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THE VECTOR FIELDS ADMITTING ONE-PARAMETER SPATIAL SYMMETRY GROUP AND THEIR REDUCTION *

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Abstract: *For a n-dimensional vector fields preserving some n-form, the following conclusion is reached by the method of Lie group. That is, if it admits an one-parameter, n-form preserving symmetry group, a transformation independent of the vector field is constructed explicitly, which can reduce not only dimesion of the vector field by one, but also make the reduced vector field preserve the corresponding* ($n-1$)*-form. In partic ular, while* $n=3$, *an important result can be directly got which is given by Mezie and Wiggins in* 1994.

Key words: vector field; symmtry group; Lie group; reduction; preserving n-form
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Introduction

With the development of Lie group theory since the end of last century, the important role played by this theory in the research of differential equations has been noted by more and more physicists and mathematicians, and has been getting continuously development both in theoretic and applied researches. In [1], Olver introduced systematically some elementary concepts, theorems and important applications about Lie group. Today, Lie methods have gotten into all research fields of differential equations, such as the integrability of equations. In [2], Sen and Tabor got first integrals of Lorenz model just by Lie methods. For high dimensional differential systems, Lie methods are more important due to its role in reducing dimension of systems.

It is well-known that a n-dimensional first-order ordinary differential equations (ODEs) admitting an one-parameter symmetry group can be reduced into a ($n-1$)-dimensional system of ODE, and the solutions of the original equations may be obtained by integrating the solutions of the reduced equations. However, do there exist other connections between the original system and the reduced system? Especially, if the original n-dimensional system is of some properties, one

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Biography: Huang Debin (1972 ~), Doctor may ask that whether the reduced system still preserve these properties. This is a very significant problem in theory and applications. To answer this question, the key is to know about what symmetric group the system should admit, so that the reduction procedure would not destory the properties of the original system. In past, because of the importance of Hamiltonian systems, reduction of Hamiltonian systems on symplectic manifolds was one of active research subjects of differential dynamical systems. Lie method was one of modern reduction methods for symplectic manifolds, which first appeared in the paper of Smale[3]. Further developments due to Meyer[4], etc., led to the fully developed Marsden and Weinstein[5], i.e. any n-dimensional Hamiltonian system admitting an one-parameter Hamiltonian symmetry group can be reduced into a $(n-2)$-dimensional Hamiltonian system (refer to [1, 6]). Recently, Mezic and Wiggins[7] got such a result that 3-dimensional volume-preserving vector fields admitting one-parameter, spatial, volume-preserving symmetry group could be reduced into a Hamiltonian system with one degree of freedom. Thus the dynamical study on this kind of 3-dimensional systems was simplified.

There exists a common point in the works mentioned above, that is, they are all focused on the study of 2-form-preserving Hamiltonian systems. But a kind of more wide-ranging n-dimensional systems preserving some n-form (here, it means divergence free) largely emerges in mathematical models of physics, atmospheric dynamics, biology, etc.. Thus the research on reduction of these systems is very important too. With the aid of Lie method, the main result of this paper shows that for an n-dimensional vector field preserving n-from, if it admits an one-parameter, spatial, n-form-preserving symmetric group, then it can be reduced into a $(n-1)$-dimensional vector field preserving the corresponding $(n-1)$-form. Particularly, when $n=3$, we can deduce directly one of the main results obtained in [7]. In addition, by using the main theorem in this paper, some previous related results can be easily obtained. In the meantime, it is well-known that the classical Hamiltonian systems are divergence free, so, in a sense, the main results obtained in this paper are a geometric generalization in the spirit of symplectic reduction for Hamiltonian systems (to see [5]).

1 Basic Concepts and Preparatory Theorems

For later consulting, in this section, we introduce some definitions, notations and fundamental results. Symbols used in this paper are the same as those in Olver[1].

