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# First-Passage Time of Duffing Oscillator under Combined Harmonic and White-Noise Excitations

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**Abstract.** The first-passage time of Duffing oscillator under combined harmonic and white-noise excitations is studied. The equation of motion of the system is first reduced to a set of averaged Itô stochastic differential equations by using the stochastic averaging method. Then, a backward Kolmogorov equation governing the conditional reliability function and a set of generalized Pontryagin equations governing the conditional moments of first-passage time are established. Finally, the conditional reliability function, and the conditional probability density and moments of first-passage time are obtained by solving the backward Kolmogorov equation and generalized Pontryagin equations with suitable initial and boundary conditions. Numerical results for two resonant cases with several sets of parameter values are obtained and the analytical results are verified by using those from digital simulation.

**Keywords:** Duffing oscillator, combined harmonic and white noise excitations, stochastic averaging, first-passage time, reliability.

## 1. Introduction

The first-passage time is related to the state transition of multi-steady-state physical systems and to the reliability of structural systems under random excitation. Thus, it is significant to evaluate the probability and/or statistics of the first-passage time. On the other hand, the first-passage time is among the most difficult problems in the theory of stochastic dynamics. At present, a mathematical exact solution is possible only if the random phenomenon in question can be treated as a diffusive process. Still, known solutions are limited to one-dimensional case [1, 2]. A powerful way to study the first passage time of higher dimensional systems is the combination of the stochastic averaging method and diffusion process method of first-passage time. This combination approach has been successfully applied by many researchers [3–15].

Physical and engineering systems are often subjected to combined harmonic and random excitations. Linear and quasi-linear systems under combined harmonic and white-noise or wide-band random excitations have been studied by using the classical stochastic averaging method for obtaining the conditions of moment stability [16–19] or for obtaining the response probability density [20, 21]. Recently, a stochastic averaging method using generalized harmonic functions has been developed for studying the response of strongly nonlinear oscillators under combined harmonic and white-noise excitations [22].

In the present paper, the later version of the stochastic averaging method and the diffusion process method of first-passage time are applied to study the first-passage time of Duffing oscillator under combined harmonic and white-noise excitations. The effects of damping, white noise intensity and detuning parameter on the probability and statistics of first passage time are examined. The analytical results are compared with those from digital simulation.

## 2. Averaged Itô Equations

Consider a Duffing oscillator subject to external harmonic excitation and external and parametric white- noise excitations. The equation of motion of the system is of the form

$$\ddot{X} + \omega^2 X + \alpha X^3 = -\beta \dot{X} + E\cos\Omega t + \xi_1(t) + X\xi_2(t), \tag{1}$$

where $\omega$, $\alpha$, $\beta$, $E$ and $\Omega$ are positive constants denoting the natural frequency of degenerate linear oscillator, intensity of nonlinearity, damping coefficient, amplitude and frequency of harmonic excitation, respectively; $\xi_k(t)$ $(k = 1, 2)$ are independent Gaussian white noises in the sense of Stratonovich with intensities $2D_k$. $\beta$, $E$ and $D_k$ are assumed of the same small order. The conservative Duffing oscillator (without damping and excitations) has a family of periodic solutions in whole phase plane $(x, \dot{x})$ surrounding (0, 0) with instantaneous period

$$\nu^{-1}(a,\varphi) = [(\omega^2 + 3\alpha a^2/4)(1 + \lambda\cos 2\varphi)]^{-1/2} = \sum_{n=0}^{\infty} C_{2n}(a)\cos 2n\varphi,$$

$$C_{2n}(a) = \frac{1}{2\pi}\int_0^{2\pi} \nu^{-1}(a,\varphi)\cos 2n\varphi \,\mathrm{d}\varphi,$$

$$\lambda = \alpha a^2/4(\omega^2 + 3\alpha a^2/4). \tag{2}$$

The averaged Itô equations for system (1) can be obtained by using the procedure developed in [22]. Two cases are considered in the following:

*Case 1: Primary external resonance*. In this case,

$$\Omega/\omega(a) = 1 + \sigma, \tag{3}$$

where $\omega(a) = 1/C_0(a)$ is averaged frequency of conserved Duffing oscillator and $\sigma$ is a small parameter denoting detuning. By using the generalized Van der Pol transformations, Equation (3) is converted into

