Applied Mathematics and Mechanics
(English Edition, Vol. 21, No. 2, Feb 2000)

Published by Shanghai University,
Shanghai, China

Article ID: 0253-4827(2000)02-0165-08

A SECOND-ORDER DYNAMIC SUBGRID-SCALE STRESS MODEL *

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(Communicated by Li Jiachum)

Abstract: *A second-order dynamic model based on the general relation between the subgrid-scale stress and the velocity gradient tensors was proposed. A priori test of the second-order model was made using moderate resolution direct numerical simulation date at high Reynolds number (Taylor microscale Reynolds number* R_λ = 102 ~ 216) *for homogeneous, isotropic forced flow, decaying flow, and homogeneous rotating flow. Numerical testing shows that the second-order dynamic model significantly improves the correlation coefficient when compared to the first-order dynamic models.*

Key words: turbulent flow; dynamic model; subgrid-scale stress model; Smagorinsky model

CLC numbers: O357.5; O302   **Document code**: A

Introduction

In LES, the large scales in the flow are computed explicitly and the subgrid-scales (SGS), which are filtered out by an average process operation on the Navier-Stokes equations, are modeled. Since LES requires less computer time than the direct numerical simulations and uses simpler and more universal models than standard Reynolds stress model, LES has become an important method for simulating turbulent flows. In LES, the objective is to model the effect of subgrid-scale stress on the large scale motion. Intensive efforts have been made to model the subgrid-scale stress using information from the resolved large scales. Most SGS models are based on eddy viscosity assumptions. The most commonly used model is the Smagorinsky model. Assuming that energy production is balanced by dissipation, Smagorinsky obtained the following

* **Received date**: 1998-08-10
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expressions for the subgrid-scale stress and eddy viscosity:

$$\tau_{ij} - (1/3)\delta_{ij}\tau_{kk} = -2\nu_t\bar{S}_{ij}, \tag{1}$$

$$\nu_t = (c_s\bar{\Delta})^2 \mid \bar{S} \mid, \tag{2}$$

where

$$\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)$$

is the resolved strain rate tensor, $\mid \bar{S} \mid = \sqrt{2\bar{S}_{ij}\bar{S}_{ij}}$ is the magnitude of strain rate tensor, c_s is the Smagorinsky constant with a value between 0.1 and 0.27 and $\bar{\Delta}$ is the length scale for the grid filter.

The Smagorinsky model, though very successful in the LES of turbulent flow, has some notable drawbacks. For example, it requires an empirical flow-dependent model constant and an ad hoc damping function to predict the proper asymptotic behavior near a wall or in laminar flow. Moreover, this model can not account for energy backscatter, a desired property of LES (Leith[1]). A remarkable improvement to the Smagorinsky model was made by Germano et al.[2] and Lilly[3] using a dynamic subgrid-scale eddy viscosity model. In their dynamic model, two filters are used and the model coefficient is calculated dynamically using an algebraic identity to relate the subgrid scale stresses at two filter levels and the resolved relative subgrid stress. This model gives the proper asymptotic behavior near any wall and is able to predict the energy backscatter since the dynamically determined model coefficient could be negative. The advantages of this dynamic model over the traditional Smagorinsky model have drawn increasing attention. This model has found a wide acceptance in LES. This dynamic subgrid model was used by Yang and Ferziger[4] in the simulation of turbulent obstacle flow, by Zang et al.[5] in recirculating flow, and by Piomelli[6] in a channel flow at high Reynolds number. Moin et al.[7] extended it to the simulation of compressible flow. A notable problem associated with the dynamic model is numerical instability when the model coefficient becomes negative.

It should be pointed out that both the Smagorinsky model and the dynamic model are eddy viscosity models in which the subgrid-scale stress is only proportional to the strain rate tensor. It is expected that a model would be more accurate if more information is included in addition to the strain rate tensor. Lund and Novikov[8] suggested that a dependence on the rotation rate tensor be included because it is believed that the vortex stretching is the dominant mechanism by which turbulence transfers energy from large scales to small ones. They expressed the subgrid-scale stress as a function of strain and rotation rate tensors in the form of a series expansion involving products of the strain and rotation tensors.

