W. Q. Zhu
Professor

M. L. Deng
Graduate Student

Z. L. Huang
Associate Professor

Department of Mechanics,
Zhejiang University,
Hangzhou 310027, P. R. China and
State Key Laboratory of Nonlinear Mechanics,
Institute of Mechanics,
Chinese Academy of Science,
Beijing 100008, P. R. China

First-Passage Failure of Quasi-Integrable Hamiltonian Systems

The first-passage failure of quasi-integrable Hamiltonian systems (multidegree-of-freedom integrable Hamiltonian systems subject to light dampings and weakly random excitations) is investigated. The motion equations of such a system are first reduced to a set of averaged Itô stochastic differential equations by using the stochastic averaging method for quasi-integrable Hamitonian systems. Then, a backward Kolmogorov equation governing the conditional reliability function and a set of generalized Pontryagin equations governing the conditional moments of first-passage time are established. Finally, the conditional reliability function, and the conditional probability density and moments of first-passage time are obtained by solving these equations with suitable initial and boundary conditions. Two examples are given to illustrate the proposed procedure and the results from digital simulation are obtained to verify the effectiveness of the procedure.
[DOI: 10.1115/1.1460912]

Introduction

In the theory of random vibration or stochastic structural dynamics, usually two failure models are studied: first-passage (first-excursion) failure and fatigue failure. In recent years, fatigue failure is treated as the propagation of a dominant crack to a critical size. Thus, fatigue failure becomes a special kind of first-passage failure. The first-passage failure is among the most difficult problems in the theory of random vibration or stochastic structural dynamics. At present, a mathematical exact solution is possible only if the random phenomenon in question can be treated as a diffusive Markov process. Still, known solutions are limited to the one-dimensional case ([1,2]).

The state space of a mechanical or structural system model is generally two-dimensional or higher. For such a system subject to Gaussian white noise excitation, the response is a vector diffusive Markov process, and a backward Kolmogorov equation governing the conditional reliability function and a set of generalized Pontryagin equations governing the conditional moments of first-passage time can be set up. However, these equations can usually be solved only numerically. For this purpose, a variety of numerical methods, such as finite element procedure and generalized cell mapping approach have been developed ([3–6]). Unfortunately, at present, the problems can be solved in this way are limited to two or three dimensional.

The response quantities of a quasi-Hamiltonian system (a linear or nonlinear conservative system subject to light dampings and weakly random excitations) can be divided into two categories: rapidly varying processes and slowly varying processes. Usually the slowly varying processes are much more significant for characterizing the long-term behavior of the system. Stochastic averaging is a method to derive the equations governing the slowly varying processes from the original equations of the system. The vector of slowly varying processes after averaging are (approximately) diffusive Markov process and the dimension of the averaged equations is usually much less than that of the original equations. Furthermore, the averaged equations are much more regular than the original equations since there is only one time scale in the former equations while there are two time scales in the later equations. Thus, the stochastic averaging method is a powerful approximate procedure to deal with quasi-Hamiltonian systems.

The first-passage failure of mechanical and structural system usually occurs rarely. It is a long-term behavior and the stochastic averaging method is suitable for studying it. The classical stochastic averaging method has been applied by many researchers to study the first-passage problem of single-degree-of-freedom oscillators with linear restoring force and with nonlinear restoring force ([7–17]). Recently, the stochastic averaging method for quasi-Hamiltonian systems has been developed ([18–20]). Except for response prediction, it has been applied to study the stochastic stability and bifurcation ([20–23]), the first-passage failure of quasi-non-integrable Hamiltonian systems ([24]) and the nonlinear stochastic optimal control ([25–29]).

In the present paper, the stochastic averaging method for quasi-integrable Hamiltonian systems is first reviewed briefly. Then the backward Kolmogorov equation governing the conditional reliability function and the generalized Pontryagin equations governing the conditional moments of first-passage time are derived from the averaged equations of quasi-integrable Hamiltonian systems, and the initial and boundary conditions are formulated. Finally, two examples are worked out and the results obtained by using the proposed procedure are compared with those from digital simulation and with those obtained by using the procedure for quasi-non-integrable Hamiltonian systems ([24]).

Stochastic Averaging of Quasi-Integrable Hamiltonian Systems

The stochastic averaging method for quasi-integrable Hamiltonian systems has been developed for nonresonant and resonant cases, and for white noise and wide-band excitations ([19,23]). Here, only the method for nonresonant case and for white noise excitation is briefly reviewed. Consider a quasi-Hamiltonian system of n-degree-of-freedom governed by the following equations of motion:

$$\dot{Q}_i=\frac{\partial \overline{H}}{\partial P_i}$$

$$\dot{P}_i=-\frac{\partial \overline{H}}{\partial Q_i}-\varepsilon c_{ij}\frac{\partial \overline{H}}{\partial P_j}+\varepsilon^{1/2}f_{ik}W_k(t) \tag{1}$$

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS. Manuscript received by the ASME Applied Mechanics Division, Feb. 28, 2001; final revision, Sept. 27, 2001. Associate Editor: N. C. Perkins. Discussion on the paper should be addressed to the Editor, Professor Lewis T. Wheeler, Department of Mechanical Engineering, University of Houston, Houston, TX 77204-4792, and will be accepted until four months after final publication of the paper itself in the ASME JOURNAL OF APPLIED MECHANICS.

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$$i,j=1,2,\ldots,n; \quad k=1,2,\ldots,m$$

where Q_i and P_i are generalized displacements and momenta, respectively; $H=H(\mathbf{Q},\mathbf{P})$ is twice differentiable Hamiltonian; $c_{ij}=c_{ij}(\mathbf{Q},\mathbf{P})$ are functions representing quasi-linear damping coefficients; $f_{ik}=f_{ik}(\mathbf{Q},\mathbf{P})$ are functions representing excitation amplitudes; ε is a small positive parameter; $W_k(t)$ are Gaussian white noises in the sense of Stratonovich with correlation functions $E[W_k(t)W_l(t+\tau)]=2D_{kl}\delta(\tau)$.

Equation (1) can be modeled as the following set of Itô stochastic differential equations:

$$dQ_i=\frac{\partial \overline{H}}{\partial P_i}dt \tag{2a}$$

$$dP_i=-\left(\frac{\partial \overline{H}}{\partial Q_i}+\varepsilon c_{ij}\frac{\partial \overline{H}}{\partial P_j}-\varepsilon D_{kl}f_{jl}\frac{\partial f_{ik}}{\partial P_j}\right)dt+\varepsilon^{1/2}\sigma_{ik}dB_k(t) \tag{2b}$$
$$i,j=1,2,\ldots,n; \quad k=1,2,\ldots,m$$

where $B_k(t)$ are the independent unit Wiener processes and $\boldsymbol{\sigma\sigma}^T=2\mathbf{fDf}^T$. The double summation terms on the right-hand side of Eq. (2b) are known as the Wong-Zakai correction terms. These terms usually can be split into two parts: one having the effect of modifying the conservative forces and another modifying the damping forces. The first part can be combined with $-\partial \overline{H}/\partial Q_i$ to form an overall effective conservative forces $-\partial H/\partial Q_i$ with a modified Hamiltonian $H=H(\mathbf{Q},\mathbf{P})$ and with $\partial \overline{H}/\partial P_i=\partial H/\partial P_i$. The second part can be combined with $-\varepsilon c_{ij}\partial \overline{H}/\partial P_j$ to constitute an effective damping forces $-\varepsilon m_{ij}\partial H/\partial P_j$ with $m_{ij}=m_{ij}(\mathbf{Q},\mathbf{P})$. With these accomplished, Eqs. (2a) and (2b) can be rewritten as

$$dQ_i=\frac{\partial H}{\partial P_i}dt \tag{3a}$$

$$dP_i=-\left(\frac{\partial H}{\partial Q_i}+\varepsilon m_{ij}\frac{\partial H}{\partial P_j}\right)dt+\varepsilon^{1/2}\sigma_{ik}dB_k(t) \tag{3b}$$
$$i,j=1,2,\ldots,n; \quad k=1,2,\ldots,m.$$