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= F_1(A,\Phi,\Omega t) + h_{11}(A,\Phi)\xi_1(t) + h_{12}(A,\Phi)\xi_2(t),\\ \frac{\mathrm{d}\Theta}{\mathrm{d}t} &= F_2(A,\Phi,\Omega t) + h_{21}(A,\Phi)\xi_1(t) + h_{22}(A,\Phi)\xi_2(t),\end{aligned} \tag{4}$$

where

$$\begin{aligned}F_1 &= -\frac{A}{g(A)}[\beta A\nu(A,\Phi)\sin\Phi + E\cos\Omega t]\nu(A,\Phi)\sin\Phi,\\ F_2 &= -\frac{1}{g(A)}[\beta A\nu(A,\Phi)\sin\Phi + E\cos\Omega t]\nu(A,\Phi)\cos\Phi,\end{aligned}$$

$$
\begin{aligned}
h_{11} &= -\frac{A}{g(A)}\nu(A,\Phi)\sin\Phi, \quad h_{12} = -\frac{A^2}{g(A)}\nu(A,\Phi)\sin\Phi\cos\Phi, \\
h_{21} &= -\frac{1}{g(A)}\nu(A,\Phi)\cos\Phi, \quad h_{22} = -\frac{A}{g(A)}\nu(A,\Phi)\cos^2\Phi. \qquad (5)
\end{aligned}
$$

Equation (4) can be modeled as the following Itô stochastic differential equations by adding Wong–Zakai correction terms:

$$
\begin{aligned}
\mathrm{d}A &= m_1(A,\Phi,\Omega t)\,\mathrm{d} + \sigma_{1r}(A,\Phi)\,\mathrm{d}B_r(t), \\
\mathrm{d}\Theta &= m_2(A,\Phi,\Omega t)\,\mathrm{d} + \sigma_{2r}(A,\Phi)\,\mathrm{d}B_r(t), \quad r = 1,2. \qquad (6)
\end{aligned}
$$

where

$$
\begin{aligned}
m_i &= F_i + D_k\left(h_{1k}\frac{\partial h_{ik}}{\partial A} + h_{2k}\frac{\partial h_{ik}}{\partial \Phi}\right), \\
b_{ij} &= \sigma_{ir}\sigma_{jr} = 2D_k h_{ik} h_{jk}, \quad i,j,k = 1,2. \qquad (7)
\end{aligned}
$$

Introducing new variable

$$
\Gamma = \sigma\tau - \Theta, \qquad (8)
$$

Equation (6) is transformed into

$$
\begin{aligned}
\mathrm{d}A &= m_1(A,\Phi,\Psi+\Gamma)\,\mathrm{d}t + \sigma_{1r}(A,\Phi)\,\mathrm{d}B_r(t), \\
\mathrm{d}\Gamma &= [-m_2(A,\Phi,\Psi+\Gamma) + (\Omega/\omega(A)-1)\nu(A,\Phi)]\,\mathrm{d}t - \sigma_{2r}(A,\Phi)\,\mathrm{d}B_r(t). \qquad (9)
\end{aligned}
$$

Averaging the drift and diffusion coefficients in Itô equation (9) with respect to $\Phi$ leads to

