conserved quantities is 5, and the hydrogen atom is a completely degenerate system. The existence of the Runge-Lenz vector implies that the Coulomb potential has a higher dynamical symmetry SO_4 than its geometric symmetry SO_3 [7]. A similar situation exists for an isotropic harmonic oscillator.

In quantum mechanics, for bound states, both the angular momentum and energy are discrete. It is interesting to note that the Coulomb potential and isotropic harmonic oscillator are the only central potentials for which a radial Schrödinger equation can be factorized to yield *both* energy and angular momentum raising and lowering operators connecting neighboring simultaneous eigenstates of energy and angular momentum [8–11]. It was shown that there exists an intimate relation between the raising and lowering operators in quantum mechanics, on the one hand, and the conserved quantities responsible for the closeness of classical orbits on the other hand, and that both are physically connected with the dynamical symmetry of the system considered [12,13].

A careful examination of the arguments to derive Bertrand's theorem shows that the form of the central potential is assumed to be a power-law function of r [2]. We believe that Bertrand's theorem does hold for a power-law central potential. However, if the restriction of a power-law form of the central potential is relaxed, Bertrand's theorem may be extended. It was shown that there exists an infinite number of closed orbits (rather than elliptic orbits) for a particle with suitable discrete angular momenta in the screened Coulomb potential and isotropic harmonic oscillator [13,14]. In this case, it was found that *only* the energy (but not angular momentum) raising and lowering operators can be constructed from a factorization of the radial Schrödinger equation. General consideration shows that when the Coulomb potential or isotropic harmonic oscillator is screened, the dynamical symmetry [SO($N+1$) for N-dimensional (ND) hydrogen atom bound states or SU(N) for an ND isotropic harmonic oscillator] is broken, as a result the closeness of classical orbits is lost in general. The revival of closeness of some classical orbits may be an indication of the recurrence of the dynamical symmetry. In this paper, the dynamical symmetry of the screened Coulomb potential and isotropic harmonic oscillator will be investigated.

In quantum mechanics, the Schrödinger equation for the Coulomb potential and isotropic harmonic oscillator in arbitrary dimensions can be solved exactly. In classical mechanics, the orbits of a particle in a central potential, due to the conservation of angular momentum $\mathbf{L}$, *always lie in a 2D plane* perpendicular to $\mathbf{L}$. For clarity and simplicity, to ex-

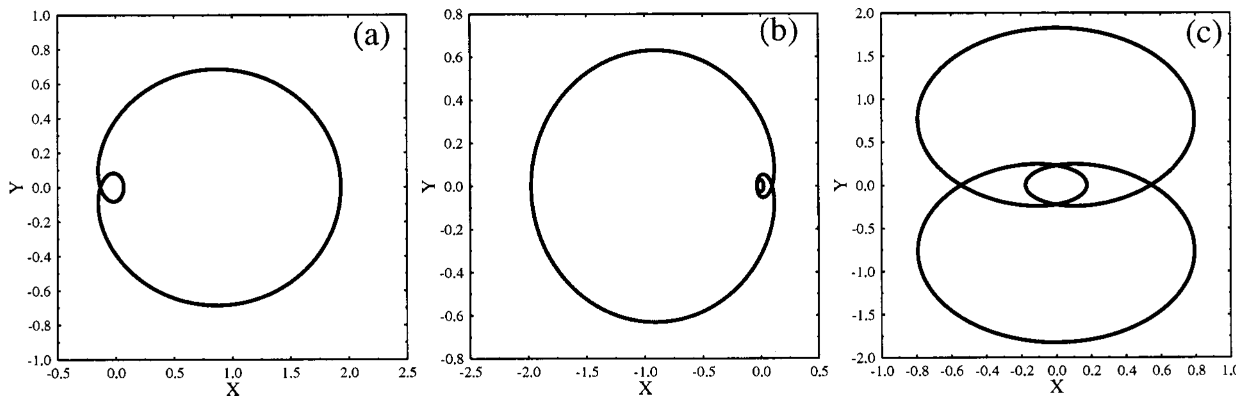

FIG. 1. Closed orbits of a particle in the screened 2D Coulomb potential of Eq. (1), with $\lambda=0.2$ and $E=-0.5$. (a) $\kappa=1/2$, (b) $\kappa=1/3$, and (c) $\kappa=2/3$.

pose the breaking and recurrence of the dynamical symmetry, we first discuss the 2D case; the extension to the 3D case is addressed later.