Assume that the Hamiltonian system with Hamiltonian H is integrable and nonresonant. That is, in the Hamiltonian system there exist n independent first integrals (conserved quantities) $H_1,H_2,\ldots,H_n$, which are in involution. The words "in involution" implies that the Poisson bracket of any two of $H_1,H_2,\ldots,H_n$ vanishes. In principle, n pairs of action-angle variables I_i,θ_i can be introduced for an integrable Hamiltonian system of n-degrees-of-freedom. Non-resonance means that the n frequencies, $\omega_i=d\theta_i/dt$, do not satisfy the following resonant relation:

$$k_i^u\omega_i=0(\epsilon) \tag{4}$$

where k_i^u are integers with $\Sigma_{i=1}^n|k_i^u|<4$.

Introduce transformations

$$H_r=H_r(\mathbf{Q},\mathbf{P}), \quad r=1,2,\ldots,n. \tag{5}$$

The Itô stochastic differential equations for H_r are obtained from Eqs. (3a) and (3b) by using Itô differential rule as follows:

$$dH_r=\varepsilon\left(-m_{ij}\frac{\partial H}{\partial P_j}\frac{\partial H_r}{\partial P_i}+\frac{1}{2}\sigma_{ik}\sigma_{jk}\frac{\partial^2 H_r}{\partial P_i\partial P_j}\right)dt+\varepsilon^{1/2}\frac{\partial H_r}{\partial P_i}\sigma_{ik}dB_k(t) \tag{6}$$
$$r,i,j=1,2,\ldots,n; \quad k=1,2,\ldots,m$$

where P_i are replaced by H_s in terms of Eq. (5). Now the system is governed by Eqs. (3a) and (6) and the state variables are Q_i and H_r. It is seen from these equations that Q_i are rapidly varying processes while H_r are slowly varying processes. According to the Khasminskii theorem ([30]), H_r converge weakly to an n-dimensional vector diffusion processes as $\varepsilon\to 0$ in a time interval $0\leqslant t\leqslant T$, where $T\sim 0(\varepsilon^{-1})$. For simplicity, the same symbols H_r are used to denote r components of this diffusion process.

The Itô stochastic differential equations for this n-dimensional vector diffusion process can be obtained by applying time averaging to Eq. (6). The result is

$$dH_r=a_r(\mathbf{H})dt+\overline{\sigma}_{rk}(\mathbf{H})d\overline{B}_k(t) \tag{7}$$
$$r=1,2,\ldots,n; \quad k=1,2,\ldots,m$$

where $\mathbf{H}=[H_1H_2\ldots H_n]^T$; $\overline{B}_k(t)$ are independent unit Wiener processes;

$$a_r(\mathbf{H})=\varepsilon\left\langle -m_{ij}\frac{\partial H}{\partial P_j}\frac{\partial H_r}{\partial P_i}+\frac{1}{2}\sigma_{ik}\sigma_{jk}\frac{\partial^2 H_r}{\partial P_i\partial P_j}\right\rangle_t$$
$$b_{rs}(\mathbf{H})=\overline{\sigma}_{rk}(\mathbf{H})\overline{\sigma}_{sk}(\mathbf{H})=\varepsilon\left\langle \sigma_{ik}\sigma_{jk}\frac{\partial H_r}{\partial P_i}\frac{\partial H_s}{\partial P_j}\right\rangle_t \tag{8}$$
$$\langle[\cdot]\rangle_t=\lim_{T\to\infty}\frac{1}{T}\int_T^{t_0+T}[\cdot]dt.$$

Note that H_r are kept constant in performing the time averaging.

The time averaging in Eq. (8) may be replaced by space averaging. For example, suppose that the Hamiltonian is separable and equal to the sum of n independent first integers, i.e.,

$$H(\mathbf{q},\mathbf{p})=\sum_{r=1}^n H_r(q_r,p_r) \tag{9}$$

and for each H_r there is a periodic orbit with period T_r. Then the averaged drift and diffusion coefficients of Eq. (7) can be obtained as follows:

$$a_r(\mathbf{H})=\frac{\varepsilon}{T}\oint\left(-m_{rj}\frac{\partial H_j}{\partial P_j}\frac{\partial H_r}{\partial P_r}+\frac{1}{2}\sigma_{rk}\sigma_{rk}\frac{\partial^2 H_r}{\partial P_r^2}\right)\times\prod_{\mu=1}^n\left(1\bigg/\frac{\partial H_\mu}{\partial P_\mu}\right)dq_\mu$$
$$b_{rs}(\mathbf{H})=\frac{\varepsilon}{T}\oint\left(\sigma_{rk}\sigma_{sk}\frac{\partial H_r}{\partial P_r}\frac{\partial H_s}{\partial P_s}\right)\times\prod_{\mu=1}^n\left(1\bigg/\frac{\partial H_\mu}{\partial P_\mu}\right)dq_\mu \tag{10}$$

where $\oint[\cdot]\Pi_{\mu=1}^n(\cdot\cdot)dq_\mu$ represents an n-fold loop integral and

$$T=T(\mathbf{H})=\prod_{\mu=1}^n T_\mu=\oint\prod_{\mu=1}^n\left(1\bigg/\frac{\partial H_\mu}{\partial P_\mu}\right)dq_\mu. \tag{11}$$

In the case where action-angle variables I_i, θ_i are available, H_r can be replaced by I_r and averaged Itô Eq. (7) by

$$dI_r=\overline{a}_r(\mathbf{I})dt+\bar{\bar{\sigma}}_{rk}(\mathbf{I})d\bar{\bar{B}}_k(t) \tag{12}$$
$$r=1,2,\ldots,n; \quad k=1,\ldots,m$$

where $\mathbf{I}=[I_1I_2\ldots I_n]^T$;

$$\overline{a}_r(\mathbf{I})=\frac{\varepsilon}{(2\pi)^n}\int_0^{2\pi}\left(-m_{ij}\frac{\partial H}{\partial P_j}\frac{\partial I_r}{\partial P_i}+\frac{1}{2}\sigma_{ik}\sigma_{jk}\frac{\partial^2 I_r}{\partial P_i\partial P_j}\right)d\boldsymbol{\theta}$$
$$b_{rs}(\mathbf{I})=\bar{\bar{\sigma}}_{rk}(\mathbf{I})\bar{\bar{\sigma}}_{sk}(\mathbf{I})=\frac{\varepsilon}{(2\pi)^n}\int_0^{2\pi}\left(\sigma_{ik}\sigma_{jk}\frac{\partial I_r}{\partial P_i}\frac{\partial I_s}{\partial P_j}\right)d\boldsymbol{\theta} \tag{13}$$

in which $\boldsymbol{\theta}=[\theta_1\theta_2\ldots\theta_n]^T$; $\int_0^{2\pi}[\cdot]d\boldsymbol{\theta}$ denotes an n-fold integral.