$$
\begin{aligned}
\mathrm{d}\bar{A} &= \bar{m}_1(\bar{A},\bar{\Gamma})\,\mathrm{d}t + \bar{\sigma}_{1r}(\bar{A})\,\mathrm{d}B_r(t), \\
\mathrm{d}\bar{\Gamma} &= \bar{m}_2(\bar{A},\bar{\Gamma})\,\mathrm{d}t + \bar{\sigma}_{2r}(\bar{A})\,\mathrm{d}B_r(t), \quad r = 1,2, \qquad (10)
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{m}_1(\bar{A},\bar{\Gamma}) &= -\beta\bar{A}(\omega^2 + 5\alpha\bar{A}^2/8)/2(\omega^2+\alpha\bar{A}^2) + E\sin\bar{\Gamma} \\
&\quad\times \left\langle \nu(\bar{A},\Phi)\sin\Phi\sin\left(\Phi + \Omega\sum_{n=1}^{\infty}\frac{1}{n}C_n(\bar{A})\sin n\Phi\right)\right\rangle_{\Phi} \Big/ (\omega^2+\alpha\bar{A}^2) \\
&\quad - \alpha D_1\bar{A}(3\omega^2 + 3\alpha\bar{A}^2/2)/4(\omega^2+\alpha\bar{A}^2)^3 \\
&\quad + D_1(\omega^2 + 7\alpha\bar{A}^2/8)/2\bar{A}(\omega^2+\alpha\bar{A}^2)^2 \\
&\quad + D_2\omega^2\bar{A}(\omega^2 + \alpha\bar{A}^2/2)/8(\omega^2+\alpha\bar{A}^2)^3 \\
&\quad + D_2\bar{A}(\omega^2 + 7\alpha\bar{A}^2/8)/4(\omega^2+\alpha\bar{A}^2)^2, \\
\bar{m}_2(\bar{A},\bar{\Gamma}) &= E\cos\bar{\Gamma} \\
&\quad\times \left\langle \nu(\bar{A},\Phi)\cos\Phi\cos\left(\Phi + \Omega\sum_{n=1}^{\infty}\frac{1}{n}C_n(\bar{A})\sin n\Phi\right)\right\rangle_{\Phi} \Big/ \bar{A}(\omega^2+\alpha\bar{A}^2) \\
&\quad + [\Omega C_0(\bar{A}) - 1]\langle\nu(\bar{A},\Phi)\rangle_{\Phi},
\end{aligned}
$$

$$
\begin{aligned}
\bar{b}_{11}(\bar{A}) &= \bar{\sigma}_{1r}\bar{\sigma}_{1r} \\
&= D_1(\omega^2 + 5\alpha\bar{A}^2/8)/(\omega^2 + \alpha\bar{A}^2)^2 + D_2 A^2(\omega^2 + 3\alpha\bar{A}^2/4)/4(\omega^2 + \alpha\bar{A}^2)^2,
\end{aligned}
$$

$$
\begin{aligned}
\bar{b}_{22}(\bar{A}) &= \bar{\sigma}_{2r}\bar{\sigma}_{2r} \\
&= D_1(\omega^2 + 7\alpha\bar{A}^2/8)/\bar{A}^2(\omega^2 + \alpha\bar{A}^2)^2 \\
&\quad + 2D_2(3\omega^2/8 + 11\alpha\bar{A}^2/32)/(\omega^2 + \alpha\bar{A}^2)^2,
\end{aligned}
$$

$$
\bar{b}_{12}(\bar{A}) = \bar{b}_{21}(\bar{A}) = \bar{\sigma}_{1r}\bar{\sigma}_{2r} = 0. \tag{11}
$$

*Case 2: Primary parametric resonance*. In this case,

$$
\Omega/\omega(a) = 2 + \sigma. \tag{12}
$$

Introduce new variable

$$
\Gamma = \sigma\tau - 2\Theta. \tag{13}
$$

A similar derivation leads to the following averaged Itô equations:

$$
\begin{aligned}
\mathrm{d}\bar{A} &= \bar{m}_1(\bar{A}, \bar{\Gamma})\,\mathrm{d}t + \bar{\sigma}_{1r}(\bar{A})\,\mathrm{d}B_r t, \\
\mathrm{d}\bar{\Gamma} &= \bar{m}_2(\bar{A}, \bar{\Gamma})\,\mathrm{d}t + \bar{\sigma}_{2r}(\bar{A})\,\mathrm{d}B_r t, \quad r = 1, 2,
\end{aligned} \tag{14}
$$

