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Influence of Threshold Stress on the Estimation of the Weibull Statistics

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The influence of threshold stress on the estimation of the Weibull statistics is discussed in terms of the Akaike information criterion. Numerical simulations show that, if sample data are limited in number and threshold stress is not too large, the two-parameter Weibull distribution is still a preferred choice. For example, the fit of strength data of glass and ceramics to the two- and three-parameter Weibull distributions is compared.

I. Introduction

Similar to other brittle materials, measurement of the strength of a series of nominally identical ceramic specimens typically produces considerable scatter in the results. It is thus desirable to have some means of describing such a behavior quantitatively and further incorporating it into the assessment of reliability.[1–3] Weibull first proposed such a means based mainly on the weakest-link hypothesis and a simple empirical function, which has been and is still widespreadly used in the description of fracture data.[4–6] According to the Weibull statistics, the cumulative failure probability $F(\sigma)$ of a material subjected to a stress σ is given by

$$F(\sigma) = 1 - \exp\left[-\left(\frac{\sigma - \sigma_{th}}{\sigma_0}\right)^m\right] \quad (1)$$

where σ_0 is a normalized factor known as the characteristic strength, σ_{th} is the threshold stress (below which no failure will occur), and m is the Weibull modulus. Here, the Weibull modulus m is a measure of strength diversity, and is also called the shape factor. In most cases, for the sake of simplicity σ_{th} is usually assumed to be 0, and then the Weibull distribution can be reduced to a flexible two-parameter analytic form, such as

$$F(\sigma) = 1 - \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m\right] \quad (2)$$

Up to now, there has been a lack of clear understanding of the effects of the stress threshold, and this can lead to an overestimation of the probability of failure when we adopt the fitted simple distribution in the prediction of failure of ceramic components.[7] In this paper, we will suggest a simple quantitative procedure that can be applied to highlight the influence of the stress threshold on the estimation of the Weibull statistics.

II. Akaike Information Criterion

The best estimate of the unknown parameters in the Weibull distribution is the maximum likelihood method, which shows the smallest coefficient of variation while it is more cumbersome than the usually used linear-regression approach.[8] Here, the log-likelihood for a given probability density function is defined as $\ln L = \sum_{i=1}^{N} \ln f(\sigma_i)$, where σ_i is the strength of the ith specimen, N is the number of measurements, and $f(\sigma)$ is the probability density function, i.e., $f(\sigma) = dF(\sigma)/d\sigma$. Thus, for the three-parameter Weibull distribution, we have the following log-likelihood function:

$$\ln L = \sum_{i=1}^{N} \ln\left\{\frac{m}{\sigma_0}\left(\frac{\sigma_i - \sigma_{th}}{\sigma_0}\right)^{m-1} \exp\left[-\left(\frac{\sigma_i - \sigma_{th}}{\sigma_0}\right)^m\right]\right\} \quad (3)$$

The solution is found by maximizing the log-likelihood function and solving for $(\hat{m}, \hat{\sigma}_0, \hat{\sigma}_{th})$ so that $\partial \ln L/\partial m = 0$, $\partial \ln L/\partial \sigma_0 = 0$, and $\partial \ln L/\partial \sigma_{th} = 0$.

The likelihood approach can be extended to making comparisons between models by the Akaike information criterion (AIC),[9,10] which starts by linking the likelihood to a distance between the true and estimated distributions, and is defined as

$$\text{AIC} = -2(\ln \hat{L} - k) \quad (4)$$

where $\ln \hat{L}$ is the maximum log-likelihood of a given model, k is the number of parameters to be fitted in the model, and the additional factor 2 is a sop to historical precedents and could be omitted.

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Table I. Δ(AIC) Values Calculated for $m_{3p} = 10$, $\sigma_{03p} = 1000$ MPa

	Δ(AIC)				
σ_{th}/σ_{03p}	$N = 20$	$N = 40$	$N = 60$	$N = 80$	$N = 100$
0.1	−1.99	−1.97	−2.00	−1.80	−1.75
0.2	−1.95	−1.89	−2.00	−1.65	−1.57
0.3	−1.92	−1.89	−1.99	−1.53	−1.42
0.4	−1.90	−1.90	−1.98	−1.43	−1.30
0.5	−1.88	−1.87	4.06	5.71	8.26

Generally speaking, it is easier to fit a data set using a complex model with more parameters than a simple one with few parameters. The AIC represents a rough way of compensating for additional parameters and is a useful heuristic measure of the relative effectiveness of different models. It is, however, worth noting that the AIC values obtained here should be used with some caution since the amount of experimental data is usually not very large (e.g., standard requests $N \geq 30$). In typical cases, model differences which would be significant at around the 5% confidence level correspond to differences in AIC values of around 1.5–2.[10–12] For example, in comparing the Weibull distribution with three parameters ($k = 3$) against the Weibull distribution with two parameters ($k = 2$), the former must demonstrate a significantly better fit to justify the additional parameter.