Note that averaged Eq. (7) or (12) is much simpler than original Eq. (1). The dimension of the former equation is only a half of that of the later equation. Equations (7) and (12) contain only slowly varying process $\mathbf{H}(t)$ and $\mathbf{I}(t)$, respectively, and they are suitable for studying the long-term behavior of the system, such as the first-passage failure.

Backward Kolmogorov Equation and Generalized Pontryagin Equations

For most mechanical/structural systems Hamiltonian H represents the total energy of the system, and H_r the energy of the rth degree-of-freedom of the system. H_r may vary between H_{r0} and ∞, where H_{r0} is a constant, such as H for a Duffing oscillator with hardening spring, between $-\infty$ and H_{r0}, such as H for a Duffing oscillator with softening spring, or between H_{r0} and H_{rm}, where H_{rm} is a constant, such as H for a pendulum. The state of the averaged system of a quasi-integrable Hamiltonian system varies randomly in the n-dimensional domain defined by the direct product of the H_r intervals and the safety domain Ω is a bounded region with boundary Γ within the n-dimensional H_r domain. Suppose that the lower boundary of a safety domain for each H_r is at zero (it is always possible to make so by using coordinate transformation). Then the boundary Γ consists of Γ_0 (at least one of H_r vanishes) and critical boundary Γ_c. The first-passage failure occurs when $\mathbf{H}(t)$ crosses Γ_c for the first time, and it is characterized by the conditional reliability function, the conditional probability density or conditional moments of first-passage time, where the word "conditional" means under the given initial condition in the safety domain.

The conditional reliability function, denoted by $R(t|\mathbf{H}_0)$, is defined as the probability of $\mathbf{H}(t)$ being in safety domain Ω within time interval $(0,t]$ given initial state $\mathbf{H}_0=\mathbf{H}(0)$ being in Ω, i.e.,

$$R(t|\mathbf{H}_0)=P\{\mathbf{H}(\tau)\in\Omega,\tau\in(0,t]|\mathbf{H}_0\in\Omega\}. \tag{14}$$

It is the integral of the conditional transition probability density in Ω. The conditional transition probability density is the transition probability density of the sample functions which remain in Ω in time interval $[0,t]$. For an averaged system, the conditional transition probability density satisfies the backward Kolmogorov equation with drift and diffusion coefficients defined by Eqs. (8), (10), or (13). Thus, the following backward Kolmogorov equation can be derived for the conditional reliability function:

$$\frac{\partial R}{\partial t}=a_r(\mathbf{H}_0)\frac{\partial R}{\partial H_{r0}}+\frac{1}{2}b_{rs}(\mathbf{H}_0)\frac{\partial^2 R}{\partial H_{r0}\partial H_{s0}} \qquad r,s=1,2,\ldots,n \tag{15}$$

where $a_r(\mathbf{H}_0)$ and $b_{rs}(\mathbf{H}_0)$ are defined by Eqs. (8) or (10) with $\mathbf{H}$ replaced by $\mathbf{H}_0$. The initial condition is

$$R(0|\mathbf{H}_0)=1, \quad \mathbf{H}_0\in\Omega \tag{16}$$

which implies that the system is initially in the safety domain. The boundary conditions are

$$R(t|\Gamma_0)=finite \tag{17}$$

$$R(t|\Gamma_c)=0. \tag{18}$$

Equations (17) and (18) imply that Γ_0 is a reflecting boundary while Γ_c is the absorbing boundary.

The first-passage time T is defined as the time when the system reaches critical boundary Γ_c for the first time given $\mathbf{H}_0$ being in Ω. Noting that the conditional probability of the first-passage failure $F(t|\mathbf{H}_0)=1-R(t|\mathbf{H}_0)$, the conditional probability density of the first-passage time can be obtained from the conditional reliability function as follows:

$$p(T|\mathbf{H}_0)=\left.\frac{-\partial R(t|\mathbf{H}_0)}{\partial t}\right|_{t=T}. \tag{19}$$

The conditional moments of first-passage time are defined as

$$\mu_l(\mathbf{H}_0)=\int_0^\infty T^l p(T|\mathbf{H}_0)dT, \quad l=1,2,\ldots. \tag{20}$$

The equations governing the conditional moments of first-passage time can be obtained from Eq. (15) in terms of relationships (19) and (20) as follows:

$$\frac{1}{2}b_{rs}(\mathbf{H}_0)\frac{\partial^2\mu_{l+1}}{\partial H_{r0}\partial H_{s0}}+a_r(\mathbf{H}_0)\frac{\partial\mu_{l+1}}{\partial H_{r0}}=-(l+1)\mu_l \qquad r,s=1,2,\ldots,n; \quad l=0,1,2,\ldots. \tag{21}$$

It is easily seen from Eq. (20) that $\mu_0=1$. The boundary conditions associated with Eq. (21) are obtained from Eqs. (17) and (18) in terms of Eqs. (19) and (20). They are

$$\mu_l(\Gamma_0)=finite \tag{22}$$

$$\mu_l(\Gamma_c)=0. \tag{23}$$

Note that both boundary conditions (17) and (22) are qualitative rather than quantitative. They can be made to be quantitative by using Eqs. (15) and (21), respectively, based on the limiting behavior of the drift and diffusion coefficients in Eqs. (15) and (21) at boundary Γ_0 and it will be illustrated with the followingj examples.

The conditional reliability function is obtained from solving backward Kolmogorov Eq. (15) together with initial condition (16) and boundary conditions (17) and (18). The conditional probability density of first-passage time is obtained from the conditional reliability function by using Eq. (19). The conditional moments of first-passage time are obtained either from the conditional probability density of first-passage time by using definition (20) or directly from solving generalized Pontryagin Eq. (21) together with boundary conditions (22) and (23).

Examples

Example 1. Consider linearly and nonlinearly coupled two linear oscillators subject to external and parametric excitations of Gaussian white noises. The equations of motion of the system are of the form

$$\begin{aligned}\ddot{X}_1+\alpha_{11}\dot{X}_1+\alpha_{12}\dot{X}_2+\beta_1(X_1^2+X_2^2)\dot{X}_1+\omega_1^2X_1&=W_1(t)+X_1W_3(t)\\ \ddot{X}_2+\alpha_{21}\dot{X}_1+\alpha_{22}\dot{X}_2+\beta_2(X_1^2+X_2^2)\dot{X}_2+\omega_2^2X_2&=W_2(t)+X_2W_4(t)\end{aligned} \tag{24}$$

where α_{ij}, β_i, and $\omega_i (i,j=1,2)$ are constants; $W_k(t) (k=1,2,3,4)$ are independent Gaussian white noises with intensities $2D_k$; α_{ij}, β_i, and D_k are assumed of the same order of ε. The response of system (24) in both nonresonant and resonant cases with external excitations only has been studied by using the stochastic averaging method for quasi-integrable Hamiltonian systems ([19]). Here we study the first-passage failure of system (24) in a nonresonant case.