where

$$
\begin{aligned}
\bar{m}_1(\bar{A}, \bar{\Gamma}) &= -\beta\bar{A}(\omega^2 + 5\alpha\bar{A}^2/8)/2(\omega^2 + \alpha\bar{A}^2) + E\sin\bar{\Gamma} \\
&\quad \times \left\langle \nu(\bar{A}, \Phi)\sin\Phi\sin\left(2\Phi + \Omega\sum_{n=1}^{\infty}\frac{1}{n}C_n(\bar{A})\sin n\Phi\right)\right\rangle_{\Phi} \Big/ (\omega^2 + \alpha\bar{A}^2) \\
&\quad - \alpha D_1\bar{A}(3\omega^2 + 3\alpha\bar{A}^2/2)/4(\omega^2 + \alpha\bar{A}^2)^3 \\
&\quad + D_1(\omega^2 + 7\alpha\bar{A}^2/8)/2\bar{A}(\omega^2 + \alpha\bar{A}^2)^2 \\
&\quad + D_2\omega^2\bar{A}(\omega^2 + \alpha\bar{A}^2/2)/8(\omega^2 + \alpha\bar{A}^2)^3 \\
&\quad + D_2\bar{A}(\omega^2 + 7\alpha\bar{A}^2/8)/4(\omega^2 + \alpha\bar{A}^2)^2,
\end{aligned}
$$

$$
\begin{aligned}
\bar{m}_2(\bar{A}, \bar{\Gamma}) &= 2E\cos\bar{\Gamma} \\
&\quad \times \left\langle \nu(\bar{A}, \Phi)\cos\Phi\cos\left(2\Phi + \Omega\sum_{n=1}^{\infty}\frac{1}{n}C_n(\bar{A})\sin n\Phi\right)\right\rangle_{\Phi} \Big/ \bar{A}(\omega^2 + \alpha\bar{A}^2) \\
&\quad + [\Omega C_0(\bar{A}) - 2]\langle \nu(\bar{A}, \Phi)\rangle_{\Phi},
\end{aligned}
$$

$$
\begin{aligned}
\bar{b}_{11}(\bar{A}) &= \bar{\sigma}_{1r}\bar{\sigma}_{1r} \\
&= D_1(\omega^2 + 5\alpha\bar{A}^2/8)/(\omega^2 + \alpha\bar{A}^2)^2 + D_2\bar{A}^2(\omega^2 + 3\alpha\bar{A}^2/4)/4(\omega^2 + \alpha\bar{A}^2)^2,
\end{aligned}
$$

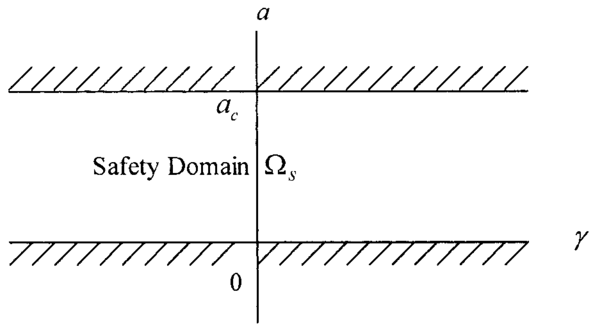


*Figure 1.* Safety domain.

$$\begin{aligned}
\bar{b}_{22}(\bar{A}) &= \bar{\sigma}_{2r}\bar{\sigma}_{2r} \\
&= 4D_1(\omega^2 + 7\alpha\bar{A}^2/8)/\bar{A}^2(\omega^2+\alpha\bar{A}^2)^2 \\
&\quad + 8D_2(3\omega^2/8 + 11\alpha\bar{A}^2/32)/(\omega^2+\alpha\bar{A}^2)^2, \\
\bar{b}_{12}(\bar{A}) &= \bar{b}_{21}(\bar{A}) = \bar{\sigma}_{1r}\bar{\sigma}_{2r} = 0.
\end{aligned} \tag{15}$$

## 3. Backward Kolmogorov Equation and Generalized Pontryagin Equations

$A(t)$ is the displacement amplitude of system (1). It is reasonable to assume that the first-passage failure occurs once $A(t)$ exceed certain critical value $A_c$ for the first time. In phase plane $(a, \gamma)$, the safe domain $\Omega_s$ is inside of the two parallel lines $a = 0$ and $a = a_c$ (Figure 1). The conditional reliability function, denoted by $R(t \mid a_0, \gamma_0)$, is defined as the probability of $(A(t), \Gamma(t))$ being in safety domain $\Omega_s$ within interval $(0, t]$ given initial state $(a_0, \gamma_0)$ being in $\Omega_s$, i.e.,

$$R(t \mid a_0, \gamma_0) = P\big\{(A(\tau), \Gamma(\tau)) \in \Omega_s,\ \tau \in (0, t] \mid (a_0, \gamma_0) \in \Omega_s\big\}. \tag{16}$$