In this paper, a second-order dynamic model based on the general model suggested by Lund and Novikov[8] is formulated and tested using DNS data. The directly simulated turbulent flows used for model testing are: homogeneous isotropic forced flow, homogeneous isotropic decaying flow and homogeneous rotation flow. In the following section, the essential features of the second-order dynamic models are described and the method used to test the models is presented. The test results and discussion are given in Section 2. The conclusions are presented in Section 3.

1 Methodology

In a dynamic eddy viscosity model, two filter operators are defined to obtain the large-scale quantities. One is the grid filter $\bar{G}$, the other one is the test filter $\tilde{G}$ with a filter width larger than $\bar{G}$. The large scale quantities are obtained by

$$\bar{f}(x_i) = \int f(x_i)\bar{G}(x_i - x_i')\mathrm{d}x_i', \tag{3}$$

$$\tilde{f}(x_i) = \int f(x_i)\tilde{G}(x_i - x_i')\mathrm{d}x_i'. \tag{4}$$

Let $\tilde{\bar{G}} = \bar{G}\tilde{G}$. Applying the filters $\bar{G}$ and $\tilde{\bar{G}}$ to the incompressible Navier-Stokes equations one obtains two sets of filtered equations

$$\frac{\partial \bar{u}_i}{\partial t} + \frac{\partial}{\partial x_j}(\bar{u}_i\bar{u}_j) = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} + \nu\frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j}, \tag{5}$$

$$\frac{\partial \tilde{\bar{u}}_i}{\partial t} + \frac{\partial}{\partial x_j}(\tilde{\bar{u}}_i\tilde{\bar{u}}_j) = -\frac{1}{\rho}\frac{\partial \tilde{\bar{p}}}{\partial x_i} - \frac{\partial T_{ij}}{\partial x_j} + \nu\frac{\partial^2 \tilde{\bar{u}}_i}{\partial x_j \partial x_j}, \tag{6}$$

where

$$\tau_{ij} = \overline{u_iu_j} - \bar{u}_i\bar{u}_j, \tag{7}$$

$$T_{ij} = \widetilde{\overline{u_iu_j}} - \tilde{\bar{u}}_i\tilde{\bar{u}}_j \tag{8}$$

are subgrid-scale stress and subtest-scale stress. The resolved stress is defined as

$$\mathscr{L}_{ij} = \widetilde{\bar{u}_i\bar{u}_j} - \tilde{\bar{u}}_i\tilde{\bar{u}}_j. \tag{9}$$

As shown by Germano[2], T_{ij}, τ_{ij}, $\mathscr{L}_{ij}$ are related by

$$\mathscr{L}_{ij} = T_{ij} - \tilde{\tau}_{ij}. \tag{10}$$

1.1 The second-order dynamic model

If it is assumed that the subgrid stress is a function of both the strain and the rotation tensors, the subgrid stress model can be expressed in general form as (Lund and Novikov[8])

$$\begin{aligned}\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = & -2c_1\bar{\Delta}^2 |\bar{S}| \bar{S}_{ij} + c_2\bar{\Delta}^2(\bar{S}_{ik}\bar{R}_{kj} - \bar{R}_{ik}\bar{S}_{kj}) + \\ & c_3\bar{\Delta}^2(\bar{S}_{ik}\bar{S}_{kj})^* + c_4\bar{\Delta}^2(\bar{R}_{ik}\bar{R}_{kj})^* + \\ & c_5\bar{\Delta}^2\frac{1}{|\bar{S}|}(\bar{S}_{ik}\bar{S}_{kl}\bar{R}_{lj} - \bar{R}_{ik}\bar{S}_{kl}\bar{S}_{lj}),\end{aligned} \tag{11}$$