Definition 1.1 Let M be an n-dimensional manifold with local coordinates $(x^1,\cdots,x^n)$. Consider the n-form Ω on M: $\Omega = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. So the divergence of any vector field Y on M maybe defined by $\mathrm{L}_Y\Omega = (\mathrm{div}\,Y)\Omega$, where L is Lie derivative, see [1,2].

Remark 1 From now on, the considered n-form refers to Ω defined above.

With the above definition, we will give an equivalent condition in the form of definition, under which the flow generated by vector field Y preserves the n-form Ω.

Definition 1.2 Let F be a vector field on M, defined by

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = f_i(x^1,\cdots,x^n,t), \qquad (x^1,\cdots,x^n) \in M,\ t \in \mathbf{R},\ i = 1,\cdots,n. \tag{1}$$

We say that the vector field F preserves the n-form Ω if and only if

$$\operatorname{div} \boldsymbol{F} = \sum_{i=1}^{n} \frac{\partial f_i(x^1, \cdots, x^n, t)}{\partial x^i} = 0.$$

Definition 1.3 Let G be an one-parameter Lie group acting on $M \times \mathbf{R}$. If G satisfies conditions: (i) G is a symmetry group of the system (1); and (ii) the infinitesimal generator V of G as follows: $V = \sum_{i=1}^{n} \xi^i(x^1, \cdots, x^n) \frac{\partial}{\partial x^i}$. Then, we call the G as a spatial symmetry group of the system (1). Furthermore, if V satisfies condition: $\sum_{i=1}^{n} \frac{\partial \xi^i(x^1, \cdots, x^n)}{\partial x^i} = 0$, we call G as an n-form-preserving, spatial symmetric group of (1).

In general, for a given vector field (1), how to find its spatial symmetry group is a nuisance. But in many cases, with the help of acknowledge of the background of systems considered, one can guess the spatial symmetry group of the system through physical or geometric methods, herein lies the art of Lie group theory. For the sake of completeness in theory, we introduce a quite useful theorem below.

Theorem 1.4 Lie group G generated by $V = \sum_{i=1}^{n} \xi^i(x^1, \cdots, x^n) \frac{\partial}{\partial x^i}$ (hereafter denote $V = (\xi^1, \cdots, \xi^n)$) is a spatial symmetry group of the system (1) if and only if $[\boldsymbol{F}, V] = 0$. Where $\boldsymbol{F} = (f_1, \cdots, f_n)$, $[\boldsymbol{F}, V]$ denotes the Lie bracket of $\boldsymbol{F}$ and V, defined in coordinates by

$$[\boldsymbol{F}, V]_i = \sum_{j=1}^{n} \left(f_j \frac{\partial \xi^i}{\partial x^j} - \xi^j \frac{\partial f_i}{\partial x^j} \right), \qquad i = 1, \cdots, n.$$

Proof This is an easy calculation applying Theorems 2.36 and 2.71 in [1].

Theorem 1.5 Let the system (1) admit an one-parameter symmetry group G whose infinitesimal generator is V. Then there exists a local transformation of variables, defined near point (x, t) at which $V|_{(x,t)} \neq 0$, given by

$$x^i = \eta_i(y^1, \cdots, y^n, s), \quad t = \phi(y^1, \cdots, y^n, s), \qquad i = 1, \cdots, n, \tag{2}$$

such that in new variables (y, s), the system (1) becomes

$$\frac{\mathrm{d} y^i}{\mathrm{d} s} = g_i(y^1, \cdots, y^{n-1}, s), \qquad i = 1, \cdots, n, \tag{3}$$

where $y^1, \cdots, y^{n-1}, s$ are a complete set of functional independent invariants of V, i.e.

$$V(y^i) = 0, \quad i = 1, \cdots, n-1; V(s) = 0, V(y^n) = 1. \tag{4}$$

Proof The main idea is to apply the "straightened out" theorem, for the details one can refer to Theorem 2.66 in [1].

Remark 2 Because the right hand of the system (3) is independent of y^n, the component y^n can be gotten by quadrature. Thus we usually call the first $n-1$ equations of the system (3) as a reduced system of the system (1) under G.