It is the integral of the conditional transition probability density in $\Omega_s$. The conditional transition probability density is the transition probability density of the sample functions which remain in safety domain $\Omega_s$ in all time interval $(0, t]$. For diffusion process $[\bar{A}, \bar{\Gamma}]^T$, the conditional transition probability density is governed by the backward Kolmogorov equation with drift and diffusion coefficients defined by Equation (11) for Case 1 or Equation (15) for Case 2. Thus, the conditional reliability function is governed by the following backward Kolmogorov equation:

$$\frac{\partial R}{\partial t} = \alpha_1\frac{\partial R}{\partial a_0} + \alpha_2\frac{\partial R}{\partial \gamma_0} + \frac{1}{2}\beta_{11}\frac{\partial^2 R}{\partial^2 a_0} + \beta_{12}\frac{\partial^2 R}{\partial a_0 \partial \gamma_0} + \frac{1}{2}\beta_{22}\frac{\partial^2 R}{\partial \gamma_0^2}, \tag{17}$$

where

$$\begin{aligned}
\alpha_i &= \alpha_i(a_0, \gamma_0) = \bar{m}_i\ (\bar{A}, \bar{\Gamma})\big|_{\bar{A}=a_0, \bar{\Gamma}=\gamma_0}, \\
\beta_{ij} &= \beta_{ij}(a_0) = \bar{\sigma}_{ir}(\bar{A})\bar{\sigma}_{jr}\ (\bar{A})\big|_{\bar{A}=a_0}.
\end{aligned} \tag{18}$$

The initial condition associated with Equation (17) is

$$R(0 \mid a_0, \gamma_0) = 1, \quad a_0 < a_c \tag{19}$$

and the boundary conditions are

$$R(t \mid 0, \gamma_0) = \text{finite}, \tag{20}$$

$$R(t \mid a_c, \gamma_0) = 0, \tag{21}$$

$$R(t \mid a_0, \gamma_0 + 2n\pi) = R(t \mid a_0, \gamma_0). \tag{22}$$

The conditional probability of first-passage failure is

$$P_f(t \mid a_0, \gamma_0) = 1 - R(t \mid a_0, \gamma_0). \tag{23}$$

The conditional probability density of the first-passage time $T$ is then the derivative of $P_t(t \mid a_0, \gamma_0)$, i.e.,

$$p(T \mid a_0, \gamma_0) = \left.\frac{\partial P_f}{\partial t}\right|_{t=T} = -\left.\frac{\partial R}{\partial t}\right|_{t=T}. \tag{24}$$

The conditional moments of the first passage time are defined as

$$\mu_n(a_0, \gamma_0) = \int_0^\infty T^n p(T \mid a_0, \gamma_0)\, \mathrm{d}T = n \int_0^\infty T^{n-1} R(T \mid a_0, \gamma_0)\, \mathrm{d}T, \quad n = 1, 2, \ldots. \tag{25}$$

It can be shown by using Equations (17), (24) and (25) that the conditional moments of the first-passage time are governed by the following generalized Pontryagin equations:

$$\frac{1}{2}\beta_{11}\frac{\partial^2 \mu_n}{\partial a_0^2} + \beta_{12}\frac{\partial^2 \mu_n}{\partial a_0 \gamma_0} + \frac{1}{2}\beta_{22}\frac{\partial^2 \mu_n}{\partial \gamma_0^2} + \alpha_1\frac{\partial \mu_n}{\partial a_0} + \alpha_2\frac{\partial \mu_n}{\partial \gamma_0} = -n\mu_{n-1}, \quad n = 1, 2, \ldots, \tag{26}$$

where $\alpha_i$ and $\beta_{ij}$ are defined by Equation (18). The boundary conditions associated with Equation (26) are obtained from Equations (20–22) as

$$\mu_n(0, \gamma_0) = \text{finite}, \tag{27}$$

$$\mu_n(a_c, \gamma_0) = 0, \tag{28}$$

$$\mu_n(a_0, \gamma_0 + 2n\pi) = \mu_n(a_0, \gamma_0). \tag{29}$$

For $n = 1$, $\mu_1$ is the mean first-passage time and Equation (26) is reduced to Pontryagin equation