III. Simulation Results and Discussion

As is well known, many potential factors in real data could affect the Weibull distribution.[13–18] But, this uncertainty can be easily resolved in numerical simulations. In the following, we imagine that the strength data of a fictitious material yield the three-parameter Weibull distribution. According to Eq. (1), the strength σ is given by

$$\sigma = \sigma_{th} + \sigma_{03p}\{-\ln[1 - F(\sigma)]\}^{1/m_{3p}} \tag{5}$$

Note here the subscript 3p of parameters is used as a label to distinguish between Eqs. (1) and (2), and similar symbols will be used below.

Obviously the probability $F(\sigma_i)$ is a uniformly distributed random number between 0 and 1, which can be generated by means of Monte Carlo simulations.[19] In each trial a sample of size N will be simulated, provided the three parameters in Eq. (5) and a random number seed are given. As a typical example, let us assume $m_{3p} = 10$, $\sigma_{03p} = 1000$ MPa, and different threshold stresses σ_{th} are realized through adjusting the ratio of σ_{th}/σ_{03p}. Thus a series of data sets can be created, and each data set will then be applied to the fit of the two- and three-parameter Weibull distributions and further to the calculation of AIC values, as shown in Eq. (4).

Table I lists the difference of AIC values calculated by the two- and three-parameter Weibull distributions, that is, $\Delta(\text{AIC}) = (\text{AIC})_{2p} - (\text{AIC})_{3p}$. It is obvious that, if a sample is limited in number and the threshold stress is not too large, the difference of AIC values is not large enough for us to make a clear distinction between the two distributions (see Table I, where $\Delta(\text{AIC}) < 0$ in most cases). This implies that, in most applications, the flexible two-parameter Weibull distribution is still a good choice.

Next, strength data from three brittle materials with various levels of sample size, abraded Kimble R-6 soda–lime glass and two ceramics (SiC and Si_3N_4), are assessed.[20,21] In general strength data are often arranged in ascending order, and each strength is assigned to a failure probability estimate of $F(\sigma_i) = (i - 0.5)/N$, where i is the ith strength and N is the number of measurements.[22] As shown in Table II, the three-parameter Weibull distribution cannot greatly improve the fit in all three cases although an additional parameter is introduced (where $\Delta(\text{AIC}) = (\text{AIC})_{2p} - (\text{AIC})_{3p} < 0$). This can be seen from Fig. 1, where a log–log plot of $\ln[1 - F(\sigma)]^{-1}$ versus σ for the SiC ceramic is illustrated. Here, it should be noted that different values of m_{2p} and σ_{02p} are obtained which are not comparable with the values of m_{3p} and σ_{03p}.

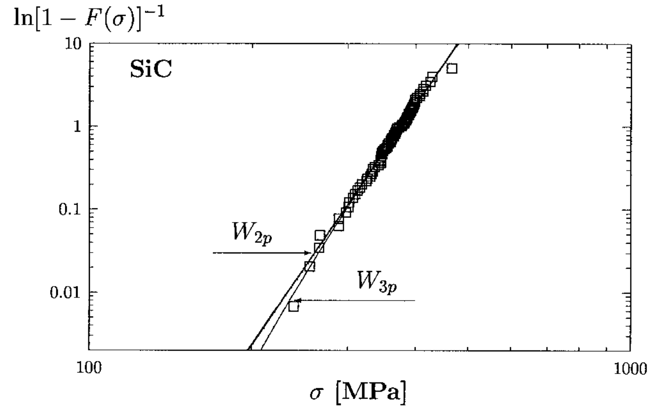

Fig. 1. Comparison of the two-parameter (W_{2p}) with three-parameter (W_{3p}) Weibull distributions for the fit of strength data of SiC.

Finally, it is worth pointing out that, as for various Weibull moduli m_{3p}, the afore-mentioned phenomena have also been found by numerical simulations. But the larger the Weibull modulus, the smaller the region where there is not a clear distinction between the two- and three-parameter Weibull distributions. In other words, the influence of threshold stress becomes more serious as the Weibull modulus m_{3p} increases, and thus threshold stress should be taken into account in the fit of strength data, or some additional examinations are needed if the two-parameter Weibull distribution is used.

IV. Conclusions

A simple quantitative procedure has been proposed in this paper, which can be implemented to highlight the effects of threshold stress on the estimation of the Weibull statistics. The results obtained from both numerical simulations and real data show that, as long as sample data are limited in number and threshold stress is not too large, the two-parameter Weibull distribution is still a preferred choice. But, as the Weibull modulus and threshold stress increase, a good compromise should be made between simplicity of the two-parameter Weibull distribution and applicability of the three-parameter Weibull distribution.

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Table II. Fitted Parameters and Calculated AIC Values by Different Distributions

Specimen	N	m_{3p}	σ_{03p}	σ_{th}	$(\text{AIC})_{3p}$	m_{2p}	σ_{02p}	$(\text{AIC})_{2p}$	Δ(AIC)
Soda-lime	24	5.01	105.70	23.25	221.96	5.74	128.70	220.86	−1.10
Si_3N_4	55	13.75	924.30	9.24	637.77	13.89	933.56	635.78	−1.99
SiC	75	8.01	310.70	65.25	779.83	9.62	376.20	778.31	−1.52

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