where $\bar{R}_{ij}$ is the resolved rotation tensor; $()^*$ indicates the trace-free part. The first term on the RHS is the Smagorinsky term associated with the strain tensor. The remaining terms depend on the rotation tensor and are of higher order than the first term. Lund and Novikov[8] evaluated this expression by calculating the correlation coefficient between the exact stress and models using direct numerical simulation for homogeneous isotropic turbulence. They found that if the model coefficients are constant and only one term is used in the right-hand side of Eq.(11), the first term is the best one with a correlation coefficient of 0.24. If more terms are included, there is not significant improvement on correlation coefficient. The correlation coefficient is only 0.26 if all terms are included. If the model coefficients are allowed to vary in space and time, the best term is still the first one with a correlation much better than that with a constant model coefficient. But the first term is not dominant. As more terms are included, the correlation increases. This result indicates that the model with a variable model coefficient performs better than the model with a constant coefficient. For the model with variable coefficients the correlation is better when more information is included. From Lund and Novikov's results, if can be seen that the best two terms are the first two terms on the right-hand side of Eq.(11). The base model used here to formulate the second-order dynamic model is

$$\tau_{ij} - \frac{1}{3}\delta_{ij}\tau_{kk} = -2c_1\bar{\Delta}^2 \mid \bar{S} \mid \bar{S}_{ij} + c_2\bar{\Delta}^2(\bar{S}_{ik}\bar{R}_{kj} - \bar{R}_{ik}\bar{S}_{kj}). \tag{12}$$

Since the model is invariant to the filtering process, the subtest stress can be expressed as

$$T_{ij} - \frac{1}{3}\delta_{ij}T_{kk} = -2c_1\tilde{\bar{\Delta}}^2 \mid \tilde{\bar{S}} \mid \tilde{\bar{S}}_{ij} + c_2\tilde{\bar{\Delta}}^2(\tilde{\bar{S}}_{ik}\tilde{\bar{R}}_{kj} - \tilde{\bar{R}}_{ik}\tilde{\bar{S}}_{kj}). \tag{13}$$

Substituting Eqs.(12) and (13) into the algebraic identity Eq.(10), one obtains

$$\mathscr{L}_{ij}^* = c_1 M_{ij} + c_2 N_{ij}, \tag{14}$$

where $\mathscr{L}_{ij}^*$ is the trace-free part of $\mathscr{L}_{ij}$, and

$$M_{ij} = -2(\tilde{\bar{\Delta}}^2 \mid \tilde{\bar{S}} \mid \tilde{\bar{S}}_{ij} - \bar{\Delta}^2 \widetilde{\mid \bar{S} \mid \bar{S}_{ij}}), \tag{15}$$

$$N_{ij} = \tilde{\bar{\Delta}}^2(\tilde{\bar{S}}_{ik}\tilde{\bar{R}}_{kj} - \tilde{\bar{R}}_{ik}\tilde{\bar{S}}_{kj}) - \bar{\Delta}^2(\widetilde{\bar{S}_{ik}\bar{R}_{kj}} - \widetilde{\bar{R}_{ik}\bar{S}_{kj}}). \tag{16}$$

Eq.(14) represents five independent equations for two unknowns. The error at each point of space can be minimized by applying a least-squares approach. Let

$$\frac{d}{dc_l}\sum_i \sum_j (\mathscr{L}_{ij}^* - c_1 M_{ij} - c_2 N_{ij})^2 = 0 \qquad (l = 1,2). \tag{17}$$

c_1 and c_2 are then given by

$$c_1 = \frac{(\mathscr{L}^*M + M\mathscr{L}^*)(2NN) - (\mathscr{L}^*N + N\mathscr{L}^*)(MN + NM)}{(2MM)(2NN) - (MN + NM)^2}, \tag{18}$$

$$c_2 = \frac{(\mathscr{L}^*N + N\mathscr{L}^*)(2MM) - (\mathscr{L}^*M + M\mathscr{L}^*)(MN + NM)}{(2MM)(2NN) - (MN + NM)^2}, \tag{19}$$

where $\mathscr{L}^*M = \mathscr{L}_{ij}^* M_{ij}$ and the same rule is applied for the remaining terms. Substituting c_1 and c_2 into Eq.(12), one obtains the second-order dynamic model.