$$\frac{1}{2}\beta_{11}\frac{\partial^2 \mu_1}{\partial a_0^2} + \beta_{12}\frac{\partial^2 \mu_1}{\partial a_0 \partial \gamma_0} + \frac{1}{2}\beta_{22}\frac{\partial^2 \mu_1}{\partial \gamma_0^2} + \alpha_1\frac{\partial \mu_1}{\partial a_0} + \alpha_2\frac{\partial \mu_1}{\partial \gamma_0} = -1. \tag{30}$$

To obtain the probability and statistics of the first-passage time, one has to solve backward Kolmogorov equation (17) with initial and boundary conditions (19–22), or to solve generalized Pontryagin equations (26) with boundary conditions (27–29). Generally, they can be solved only numerically, e.g., by using finite difference method.

Note that boundary condition (20) or (27) is qualitative and has to be made to be quantitative for solving Equation (17) or (26) numerically. In Case 1,

$$\alpha_1, \alpha_2, \beta_{22} \to \infty \quad \text{as} \quad a_0 \to 0. \tag{31}$$

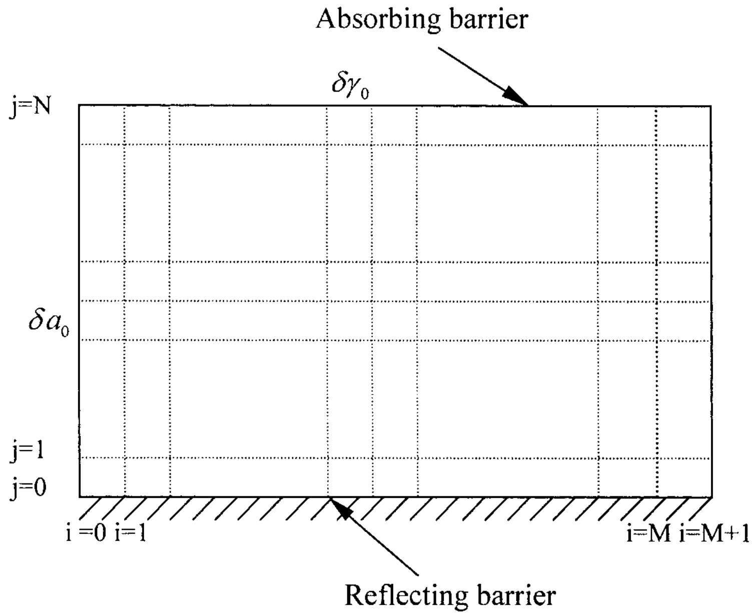


*Figure 2.* Rectangular mesh for finite difference scheme.

To satisfy boundary condition (20), it is necessary that

$$\frac{\partial R}{\partial a_0}, \frac{\partial R}{\partial \gamma_0}, \frac{\partial^2 R}{\partial \gamma_0^2} \to 0 \quad \text{as} \quad a_0 \to 0. \tag{32}$$

Similarly, boundary condition (27) should be replaced by

$$\frac{\partial \mu_n}{\partial a_0}, \frac{\partial \mu_n}{\partial \gamma_0}, \frac{\partial^2 \mu_n}{\partial \gamma_0^2} \to 0 \quad \text{as} \quad a_0 \to 0. \tag{33}$$

The backward Kolmogorov equation (17) with drift and diffusion coefficients in Equation (11) and with initial condition (19) and boundary conditions (21), (22) and (32) is solved by using the implicit finite difference method of alternate direction type. $\alpha_0$ and $\gamma_0$ are discreted as shown in Figure 2. The time axis perpendicular to plane $(\alpha_0, \gamma_0)$ is also discreted. Introduce the following notations:

$$\begin{aligned} R_{i,j}^n &= R(n\delta t \mid j\delta a_0, i\delta\gamma_0), \quad (\alpha_r)_{i,j} = \alpha_r(j\delta a_0, i\delta\gamma_0), \quad (\beta_{rs})_{i,j} = \beta_{rs}(j\delta a_0, i\delta\gamma_0), \\ r.s &= 1, 2; \quad i = 1, 2, \ldots, M; \quad j = 1, 2, \ldots, N; \quad n = 0, 1, \ldots. \end{aligned} \tag{34}$$