Let $X_1=Q_1$, $X_2=Q_2$, $\dot{X}_1=P_1$, $\dot{X}_2=P_2$. Equation (24) can be recast in the form of Eq. (1) as follows:

$$\begin{aligned}\dot{Q}_1&=P_1\\ \dot{Q}_2&=P_2\\ \dot{P}_1&=-\omega_1^2Q_1-[\alpha_{11}+\beta_1(Q_1^2+Q_2^2)]P_1-\alpha_{12}P_2+W_1(t)+Q_1W_3(t)\\ \dot{P}_2&=-\omega_2^2Q_2-[\alpha_{22}+\beta_2(Q_1^2+Q_2^2)]P_2-\alpha_{21}P_1+W_2(t)+Q_2W_4(t).\end{aligned} \tag{25}$$

Equation (25) can be modeled as Itô stochastic differential equations of the form of Eqs. (3*a*) and (3*b*). Since the Wong-Zakai

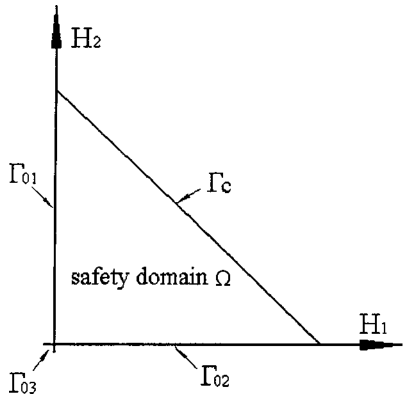

Fig. 1 Safety domain Ω and its boundary on plane H_1 and H_2 for system (24)

correction terms in this case vanish, the modified Hamiltonian associated with the Itô equations is the same as that associated with Eq. (25), i.e.,

$$H=H_1+H_2 \tag{26}$$

$$H_i=\frac{1}{2}(P_i^2+\omega_i^2Q_i^2),\quad i=1,2. \tag{27}$$

The Hamiltonian system with Hamiltonian H is integrable. Thus, system (25) is a quasi-integrable Hamiltonian system. By using the stochastic averaging method for quasi-integrable Hamiltonian systems, the following averaged Itô equations can be obtained in the nonresonant case:

$$\begin{gathered} dH_r=a_r(H_1,H_2)dt+\bar{\sigma}_{rk}(H_1,H_2)d\bar{B}_k(t) \\ r=1,2,\quad k=1,2,3,4 \end{gathered} \tag{28}$$

where

$$\begin{aligned} a_1&=-\alpha_{11}H_1-\frac{\beta_1}{2\omega_1^2}H_1^2-\frac{\beta_1}{\omega_2^2}H_1H_2+D_1+\frac{D_3}{\omega_1^2}H_1 \\ a_2&=-\alpha_{22}H_2-\frac{\beta_2}{2\omega_2^2}H_2^2-\frac{\beta_2}{\omega_1^2}H_1H_2+D_2+\frac{D_4}{\omega_2^2}H_2 \\ b_{11}&=\bar{\sigma}_{1k}\bar{\sigma}_{1k}=2D_1H_1+D_3\frac{H_1^2}{\omega_1^2} \\ b_{22}&=\bar{\sigma}_{2k}\bar{\sigma}_{2k}=2D_2H_2+D_4\frac{H_2^2}{\omega_2^2} \\ b_{12}&=b_{21}=\bar{\sigma}_{1k}\bar{\sigma}_{2k}=0. \end{aligned} \tag{29}$$

It is seen from Eq. (27) that H_i vary from 0 to ∞. So, the state of averaged system (28) varies randomly in the first quadrant of plane (H_1,H_2). Suppose that the limit state of the system is $H=H_1+H_2=H_c$, i.e.,

$$\Gamma_c:\ H_1+H_2=H_c,\quad H_1,H_2\geqslant 0. \tag{30}$$

The safety domain of the system is the inside of a right triangle with boundaries Γ_c in Eq. (30) and Γ_0 defined by

$$\begin{gathered} \Gamma_0=\Gamma_{01}+\Gamma_{02}+\Gamma_{03}, \\ \Gamma_{01}:\ H_1=0,\quad 0<H_2<H_c \\ \Gamma_{02}:\ H_2=0,\quad 0<H_1<H_c \\ \Gamma_{03}:\ H_1=H_2=0 \end{gathered} \tag{31}$$

(see Fig. 1).

Following Eq. (15), the conditional reliability function $R(t|H_{10},H_{20})$ of system (24) is governed by the following backward Kolmogorov equation:

$$\frac{\partial R}{\partial t}=a_1\frac{\partial R}{\partial H_{10}}+a_2\frac{\partial R}{\partial H_{20}}+\frac{1}{2}b_{11}\frac{\partial^2 R}{\partial H_{10}^2}+\frac{1}{2}b_{22}\frac{\partial^2 R}{\partial H_{20}^2} \tag{32}$$

where a_1, a_2, b_{11}, and b_{22} are defined by Eq. (29) with H_1, H_2 replaced by H_{10} and H_{20}, respectively. The initial condition is Eq. (16) with $\mathbf{H}_0=[H_{10}H_{20}]^T$. One boundary condition is Eq. (18) with Γ_c defined by Eq. (30). The other qualitative boundary condition, Eq. (17) with Γ_0 defined by Eq. (31), can be transformed into a quantitative one by using Eq. (32) and considering the limiting behavior of drift and diffusion coefficients in Eq. (29) at boundary Γ_0 defined by Eq. (31). It is

$$\begin{aligned} \frac{\partial R}{\partial t}=&D_1\frac{\partial R}{\partial H_{10}}+\left(D_2-\alpha_{22}H_{20}-\frac{\beta_2}{2\omega_2^2}H_{20}^2+\frac{D_4}{\omega_2^2}H_{20}\right)\frac{\partial R}{\partial H_{20}} \\ &+\left(D_2H_{20}+D_4\frac{H_{20}^2}{2\omega_2^2}\right)\frac{\partial^2 R}{\partial H_{20}^2} \end{aligned} \tag{33}$$

for boundary Γ_{01};

$$\begin{aligned} \frac{\partial R}{\partial t}=&\left(D_1-\alpha_{11}H_{10}-\frac{\beta_2}{2\omega_1^2}H_{10}^2+\frac{D_3}{\omega_1^2}H_{10}\right)\frac{\partial R}{\partial H_{10}} \\ &+D_2\frac{\partial R}{\partial H_{20}}+\left(D_1H_{10}+D_3\frac{H_{10}^2}{2\omega_1^2}\right)\frac{\partial^2 R}{\partial H_{10}^2} \end{aligned} \tag{34}$$

for boundary Γ_{02};

$$\frac{\partial R}{\partial t}=D_1\frac{\partial R}{\partial H_{10}}+D_2\frac{\partial R}{\partial H_{20}} \tag{35}$$

for boundary Γ_{03}.