For the screened 2D Coulomb potential ($m=k=1$),

$$V(\rho)=-1/\rho-\lambda/\rho^2 \quad (0<\lambda\ll 1), \tag{1}$$

the orbit equation may be expressed as

$$\frac{1}{\rho}=\frac{1}{L_z^2\kappa^2}[1+\sqrt{1+2EL_z^2\kappa^2}\cos\kappa(\theta-\theta_0)], \tag{2}$$

where $E<0$ and L_z are the energy and angular momentum, and $\kappa=\sqrt{1-2\lambda/L_z^2}<1$. In general, the orbit is not closed. However, for rational values of κ, i.e., for suitable angular momenta $L_z=\sqrt{2\lambda/(1-\kappa^2)}$, there still exists an infinite number of closed orbits (rather than elliptic orbits), whose geometry depends only on the angular momentum, but is irrespective of the energy E. Three simplest examples ($\kappa=\frac{1}{2}$, $\frac{1}{3}$, and $\frac{2}{3}$) are displayed in Fig. 1. It is seen that the directions of each aphelion vector (θ_a) and perihelion vector (θ_p) are given by

$$\theta_a-\theta_0=(2\mu+1)\pi/\kappa, \quad \theta_p-\theta_0=2\mu\pi/\kappa,$$
$$\mu=0,1,2,\ldots\ . \tag{3}$$

The closeness of a planar orbit implies that the radial frequency ω_ρ and angular frequency ω_θ are commensurate, and it is seen that

$$\omega_\rho/\omega_\theta=\kappa. \tag{4}$$

For the screened 2D Coulomb potential ($\mathbf{L}=L_z\mathbf{k}$), the usual Runge-Lenz vector $\mathbf{R}=\mathbf{p}\times\mathbf{L}-\mathbf{e}_\rho$ no longer remains conserved. What is the additional conserved quantity responsible for the closeness of the orbits? It is found that the extended Runge-Lenz vector

$$\mathbf{R}'=\mathbf{p}\times\mathbf{L}-\left(1+\frac{2\lambda}{\rho}\right)\mathbf{e}_\rho \tag{5}$$

is still conserved at the aphelion (perihelion) points ($\dot{\rho}=0$), i.e.,

$$d\mathbf{R}'/dt=\mathbf{0}. \tag{6}$$

From this one can understand why there exists an infinite number of closed orbits with angular momenta $L_z=\sqrt{2\lambda/(1-\kappa^2)}$ (κ being a rational number).

For a pure Coulomb potential ($\lambda=0$), $\mathbf{R}'$ is reduced to the usual Runge-Lenz vector $\mathbf{R}$, which remains constant at all points along the closed orbit. The quantum analog of Eq. (6) is $[\mathbf{R}',H]=\mathbf{0}$, which holds at the aphelion (perihelion) points. Moreover, it can be shown that ($\hbar=1$)

$$[L_z,R_x']=iR_y',$$
$$[L_z,R_y']=-iR_x', \tag{7}$$
$$[R_x',R_y']=(-2H)iL_z,$$

where $H=p^2/2-1/\rho-\lambda/\rho^2$ and $\mathbf{R}'=(\mathbf{p}\times\mathbf{L}-i\mathbf{p})-(1+2\lambda/\rho)\mathbf{e}_\rho$ [the quantum version of $\mathbf{R}'$ in Eq. (5)]. Equation (7) implies that (L_z,R_x',R_y') constitute an SO_3 algebra in Hilbert space spanned by degenerate states belonging to a given energy eigenvalue $E_n=-1/(2n^2)(n=1/2,3/2,5/2,\ldots)$. Because, in addition to $[L_z,H]=0$, $[\mathbf{R}',H]=\mathbf{0}$ holds at the aphelion (perihelion) points, it is seen that, in general, through the dynamical symmetry SO_3 of a 2D hydrogen atom is broken, the SO_3 symmetry may be restored at the aphelion (perihelion) points of the classical orbits.