The implicit finite difference approximation to Equation (17) with drift and diffusion coefficients in Equation (11) from $n$th time step to $(n + (1/2))$th time step is

$$\begin{aligned} \frac{R_{i,j}^{n+(1/2)} - R_{i,j}^n}{\delta t/2} &= (\alpha_1)_{i,j} \frac{R_{i,j+1}^n - R_{i,j-1}^n}{2\delta a_0} + (\alpha_2)_{i,j} \frac{R_{i+1,j}^{n+(1/2)} - R_{i-1,j}^{n+(1/2)}}{2\delta\gamma_0} \\ &\quad + \frac{1}{2}(\beta_{11})_{i,j} \frac{R_{i,j+1}^n - 2R_{i,j}^n + R_{i,j-1}^n}{(\delta a_0)^2} \\ &\quad + \frac{1}{2}(\beta_{22})_{i,j} \frac{R_{i+1,j}^{n+(1/2)} - 2R_{i,j}^{n+(1/2)} + R_{i-1,j}^{n+(1/2)}}{(\delta\gamma_0)^2}. \end{aligned} \tag{35}$$

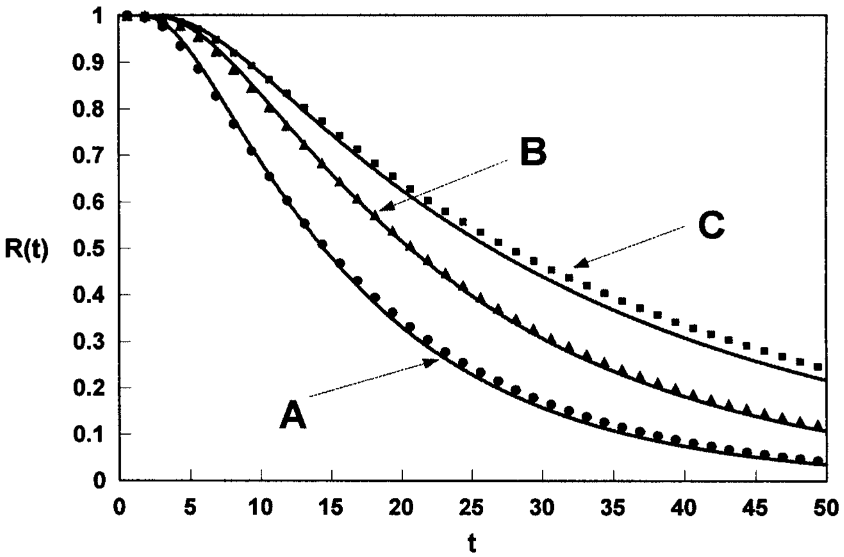


*Figure 3.* Reliability function of system (1) in primary external resonance. $\omega = 1.0$, $\Omega = 1.0$, $\alpha = 0.6$, $E = 0.01$, $D_2 = 0.04$, $a_0 = 0.24$, $\gamma_0 = 0.1257$. (A) $D_1 = 0.03$, $\beta = 0.01$; (B) $D_1 = 0.02$, $\beta = 0.01$; (C) $D_1 = 0.02$, $\beta = 0.05$; — analytical result; ● ▲ ■ result from digital simulation.