Equation (32) is a two-dimensional parabolic partial differential equation and can be solved numerically together with the initial and boundary conditions by using the Peaceman-Rachford scheme of the finite difference method to yield the conditional reliability function of system (24). The conditional probability density of the first-passage time of system (24) is then obtained from the conditional reliability function by using Eq. (19).

Similarly, the generalized Pontryagin equations for the conditional moments of the first passage time of system (24) can be derived from the averaged Itô Eq. (28) as follows:

$$\begin{aligned} &\frac{1}{2}b_{11}\frac{\partial^2\mu_{l+1}}{\partial H_{10}^2}+\frac{1}{2}b_{22}\frac{\partial^2\mu_{l+1}}{\partial H_{20}^2}+a_1\frac{\partial\mu_{l+1}}{\partial H_{10}} \\ &\quad+a_2\frac{\partial\mu_{l+1}}{\partial H_{20}} \\ &\quad=-(l+1)\mu_l \end{aligned} \tag{36}$$

where a_1, a_2, b_1, and b_2 are defined by Eq. (29) with H_1 and H_2 replaced by H_{10} and H_{20}, respectively. One boundary condition is (23) with Γ_c defined by Eq. (30). The other qualitative boundary condition, Eq. (22) with Γ_0 defined by Eq. (31), can be transformed into quantitative one by using Eq. (36) and considering the limiting behavior of the drift and diffusion coefficients in Eq. (29) at boundary Γ_0. It is

$$\begin{aligned} &\left(D_2H_{20}+D_4\frac{H_{20}^2}{2\omega_2^2}\right)\frac{\partial^2\mu_{l+1}}{\partial H_{20}^2}+D_1\frac{\partial\mu_{l+1}}{\partial H_{10}}+\left(D_2-\alpha_{22}H_{20}\right. \\ &\quad\left.-\frac{\beta_2}{2\omega_2^2}H_{20}^2+\frac{D_4}{\omega_2^2}H_{20}\right)\frac{\partial\mu_{l+1}}{\partial H_{20}}=-(l+1)\mu_l \end{aligned} \tag{37}$$

for boundary Γ_{01};

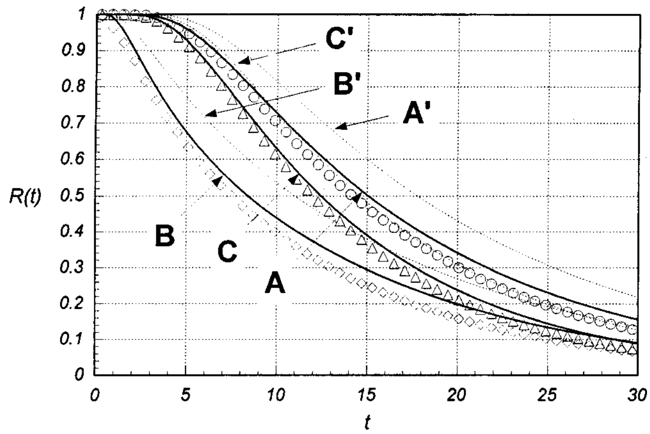

Fig. 2 Reliability function of system (24) for given initial condition. $\alpha_{11}=0.01$, $\alpha_{12}=0.03$, $\beta_1=0.1$, $\omega_1=1.0$, $\alpha_{21}=0.04$, $\alpha_{22}=0.04$, $\beta_2=0.4$, $\omega_2=0.707$, $2D_1=0.03$, $2D_2=0.01$, $H_c=0.3$. **The other parameters are** $2D_3=2D_4=0$, $H_{10}=H_{20}=0$ **for A and A′;** $2D_3=2D_4=0$, $H_{10}=0.09$, $H_{20}=0.03$ **for B and B′;** $2D_3=0.1$, $2D_4=0.01$, $H_{10}=H_{20}=0$ **for C and C′. ——analytical result by using the present proposed procedure; – – – –analytical result by using the procedure proposed in [24]; ○ ◇ △ from digital simulation.**

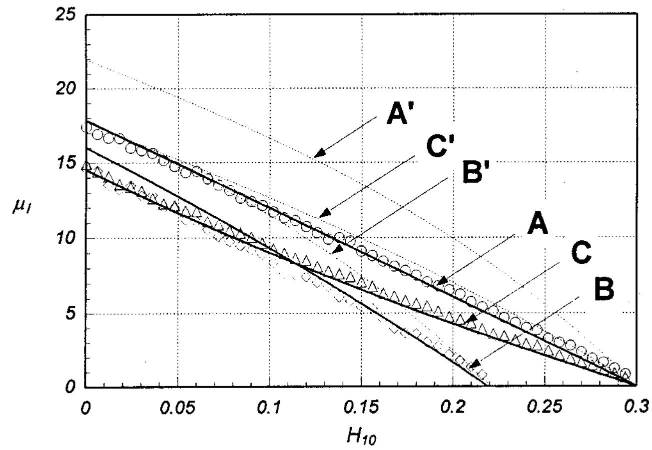

Fig. 4 Mean first-passage time of system (24) as function of H_{10} **for given** H_{20}. $2D_3=2D_4=0$, $H_{20}=0$ **for A and A′;** $2D_3=2D_4=0$, $H_{20}=0.08$ **for B and B′;** $2D_3=0.1$, $2D_4=0.01$, $H_{20}=0$ **for C and C′. The other parameters and symbols are the same as those in Fig. 2.**

$$\left(D_1H_{10}+D_3\frac{H_{10}^2}{2\omega_1^2}\right)\frac{\partial^2\mu_{l+1}}{\partial H_{10}^2}+\left(D_1-\alpha_{11}H_{10}-\frac{\beta_1}{2\omega_1^2}H_{10}^2+\frac{D_3}{\omega_1^2}H_{10}\right)\frac{\partial\mu_{l+1}}{\partial H_{10}}+D_2\frac{\partial\mu_{l+1}}{\partial H_{20}}=-(l+1)\mu_l \tag{38}$$

for boundary Γ_{02};

$$D_1\frac{\partial\mu_{l+1}}{\partial H_{10}}+D_2\frac{\partial\mu_{l+1}}{\partial H_{20}}=-(l+1)\mu_l. \tag{39}$$

Equation (36) is a two-dimensional elliptical partial differential equation and can be solved numerically together with boundary conditions by using the five-point scheme of the finite difference method to yield the conditional moments of first-passage time of system (24).

Some numerical results for the conditional reliability function, the conditional probability density and the conditional mean of the first passage time of system (24) obtained by using the above procedure are shown in Figs. 2–4. Similar results from digital simulation are also shown for comparison. It is seen that the two results are in excellent agreement. Note that the conditional reliability function is a monotonously decreasing function of time. Some results for the reliability function, the probability density, and the mean of first-passage time of system (24) as functions of the initial condition are shown in Figs. 5–7. It is seen that both the reliability and mean first-passage time are monotonously decreasing functions of H_{10} and/or H_{20}.