1.2 Testing of models

The 3D turbulent flow data used to test the models are obtained from direct numerical simulation of the Navier-Stokes equations using a spectral method. The direct simulations were performed for three different flows: forced homogeneous isotropic flow with $R_\lambda = 102$, 151 and 216; decaying homogeneous isotropic flow with initial $R_\lambda = 113$; and rotating homogeneous flow with $R_\lambda = 190$ and $R_0 = u'/2\pi\Omega = 0.2$, where u' is the rms velocity and Ω is the rate of rotation. Fig.1 shows the energy spectrum of stationary isotropic turbulence for different

Reynolds numbers. Decaying turbulence was simulated using a random initial condition with a prescribed energy spectrum. The energy spectra for decaying and rotating turbulence flows are shown in Fig.2.

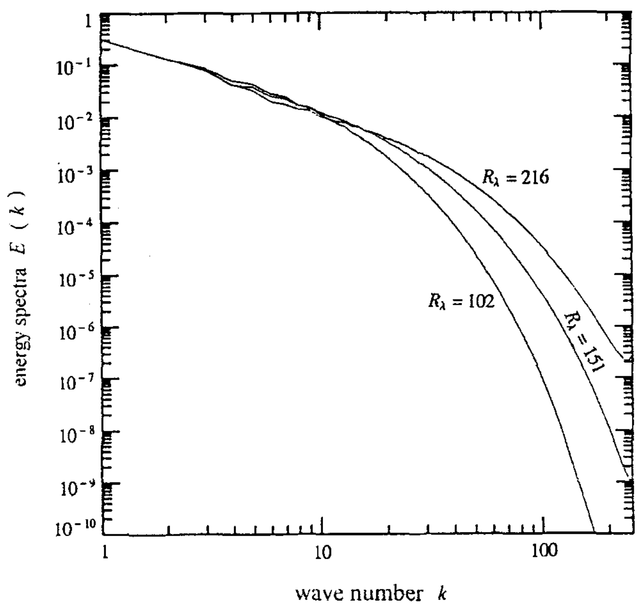

Fig.1 Energy spectra for forced flows

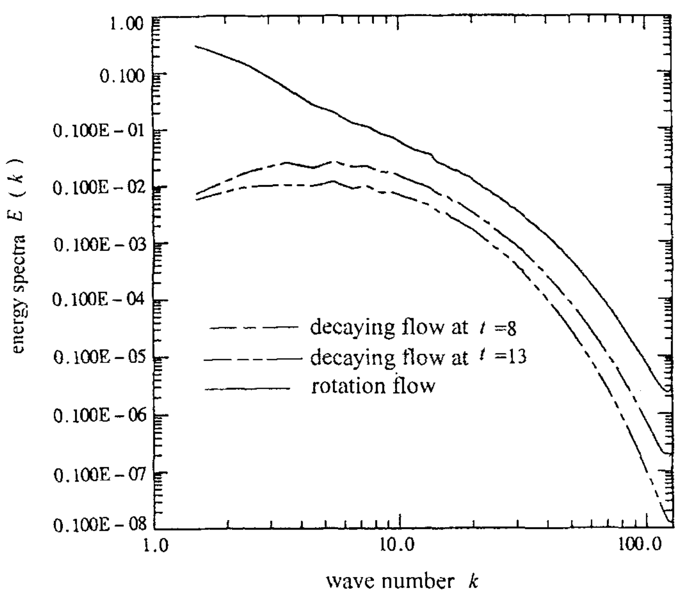

Fig.2 Energy spectra for decaying flow with R_λ = 113 and rotating flow with R_λ = 190 and R_0 = 0.2

With the velocities known at the fine grid, the large scale quantities at grid filter level such as $\bar{u}_i$, τ_{ij}, $\bar{S}_{ij}$ and $\bar{R}_{ij}$ were calculated using the filter operator in Eq.(3). Then, the quantities at test filter level T_{ij}, $\mathscr{L}_{ij}$, $\tilde{\bar{S}}_{ij}$ and $\tilde{\bar{R}}_{ij}$ were obtained using Eq.(4). In this paper, a box filter in physical space is applied to both grid filter and test filter. The grid filter width is taken as $\bar{\Delta} = 8\Delta$. Δ is the grid space for the fine grid. The test filter width $\tilde{\Delta}$ is given by the ratio $\alpha = \tilde{\Delta}/\bar{\Delta}$. The ratio $\alpha = 2$ is used in our work.