In particular, when G is a spatial symmetry group, we get the following corollary.

Corollary 1.6 Suppose G in Theorem 1.5 is a spatial symmetry group, then for the transformation (2), we can take $s = t$ and $\eta_i (i = 1, \cdots, n)$ independent of s, hence $y^i (i = 1, \cdots, n)$ is independent of t.

Proof Since G is a spatial symmetry group, the function t is just an invariant of G, and we can take $s = t$. Furthermore, since $V(y) = \sum_{i=1}^{n} \xi^i \frac{\partial y}{\partial x^i}$ where $\xi^i (i = 1, \cdots, n)$ are independent of t, the solutions to the equation $V(y) = 0$ or 1 are all independent of t. This completes the proof.

2 The Main Result and Its Proof

With the preparations in section 1, we will introduce and prove our main result in this paper.

Theorem 2.1 Let n-dimensional vector field (1) preserve the n-form Ω and admit an one-parameter spatial symmetry group G preserving Ω. Then there exists a transformation of variables such that the reduced vector field of (1) under G preserves the corresponding $(n-1)$-form.

Before giving the proof of the theorem, we first prove the following lemma.

Lemma 2.2 Suppose there exists a differentiable invertible transformation φ

$$x^i = \varphi_i(y^1, \cdots, y^n), \qquad i = 1, \cdots, n, \tag{5}$$

under which the system (1) takes the following form:

$$\frac{\mathrm{d}y^i}{\mathrm{d}t} = g_i(y^1, \cdots, y^n, t), \qquad i = 1, \cdots, n. \tag{6}$$

Then for system (1) and (6), the following relation is valid:

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x^i} = \frac{1}{|J|} \sum_{i=1}^{n} \frac{\partial(|J| g_i)}{\partial y^i}, \tag{7}$$

where J is the Jacobian matrix of the transformation φ, $|J|$ is its determinant.

Proof According to properties of partial derivative and matrix calculation, after a straight and lengthy calculation, one can prove the result.

Proof of Theorem 2.1 Let the infinitesimal generator of G be $V = (\xi^1, \cdots, \xi^n)$. Applying Theorem 1.5 and Corollary 1.6, there exists a transformation of coordinates as (5) under which the system (1) takes the following form:

$$\frac{\mathrm{d}y^i}{\mathrm{d}t} = k_i(y^1, \cdots, y^{n-1}, t), \qquad i = 1, \cdots, n. \tag{8}$$

According to the assumption of the theorem and the relation (7), we can deduce

$$\sum_{i=1}^{n} \frac{\partial(|J| k_i)}{\partial y^i} = 0. \tag{9}$$

Furthermore, applying (4) in Theorem 1.5, we find that under the transformation φ the vector field V takes form $(0, \cdots, 0, 1)$. Using Lemma 2.2 once more, we get

$$0 = \sum_{i=1}^{n} \frac{\partial \xi^i}{\partial x^i} = \frac{1}{|J|}\left(0 + \cdots + 0 + \frac{|J|}{\partial y^n}\right) \Rightarrow \frac{\partial |J|}{\partial y^n} = 0.$$

Therefore $|J|$ is independent of y^n, then (9) becomes

$$\sum_{i=1}^{n-1} \frac{\partial(|J| k_i)}{\partial y^i} = 0. \tag{10}$$

Now, consider the reduced system of the system (1), i.e. the first $n - 1$ equations of the system (8)

$$\frac{\mathrm{d}y^i}{\mathrm{d}t} = k_i(y^1,\cdots,y^{n-1},t), \qquad i = 1,\cdots,n-1. \tag{11}$$

Introduce a transformation Γ:

$$z^1 = \int |J| \, \mathrm{d}y^1, z^2 = y^2, \cdots, z^{n-1} = y^{n-1}, \tag{12}$$

under which, suppose (11) takes the form

$$\frac{\mathrm{d}z^i}{\mathrm{d}t} = g_i(z^1,\cdots,z^{n-1},t), \qquad i = 1,\cdots,n-1. \tag{13}$$

Apparently, the inverse Γ^{-1} of Γ exists. Namely, under Γ^{-1}, (13) becomes (11).