The extension to the 3D case is straightforward, but the situation is more complicated. Due to angular momentum conservation, the classical orbits still remain in a plane perpendicular to the angular momentum $\mathbf{L}$, and the orbit equation is the same as Eq. (2), which is nonclosed in general. Similarly, for rational values of $\kappa=\sqrt{1-2\lambda/L^2}$, i.e., for suitable angular momenta $L=\sqrt{2\lambda/(1-\kappa^2)}$, there still exists an infinite number of closed orbits, and it can be shown that the extended Runge-Lenz vector $\mathbf{R}'=\mathbf{p}\times\mathbf{L}-(1+2\lambda/r)\mathbf{r}/r$ is conserved at the aphelion (perihelion) points of classical or-

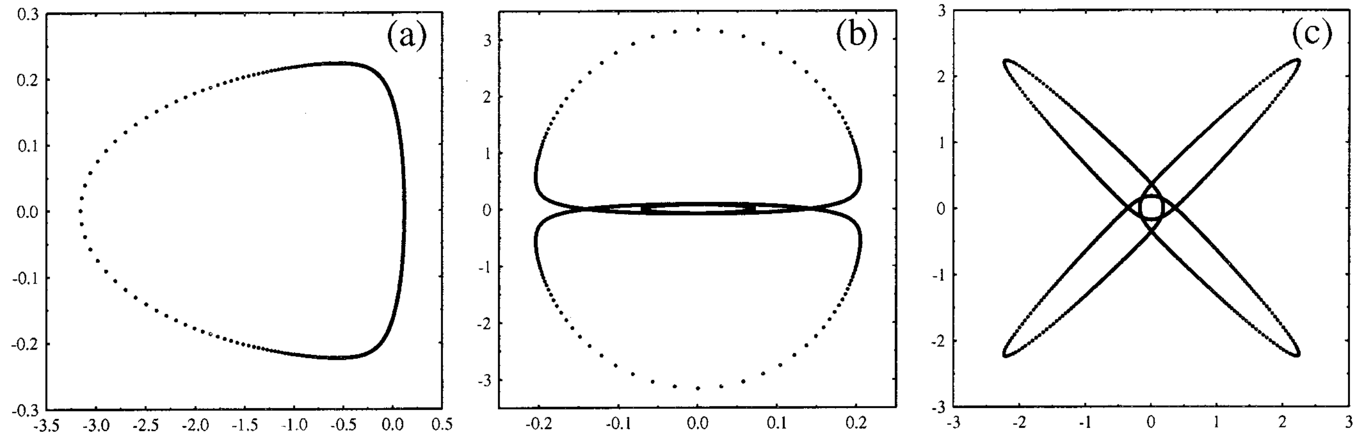

FIG. 2. Closed orbits of a particle in the screened 2D isotropic harmonic oscillator of Eq. (12), with $\lambda=0.2$ and $E=5$. (a) $\kappa=1/2$, (b) $\kappa=1/3$, and (c) $\kappa=2/3$.

bits ($\dot{r}=0$). The corresponding quantum version is $\mathbf{R}'=(\mathbf{p}\times\mathbf{L}-i\mathbf{p})-(1+2\lambda/r)\mathbf{r}/r$, and it can be shown that ($m=\hbar=1$)

$$[L_\alpha,L_\beta]=i\epsilon_{\alpha\beta\gamma}L_\gamma,$$
$$[L_\alpha,R'_\beta]=-i\epsilon_{\alpha\beta\gamma}R'_\gamma, \tag{8}$$
$$[R'_\alpha,R'_\beta]=(-2H)i\epsilon_{\alpha\beta\gamma}L_\gamma,$$

i.e., $\mathbf{L}$ and $\mathbf{R}'$ still constitute an SO_4 algebra in Hilbert space spanned by degenerate states belonging to a given energy eigenvalue $E<0$. For $\lambda=0$, $\mathbf{R}'$ is reduced to the usual Runge-Lenz vector $\mathbf{R}$. It is interesting to note that, unlike $\mathbf{R}$, $\mathbf{R}'$ is conserved *only* at the aphelion (perihelion) points of classical orbits.

Now we address the factorization of the radial Schrödinger equation for the screened 2D Coulomb potential. The energy eigenstate may be chosen as the simultaneous eigenstate of (H,L_z), $\psi\sim R_{nm'}(\rho)e^{im\theta}\sim\chi_{n,m'}(\rho)e^{im\theta}/\rho$, and $\chi_{n,m'}(\rho)$ satisfies

$$\mathcal{D}_{m'}(\rho)\chi_{n,m'}(\rho)=-2E_n\chi_{n,m'}(\rho),\quad m'=\sqrt{m^2-2\lambda},$$
$$\mathcal{D}_{m'}(\rho)=d^2/d\rho^2-(m'^2-1/4)/\rho^2-2W(\rho), \tag{9}$$
$$W(\rho)=-1/\rho,$$

which can be recast in the forms

$$\mathcal{D}_n(\rho)\chi_{n,m'}(\rho)=(m'^2-1/4)\chi_{n,m'}(\rho),$$
$$\mathcal{D}_n(\rho)=\rho^2d^2/d\rho^2+2E_n\rho^2-2W(\rho)\rho^2, \tag{10}$$