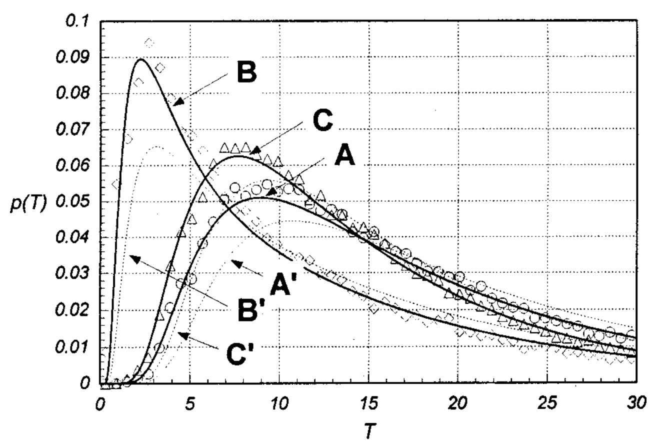

Fig. 3 Probability density of first-passage time of system (24) for given initial condition. The parameters and symbols are the same as those in Fig. 2.

As indicated above, system (24) is a quasi-integrable Hamiltonian system. However, the procedure for evaluating the conditional reliability function and the statistics of first-passage time for quasi-non-integrable Hamiltonian systems developed in [24] can also be applied to system (24). It is interesting to see if this method yields good results.

Treat system (24) as a quasi-non-integrable Hamiltonian system, the averaged Itô equation is of the form

$$dH=a(H)dt+\bar{\sigma}(H)d\bar{B}(t) \tag{40}$$

where H is defined by Eqs. (26) and (27),

$$a(H)=D_1+D_2-\frac{1}{6}(\beta_1+\beta_2)\left(\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}\right)-\frac{1}{2}\left(\alpha_{11}+\alpha_{22}-\frac{D_3}{\omega_1^2}-\frac{D_4}{\omega_2^2}\right)H$$
$$b(H)=\bar{\sigma}^2(H)=\frac{1}{3}\left(\frac{D_3}{\omega_1^2}+\frac{D_4}{\omega_2^2}\right)H^2+(D_1+D_2)H. \tag{41}$$

The conditional reliability function $R(t|H_0)$ of system (40) is governed by the following one-dimensional backward Kolmogorov equation:

$$\frac{\partial R}{\partial t}=a(H_0)\frac{\partial R}{\partial H_0}+\frac{1}{2}b(H_0)\frac{\partial^2 R}{\partial H_0^2} \tag{42}$$

where $a(H_0)$ and $b(H_0)$ are defined by Eq. (41) with H replaced by H_0. The boundary conditions are

$$R(t|H_c)=0 \tag{43}$$

$$R(t|0)=finite. \tag{44}$$

The later condition is qualitative and can be made to be quantitative by using Eq. (42) and the limiting behavior of $a(H_0)$ and $b(H_0)$ near $H_0=0$. It is

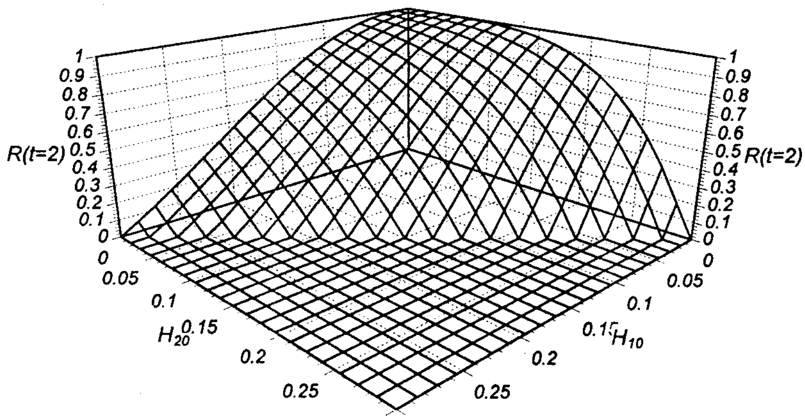

Fig. 5 Reliability of system (24) at $t=2$ (second) as function of H_{10} and H_{20}. $2D_3=0.1$, $2D_4=0.01$. The other parameters are the same as those in Fig. 2.

$$\frac{\partial R}{\partial t}=\left[D_1+D_2-\frac{1}{6}(\beta_1+\beta_2)\left(\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}\right)\right]\frac{\partial R}{\partial H_0}. \quad (45)$$

The initial condition is

$$R(0|H_0)=1. \quad (46)$$

The one-dimensional boundary-initial value problem, Eqs. (42), (43), (45), and (46), can be solved by using the finite difference method of Crank-Nicolson type. The conditional probability density of first-passage time can be obtained from $R(t|H_0)$ as follows:

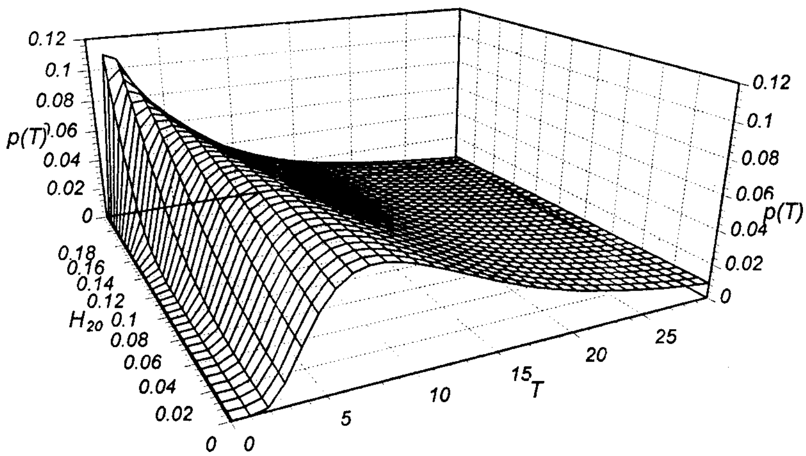

Fig. 6 Probability density of first-passage time of system (24) as function of H_{20} and t for given $H_{10}=0$. $2D_3=0.1$, $2D_4=0.01$. The other parameters are the same as those in Fig. 2.

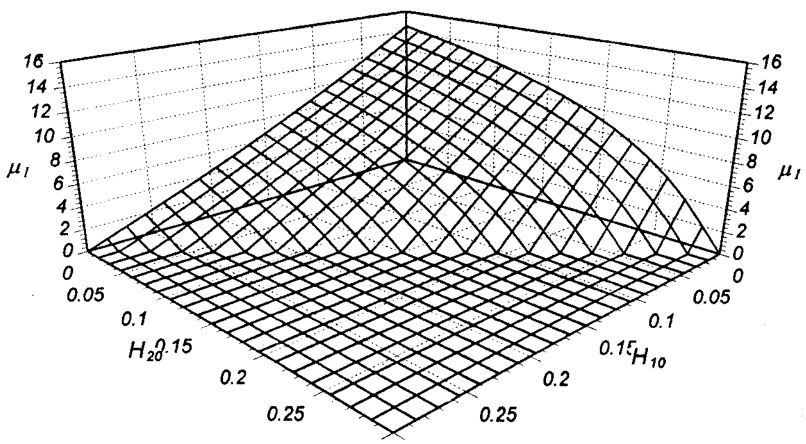

Fig. 7 Mean first-passage time of system (24) as function of H_{10} and H_{20}. $2D_3=0.1$, $2D_4=0.01$. The other parameters are the same as those in Fig. 2.