Now calculating the Jacobian matrix $\mathrm{D}Z/\mathrm{D}Y(Z = (z^1,\cdots,z^{n-1}), Y = (y^1,\cdots,y^{n-1}))$ of Γ^{-1}, we can get

$$\frac{\mathrm{D}Z}{\mathrm{D}Y} = \begin{pmatrix} |J| & \frac{\partial}{\partial y^2}\int |J|\,\mathrm{d}y^1 & \cdots & \frac{\partial}{\partial y^{n-1}}\int |J|\,\mathrm{d}y^1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{14}$$

Thus applying Lemma 2.2 and (10), we have

$$\sum_{i=1}^{n-1} \frac{\partial g_i}{\partial z^i} = \frac{1}{\left|\frac{\mathrm{D}Z}{\mathrm{D}Y}\right|} \sum_{i=1}^{n-1} \frac{\partial\left(\left|\frac{\mathrm{D}Z}{\mathrm{D}Y}\right| k_i\right)}{\partial y^i} = \frac{1}{|J|}\sum_{i=1}^{n-1}\frac{\partial(|J|\,k_i)}{\partial y^i} = 0.$$

To sum up, for the vector field (1) satisfying the conditions in Theorem 2.1, there exists a transformation of coordinates: $x \to z$, under which (1) takes the form

$$\begin{cases} \frac{\mathrm{d}z^i}{\mathrm{d}t} = g_i(z^1,\cdots,z^{n-1},t), \ \sum_{i=1}^{n-1}\frac{\partial g_i}{\partial z^i} = 0, & i = 1,\cdots,n-1, \\ \frac{\mathrm{d}z^n}{\mathrm{d}t} = g_n(z^1,\cdots,z^{n-1},t), & z^n = y^n. \end{cases} \tag{15}$$

Thus the theorem is proved.

Remark 3 From the above proof, we know that the transformation under which (1) becomes (15) does not depend on the vector field (1), but only depends on the symmetry group G. And the whole transformation preserves an n-form corresponding to Ω.

3 Corollaries and Applications

When $n = 3$, for the reduced system in Theorem 2.1, we can get directly one of the main results in [7]:

Theorem 3.1 Let the following 3-dimensional system be a volume-preserving system

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i(x_1,x_2,x_3,t), \qquad i = 1,2,3, \tag{16}$$

$$\mathrm{div} f(x) = \sum_{i=1}^{3}\frac{\partial f_i}{\partial x_i} \equiv 0.$$

Suppose further that it admits an one-parameter spatial, volume-preserving symmery group G.

Then there exists a local transformation of coordinates

$$x_i = \phi_i(z_1, z_2, z_3), \qquad i = 1,2,3,$$

such that in new variables the system (16) becomes

$$\begin{cases} \dfrac{dz_1}{dt} = \dfrac{\partial H(z_1, z_2, t)}{\partial z_2}, \\ \dfrac{dz_2}{dt} = -\dfrac{\partial H(z_1, z_2, t)}{\partial z_1}, \\ \dfrac{dz_3}{dt} = k_3(z_1, z_2, t), \end{cases} \tag{17}$$

where $H(z_1, z_2, t)$ is a certain function.

Proof From Theorem 2.1, the 2-dimensional reduced system of (16) must be a 2-form-preserving vector field, hence it can be expressed as a Hamiltonian form. So the system (16) can be turned into the form of (17).

In fluid mechanics, there are many flows in the form as (17) and are called regular duct flows (see Ottino[8]). Just because (16) can be transformed into (17), in Ref. [7], J. Mezie and S. Wiggins rewrote the system (17) in action-angle-angle variables and generalized the KAM theorem, and Melnikov method to the 3-dimensional system, which made they study the integrability and perturbation of the 3-dimensional fluid flows successfully.