where $E_n=-1/(2n^2)$, $n=n_\rho+|m'|+1/2$, and $n_\rho=0,1,2,\ldots$. Because $\Delta m'=\pm1$ does not imply $\Delta m=\pm1$, one cannot constitute the angular momentum raising and lowering operators. However, for a given value of $m'(m)$, from the factorization of Eq. (10), one may obtain the energy raising and lowering operators, as

$$\chi_{n+1,m'}(\rho)\sim\mathcal{M}\left(\frac{n}{n+1}\right)(\rho d/d\rho-\rho/n+n)\chi_{n,m'}(\rho),$$
$$\chi_{n-1,m'}(\rho)\sim\mathcal{M}\left(\frac{n}{n-1}\right)(\rho d/d\rho+\rho/n-n)\chi_{n,m'}(\rho), \tag{11}$$

where $\mathcal{M}(k)$ is a scaling operator, defined by $\mathcal{M}(k)f(\rho)=f(k\rho)$. The factorization of the radial Schrödinger equation for the 3D case is similar.

Now we discuss the screened 2D isotropic harmonic oscillator ($m=k=1$)

$$V(\rho)=\frac{1}{2}\rho^2-\lambda/\rho^2. \tag{12}$$

The orbit equation can be expressed as

$$\frac{1}{\rho^2}=\frac{1}{L_z^2\kappa^2}[E+\sqrt{E^2-L_z^2\kappa^2}\cos 2\kappa(\theta-\theta_0)], \tag{13}$$

where $\kappa=\sqrt{1-2\lambda/L_z^2}<1$. Similarly, for rational values of κ, i.e., for suitable angular momenta $L_z=\sqrt{2\lambda/(1-\kappa^2)}$, the orbits are closed (rather than elliptic orbits). The three simplest examples ($\kappa=\frac{1}{2}$, $\frac{1}{3}$, and $\frac{2}{3}$) are given in Fig. 2. It is seen that the directions of each aphelion vector (θ_a) and perihelion vector (θ_p) are given by

$$\theta_a-\theta_0=(\mu+1/2)\pi/\kappa,\quad \theta_p-\theta_0=\mu\pi/\kappa,\quad \mu=0,1,2,\ldots \tag{14}$$

and

$$\omega_\rho/\omega_\theta=2\kappa. \tag{15}$$

For a pure 2D isotropic harmonic oscillator ($\lambda=0$), the orbits are always closed (i.e. elliptic orbits), which are guaranteed by the existence of conserved quantities L_z, $Q_{xy}=xy+p_xp_y$ and $Q_1=1/2[(x^2-y^2)+(p_x^2-p_y^2)]$ [15]. In fact, the direction of the major axis and the eccentricity of the elliptiorbit are determined by Q_{xy}/Q_1 and $(Q_{xy}^2+Q_1^2)$, respectively. When the screening effect is turned on ($\lambda \neq 0$), we may define

$$\begin{aligned} Q'_{xy} &= (1+2\lambda/\rho^4)xy + p_x p_y, \\ Q'_1 &= 1/2[(1+2\lambda/\rho^4)(x^2-y^2)+(p_x^2-p_y^2)]. \end{aligned} \tag{16}$$

It can be shown that $dQ'_{xy}/dt=0$ and $dQ'_1/dt=0$ *hold only at the aphelion (perihelion) points of classical orbits* ($\dot{\rho}=0$). In quantum mechanics, L_z, Q_{xy}, and Q_1 constitute an SU_2 algebra. Similarly, it can be shown that (L_z, Q'_{xy}, Q'_1) still constitute the same SU_2 algebra as that for (L_z, Q_{xy}, Q_1), i.e.,

$$\begin{aligned} [L_z, Q'_{xy}] &= -2iQ'_1, \\ [L_z, Q'_1] &= 2iQ'_{xy}, \\ [Q'_{xy}, Q'_1] &= -2iL_z. \end{aligned} \tag{17}$$

But unlike Q_{xy} and Q_1, $[Q'_{xy}, H]=0$ and $[Q'_1, H]=0$ hold at the aphelion (perihelion) points, i.e., the SU_2 symmetry holds only at certain points along the classical orbits.