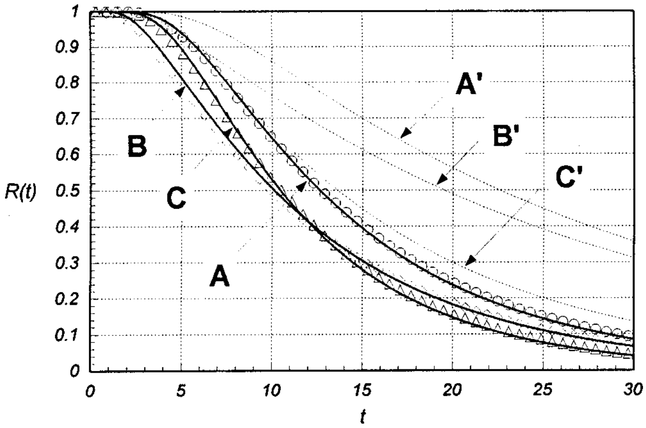

Fig. 8 Reliability function of system (52) for given initial condition. $\alpha_1=0.2$, $\alpha_2=0.1$, $\alpha_3=0.1$, $\beta_1=0.05$, $\omega=1.0$; $\alpha_4=0.4$, $\beta_2=0.1$, $k=2.0$, $2D_1=0.03$, $2D_2=0.01$, $H_c=0.3$. The other parameters are $2D_3=2D_4=0$, $H_{10}=H_{20}=0$ for A and A′; $2D_3=2D_4=0$, $H_{10}=0.04$, $H_{20}=0.02$ for B and B′; $2D_3=0.1$, $2D_4=0.05$, $H_{10}=H_{20}=0$ for C and C′. —— analytical result by using the present proposed procedure; – – – –analytical result by using the procedure proposed in [24]; ○ ◇ △ from digital simulation.

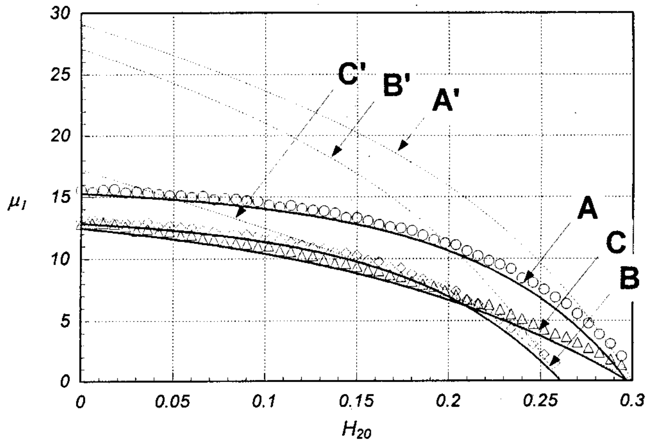

Fig. 10 Mean first-passage time of system (52) as function of H_{20} for given H_{10}. $2D_3=2D_4=0$, $H_{10}=0$ for A and A′; $2D_3=2D_4=0$, $H_{10}=0.04$ for B and B′; $2D_3=0.1$, $2D_4=0.05$, $H_{10}=0$ for C and C′. The other parameters and symbols are the same as those in Fig. 8.

$$p(T|H_0)=\left.\frac{-\partial R(t|H_0)}{\partial t}\right|_{t=T}. \tag{47}$$

Similarly, the generalized Pontryagin equations for the moments of first-passage time of system (40) can be obtained as follows:

$$\frac{1}{2}b(H_0)\frac{\partial^2\mu_{l+1}}{\partial H_0^2}+a(H_0)\frac{\partial\mu_{l+1}}{\partial H_0}=-(l+1)\mu_l. \tag{48}$$

The boundary conditions are

$$\mu_{l+1}(H_c)=0 \tag{49}$$

$$\mu_{l+1}(0)=finite. \tag{50}$$

The qualitative condition (50) can be converted into quantitative one by using Eq. (48) and the limiting behavior of $a(H_0)$ and $b(H_0)$ near $H_0=0$. It is

$$\left[D_1+D_2-\frac{1}{6}(\beta_1+\beta_2)\left(\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}\right)\right]\frac{\partial\mu_{l+1}}{\partial H_0}=-(l+1)\mu_l. \tag{51}$$

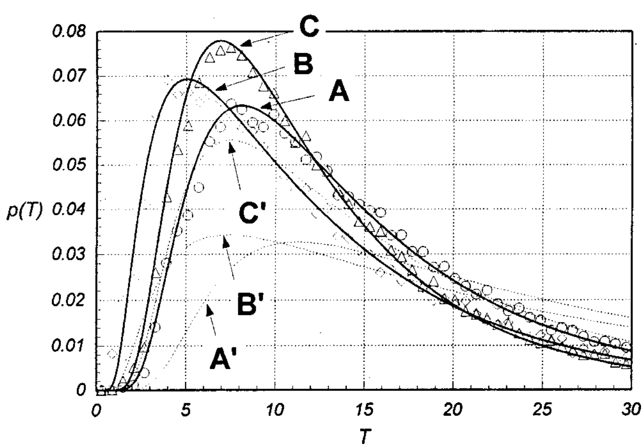

Fig. 9 Probability density of first-passage time of system (52) for given initial condition. The parameters and symbols are the same as those in Fig. 8.

The one-dimensional boundary value problem, Eqs. (48), (49), and (51), can be solved by using the Runge-Kutta method.

Obviously, for evaluating the statistics of the first-passage failure of system (24) the procedure for quasi-non-integrable Hamiltonian systems is much simpler than that for the quasi-integrable Hamiltonian system. However, the former generally yields inaccurate results as shown in Figs. 2–4. Our experience shows that it may yield good results in some very special cases, for example, the ratio of excitation intensity to damping coefficient for the first degree-of-freedom is the same as that for the second degree-of-freedom. In this case system (24) will behave like a quasi-non-integrable Hamiltonian system. On the other hand, the method proposed in this paper always yields good results for system (24) although the equations involved are more difficult to solve.