Evaluation of the accuracy of modeled quantity, M, in representing an exact quantity, X, is done in terms of a correlation coefficient

$$C(M,X) = \frac{B\langle MX\rangle}{\sqrt{\langle M^2\rangle\langle X^2\rangle}}, \tag{20}$$

where $\langle \quad \rangle$ indicates averaging over whole domain.

2 Results and Discussion

The test results for the second-order model are summarized in Tables 1 and 2. The results for Smagorinsky model and first-order model are also included for comparison. For Smagorinsky model, the correlation coefficient is about 0.2 for forced and decaying flow, shown in Table 1. This value is lower than Clark's[9] 0.27, McMillan's[10] 0.32 and Lund and Novikov's[8] 0.24 for turbulent flow at a lower Reynolds number ($R_\lambda \sim 40$) with lower resolution. The model coefficient is about 0.07 which is about the same as the value obtained by Clark[9]. The correlation for rotating flow is about 0.14. The lower coefficient for rotating flow than the others

Table 1 Summary of correlations between exact SGS stress and models

models	forced flow			decaying flow		rotational flow
	$R_\lambda = 102$	$R_\lambda = 151$	$R_\lambda = 216$	$t = 8$	$t = 13$	$R_\lambda = 190$
Smagorinsky	0.203	0.188	0.203	0.192	0.207	0.140
first-order	0.197	0.129	0.129	0.091	0.101	0.112
second-order	0.719	0.518	0.424	0.268	0.262	0.535

Table 2 Summary of model coefficients

models		forced flow			decaying flow		rotational flow
		$R_\lambda = 102$	$R_\lambda = 151$	$R_\lambda = 216$	$t = 8$	$t = 13$	$R_\lambda = 190$
Smagorinsky	c	0.053	0.064	0.075	0.083	0.084	0.072
first-order	$\bar{c}$	0.026	0.031	0.038	0.035	0.039	0.015
second-order	$\bar{c}_1$	0.024	0.030	0.038	0.038	0.044	0.014
	$\bar{c}_2$	-0.19	-0.18	-0.18	-0.14	-0.13	-0.17

may imply that more information needs to be included in the model to account for the rotation in the rotating flow.

The first-order dynamic model is essentially the Smagorinsky model with its coefficient dynamically determined. Since it has a variable model coefficient. It is anticipated that its correlation would be better than the Smagorinsky model. But it is found the correlation is only about 0.1 ~ 0.2 for all flow cases.

The correlation for second-order dynamic model is approximately 0.4 ~ 0.7 for forced flow, 0.26 for decaying flow and 0.54 for rotation flow. The overall improvement over the first-order dynamic model and base model is remarkable. This improvement might be due to the inclusion of the rotation tensor and strain tensor because the largest improvement is found for rotating flow. It is also interesting to notice that the mean of c_1, corresponding to the Smagorinsky term, is in the range 0.013 ~ 0.04, well within the Smagorinsky constant range 0.01 ~ 0.06. This means that when the coefficient is determined dynamically, its meaning still approximates the Smagorinsky constant. The mean of c_2 is almost a constant in the range between - 0.13 and - 0.18. These results imply that for the dynamic method, the model coefficients are spatially stable and consistent with Smagorinsky model.

3 Conclusions

The first-order dynamic model based on the Smagorinsky model overcome major drawbacks of its base model and has a variable coefficient which may potentially improve the model accuracy. The first-order dynamic model and its base model — the Smagorinsky model were tested by measuring the correlation coefficient between the high-resolution DNS and the model calculations. It is found that the correlations for the Smagorinsky model s about 0.2. The correlations for dynamic model based on Smagorinsky model are lower, or considerably lower in some cases, than that of its base model. To pursue a model with higher accuracy, a second-order dynamic model was formulated based on the general relation between the subgrid scale stress and the velocity gradient tensor (decomposed into strain and rotation tensors). The resulting model shows a significant improvement over the first-order dynamic model.

Finally, it should be pointed out that the results in this paper are obtained from homogeneous isotropic forced flow, decaying flow and homogeneous rotation flow. The conclusion may not be generally applied to other flows without further investigation.

Acknowledgements We thank Gary D. Doolen for useful discussions. The numerical computations were performed at the Advanced Computing Laboratory at Los Alamos National Laboratory.

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