The extension to the screened 3D isotropic harmonic oscillator is more complicated. Besides the conservative angular momentum $\mathbf{L}$, (L_x, L_y, L_z), we may define an extended quadrupole tensor

$$\begin{aligned} Q'_{xy} &= (1+2\lambda/r^4)xy + p_x p_y, \\ Q'_{yz} &= (1+2\lambda/r^4)yz + p_y p_z, \\ Q'_{zx} &= (1+2\lambda/r^4)zx + p_z p_x, \\ Q'_1 &= 1/2[(1+2\lambda/r^4)(x^2-y^2)+(p_x^2-p_y^2)] \\ &= 1/2(Q'_{xx}-Q'_{yy}), \end{aligned} \tag{18}$$

$$\begin{aligned} Q'_0 &= 1/(2\sqrt{3})[(1+2\lambda/r^4)(x^2+y^2-2z^2)+(p_x^2+p_y^2-p_z^2)] \\ &= 1/(2\sqrt{3})(Q'_{xx}+Q'_{yy}-2Q'_{zz}). \end{aligned}$$

It can be shown that at the aphelion (perihelion) points of classical orbits ($\dot{r}=0$), the extended quadrupole tensor is conserved ($dQ'_{xy}/dt=dQ'_{yz}/dt=dQ'_{zx}/dt=0$ and $dQ'_1/dt=dQ'_0/dt=0$). From this it can be seen that for the screened 3D isotropic harmonic oscillator in quantum mechanics the SU_3 symmetry no longer holds. Let us go one step further to investigate the dynamical symmetry of the screened 3D isotropic harmonic oscillator by considering the eigenvalue problem of the tenser, $Q'_{ij}=(1+2\lambda/r^4)r_i r_j + p_i p_j$. Since $\Sigma_j Q'_{ij}L_j=0$, we obtain an eigenvector $\mathbf{L}$ with an eigenvalue 0. Moreover, the real symmetrical matrix Q'_{ij} has three orthogonal eigenvectors, so we can rotate the coordinates (x,y,z) into new coordinates (ξ,η,ζ) (called eigencoordinates). The ζ coordinate axis is parallel to $\mathbf{L}$, and the $\xi-\eta$ coordinate axes lie in the orbital plane. In the eigencoordinates, we have

$$\begin{aligned} [L_\zeta, Q'_{\xi\eta}] &= -2iQ'_1, \\ [L_\zeta, Q'_1] &= 2iQ'_{\xi\eta}, \\ [Q'_{\xi\eta}, Q'_1] &= -2iL_\zeta. \end{aligned} \tag{19}$$

Thus L_ζ, $Q'_{\xi\eta}$, and Q'_1 constitute an SU_2 algebra, i.e., the screened 3D isotropic harmonic oscillator has an SU_2 symmetry at the aphelion (perihelion) points of classical orbits in the eigencoordinate system.

The factorization of the radial Schrödinger equations (9) and (10) for a screened 2D isotropic harmonic oscillator $[W(\rho)=\frac{1}{2}\rho^2]$, $E_n=n+1$, $n=2n_\rho+|m'|$, $n_\rho=0,1,2,\ldots$, is, similar to Eq. (11),

$$\begin{aligned} \chi_{n+2,m'}(\rho) &\sim [\rho d/d\rho - \rho^2 + (n+3/2)]\chi_{n,m'}(\rho), \\ \chi_{n-2,m'}(\rho) &\sim [\rho d/d\rho + \rho^2 - (n+1/2)]\chi_{n,m'}(\rho), \end{aligned} \tag{20}$$

where the raising and lowering operators for energy (but not for angular momentum) are constructed. The factorization of the radial Schrödinger equation for the 3D case is similar.

In summary, we have shown that, for the screened Coulomb potential and isotropic harmonic oscillator, there exists an infinite number of closed orbits for suitable angular momentum values. At the aphelion (perihelion) points of classical orbits, an extended Runge-Lenz vector for the screened Coulomb potential and an extended quadrupole tensor for the screened isotropic harmonic oscillator are still conserved. For the screened 2D Coulomb potential and isotropic harmonic oscillator, the dynamical symmetries SO_3 and SU(2) are still preserved at the aphelion (perihelion) points of classical orbits, respectively. For the screened 3D Coulomb potential, the dynamical symmetry SO_4 is also preserved at the aphelion (perihelion) points of classical orbits. But for the screened 3D isotropic harmonic oscillator, the dynamical symmetry SU(2) is only preserved at the aphelion (perihelion) points of classical orbits in the eigencoordinate system. For the screened Coulomb potential and isotropic harmonic oscillator, only the energy (but not angular momentum) raising and lowering operators can be constructed from a factorization of the radial Schrödinger equation.

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