Example 2. Consider a van der Pol oscillator nonlinearly coupled with a Duffing oscillator subject to external and parametric excitations of Gaussian white noises. The equations of motion of the system are of the form

$$\ddot{X}_1+(-\beta_1+\alpha_1X_1^2+\alpha_2X_2^4+\alpha_3\dot{X}_2^2)\dot{X}_1+\omega^2X_1=W_1(t)+X_1W_3(t)$$
$$\ddot{X}_2+(\beta_2+\alpha_4X_1^2)\dot{X}_2+kX_2^3=W_2(t)+X_2W_4(t) \tag{52}$$

where α_1, α_2, α_3, α_4, β_1, β_2, ω, k are constants; $W_k(t)(k=1,2,3,4)$ are independent Gaussian white noises with intensity $2D_k$. The response of system (52) with external excitations only has been studied by using the stochastic averaging method for quasi-integrable Hamiltonian systems ([19]). Let $X_1=Q_1$, $X_2=Q_2$, $\dot{X}_1=P_1$, $\dot{X}_2=P_2$, Eq. (52) can be rewritten as a quasi-Hamiltonian system of the form of Eq. (1), i.e.,

$$\dot{Q}_1=P_1$$
$$\dot{Q}_2=P_2$$
$$\dot{P}_1=-\omega^2Q_1-(-\beta_1+\alpha_1Q_1^2+\alpha_2Q_2^4+\alpha_3P_2^2)P_1+W_1(t)+Q_1W_3(t) \tag{53}$$
$$\dot{P}_2=-kQ_2^3-(\beta_2+\alpha_4Q_1^2)P_2+W_2(t)+Q_2W_4(t).$$

Equation (53) can be modeled as Itô equations. Since the Wong-Zakai correction terms for this example vanish, the modified Hamiltonian is the same as that associated with Eq. (53), i.e.,

$$H=H_1+H_2 \tag{54}$$

$$H_1 = \tfrac{1}{2}(P_1^2 + \omega^2 Q_1^2) \tag{55}$$

$$H_2 = \tfrac{1}{2}(P_2^2 + \tfrac{1}{2}kQ_2^4). \tag{56}$$

Hamiltonian H is separable and so Eq. (53) governs a quasi-integrable Hamiltonian system. Suppose that the Hamiltonian system is nonresonant. The averaged Itô equations can be obtained from Eq. (53) by using the stochastic averaging method for quasi-integrable Hamiltonian systems ([19]). It is of the same form of Eq. (28) with the following drift and diffusion coefficients:

$$a_1 = \beta_1 H_1 - \frac{\alpha_1}{2\omega^2} H_1^2 - \frac{4\alpha_2}{3k} H_1 H_2 - \frac{4\alpha_3}{3} H_1 H_2 + D_1 + \frac{H_1}{\omega^2} D_3$$

$$a_2 = -\frac{4}{3}\beta_2 H_2 - \frac{4\alpha_4}{3\omega^2} H_1 H_2 + D_2 + \frac{8\Gamma^2\left(\frac{7}{4}\right)}{9\Gamma^2\left(\frac{5}{4}\right)} \sqrt{\frac{H_2}{k}} D_4$$

$$b_{11} = 2D_1 H_1 + \frac{H_1^2}{\omega^2} D_3 \tag{57}$$

$$b_{22} = \frac{8}{3} D_2 H_2 + \frac{64\Gamma^2\left(\frac{7}{4}\right)}{45\Gamma^2\left(\frac{5}{4}\right)} H_2 \sqrt{\frac{H_2}{k}} D_4$$

$$b_{12} = b_{21} = 0.$$

Since $H_i(i=1,2)$ vary from 0 to ∞ under the condition $k>0$, the safety domain of system (52) may be of the same form as that in Fig. 1. The backward Kolmogorov equation for the conditional reliability function, the generalized Pontryagin equations for the conditional moments of first-passage time, and their associated initial and boundary conditions for system (52) can be formulated and solved as for example 1. The only difference is that the drift and diffusion coefficients for this example are defined by Eq. (57) with H_1 and H_2 replaced by H_{10} and H_{20}, respectively.

The procedure for evaluating the statistics of first-passage failure of quasi-non-integrable Hamiltonian systems ([24]) can also be applied to applied to systems (52). The mathematical formulation is the same as that for example one, i.e., Eqs. (40)–(51), except the drift and diffusion coefficients. For this example, there coefficients are

$$a(H) = D_1 + D_2 + 0.5484 D_4 \sqrt{\frac{H}{k}} + \frac{4}{7}\left(\beta_1 - \beta_2 + \frac{D_3}{\omega^2}\right)H - \left(\frac{16}{17}\frac{\alpha_1}{k} + \frac{16}{17}\frac{\alpha_2}{\omega^2} + \frac{5}{77}\alpha_3 + \frac{16}{17}\frac{\alpha_4}{\omega^2}\right)H^2 \tag{58}$$

$$b(H) = \overline{\sigma^2}(H) = 0.4876 D_4 H \sqrt{\frac{H}{k}} + \frac{8}{7}(D_1 + D_2)H + \frac{32}{17}\frac{D_3}{\omega^2} H^2.$$

Some numerical results for the conditional reliability function, the conditional probability density, and mean of first-passage time of system (52) are shown in Figs. 8–10. Some figures for this example similar to Figs. 5–7 are not given due to limited space. The same observations as those for example 1 can be made from these figures.

Conclusions

In the present paper a procedure for evaluating the statistics of the first passage failure, i.e., the conditional reliability function and the conditional probability density and moments of the first-passage time of quasi-integrable Hamiltonian systems has been proposed based on the stochastic averaging method for quasi-integrable Hamiltonian systems. Using the stochastic averaging method reduces the dimensions of the backward Kolmogorov equations governing the conditional reliability function and the generalized Pontryagin equations governing the conditional moments of first-passage time by a half when the associated Hamiltonian system is nonresonant. Furthermore, the backward Kolmogorov equation and generalized Pontryagin equations of an averaged system are nonsingular and much simpler than those for the original system. Applications of the proposed procedure to two examples show that the proposed procedure yields quite accurate results. Thus, the proposed procedure is promising and deserves further development and application.

The results for the two examples indicate that both the reliability and mean first-passage time are monotonously decreasing functions of initial energy of each degree-of-freedom of the system. This property will be used in the study of nonlinear stochastic optimal control of first-passage failure of quasi-integrable Hamiltonian systems.

The procedure for evaluating the statistics of the first-passage failure of quasi-non-integrable Hamiltonian systems has also been applied to the two examples. The numerical results showed that it generally yields inaccurate result for quasi-integrable Hamiltonian systems although it is much simpler than the procedure proposed in this paper. Experience shows that only in some very special cases it may yield good results.

It is remarked that the criteria for the failure considered in this paper are functions of the first integrals (energies) of the individual oscillators. The stochastic averaging method is the most effective for this kind of first-passage failure problem. If the failure criterion is given in terms of other physical quantity, such as the displacement, the first-passage failure problem will be much more difficult to solve. For such a kind of a first-passage failure problem of a single-degree-of-freedom quasi-Hamiltonian system, Roberts [31] developed an integral equation for evaluating the conditional transition probability density in the safety domain (the integral of which is the reliability function) by using the unconditional transition probability density obtained from solving the averaged FPK equation. Maybe this method can be extended to a multi-degree-of-freedom quasi-integrable Hamiltonian system but much more computational work is involved and some difficulties have to be solved. This will be the subject for our future research.

Acknowledgment

The work reported in this paper was supported by the National Natural Science Foundation of China under Grants No. 19972059 and 10002015 and the Cao Guang Biao Hi-Science-Technology Foundation of Zhejiang University.

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