If the system (16) is autonomous, then the function H is its first integral. Thus it is easy to get the following corollary.

Corollary 3.2 Let the system (16) be a volume-preserving autonomous vector field and admit an one-parameter, spatial, volume-preserving symmetry group G. Then the system (16) must exist a first integral.

Combining Corollary 3.2 with the result obtained in [9], that is, for any system described by three autonomous ordinary differential equations, if it admits an autonomous invariant, then this system can be written as a generalized Hamiltonian system (see [6] for the definition) and its Hamiltonian just is the invariant, we can easily get the following conclusion.

Corollary 3.3 The system satisfying the condition in Corollary 3.2 can be transformed into a generalized Hamiltonian system.

In [10], Zhang Jinyan investigated the global periodicity of 3-dimensional gradient conjugate systems (the gradient conjugate system is an autonomous system which is divergence-free and admits a first integral) and obtained the following result:

Proposition 3.4 Suppose the 3-dimensional gradient conjugate system $\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x})$ are analytic and admit a normal analytic first integral $G(\boldsymbol{x})$. Then the fixed points of the 2-dimensional restricted system on level manifold $G(\boldsymbol{x}) = c (c > 0)$ must be centers or (generalized) saddles. Further, if the number of fixed points on $G(\boldsymbol{x}) = c$ is finite, then orbits of the system lain on the integral surface $G(\boldsymbol{x}) = c$ all are closed orbits except centers and finite number of joint orbits between saddles. The above properties refer to global periodicity.

From Proposition 3.4, it follows that the 3-dimensional system satisfying condition in Theorem 3.2 must be a gradient conjugate system. On the other hand, from the proof of Theorem 3.2, under some transformations if necessary, the system restricted on its integral surface is a 2-

dimensional Hamiltonian system. Thus the global periodicity of the system follows from the intrinsic property of 2-dimensional Hamiltonian system.

Finally, to illustrate the application of the main theorem in this paper, we consider the Euler flow generated by the velocity field V:

$$\begin{cases} \dfrac{dx_1}{dt} = ax_1 + ax_2, \\ \dfrac{dx_2}{dt} = ax_1 + ax_2, \\ \dfrac{dx_3}{dt} = bx_1^2 - bx_2^2 - 2ax_3. \end{cases} \tag{18}$$

From fluid mechanics we know that the corresponding verticity field is $W = (-2bx_2, -2bx_1, 0)$. It is not difficult to check that the Lie bracket of V and W is zero, i.e. $[V, W] = 0$. Thus from Theorem 1.4, the group generated by W is a divergence-free, spatial, symmetry group of (18).

After some calculation, we can derive that under the following changes of variables (let $b \neq 0, x_2 \neq 0$)

$$y_1 = x_1^2 - x_2^2,\ y_2 = x_2,\ y_3 = -\frac{1}{2b}\arctan\left(\frac{x_1}{x_2}\right),$$

the system (18) takes the form

$$\begin{cases} \dfrac{dy_1}{dt} = 2ay_1 = \dfrac{\partial H}{\partial y_2}, \\ \dfrac{dy_2}{dt} = by_1 - 2ay_2 = -\dfrac{\partial H}{\partial y_1},\ H(y_1, y_2) = 2ay_1y_2 - \dfrac{b}{2}y_1^2, \\ \dfrac{dy_3}{dt} = -\dfrac{a}{2b}. \end{cases} \tag{19}$$

Substituting y with x in $H(y_1, y_2) = 2ay_1y_2 - \frac{b}{2}y_1^2$, we get a first integral of the system (18)

$$G(x) = 2a(x_1^2 - x_2^2)x_3 - \frac{b}{2}(x_1^2 - x_2^2)^2.$$

In this paper, we only study one-parameter symmetry group. For the case of multi-parameter group, we also can study the corresponding problem similarly, but it will be more difficult and complex. The related issues will be the topic of our following publications. For some reduction results related with multi-parameter symmetry group, the readers can refer to Olver[1].

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