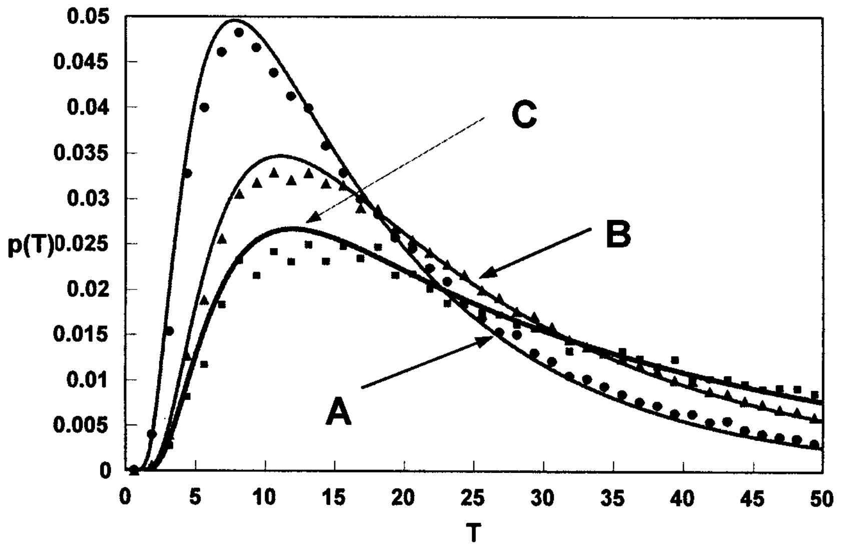


*Figure 4.* Probability density of first-passage time of system (1) in primary external resonance. The parameters are the same as those in Figure 3; — analytical result; ● ▲ ■ result from digital simulation.

The associated boundary conditions are approximated as

$$R_{i,N}^{n+(1/2)} = 0, \quad R_{i,0}^{n+(1/2)} = R_{i,1}^{n+(1/2)}, \quad R_{0,j}^{n+(1/2)} = R_{M,j}^{n+(1/2)}, \quad R_{1,j}^{n+(1/2)} = R_{M+1,j}^{n+(1/2)}. \tag{36}$$

The implicit finite difference approximation to Equation (17) with drift and diffusion coefficients in Equation (11) from $(n + (1/2))$th time step to $(n + 1)$th time step is

$$\frac{R_{i,j}^{n+1} - R_{i,j}^{n+(1/2)}}{\delta t/2} = (\alpha_1)_{i,j} \frac{R_{i,j+1}^{n+1} - R_{i,j-1}^{n+1}}{2\delta a_0} + (\alpha_2)_{i,j} \frac{R_{i+1,j}^{n+(1/2)} - R_{i-1,j}^{n+(1/2)}}{2\delta \gamma_0}$$

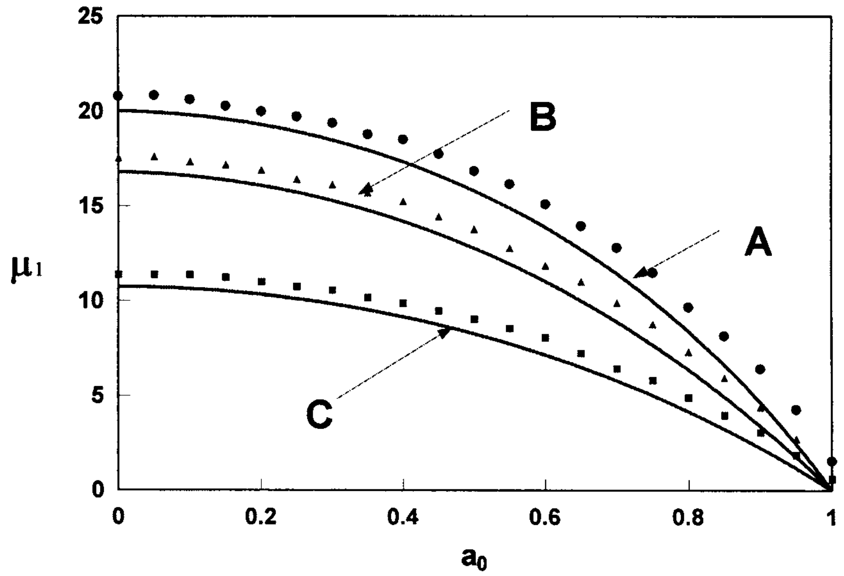


*Figure 5.* Mean first-passage time of system (1) in primary external resonance. $\omega = 1.2$, $\Omega = 1.2$, $\alpha = 0.7$, $E = 0.01$, $D_2 = 0.04$, $\gamma_0 = 2.514$. (A) $D_1 = 0.05$, $\beta = 0.05$; (B) $D_1 = 0.05$, $\beta = 0.01$; (C) $D_1 = 0.08$, $\beta = 0.01$; — analytical result; ● ▲ ■ result from digital simulation.

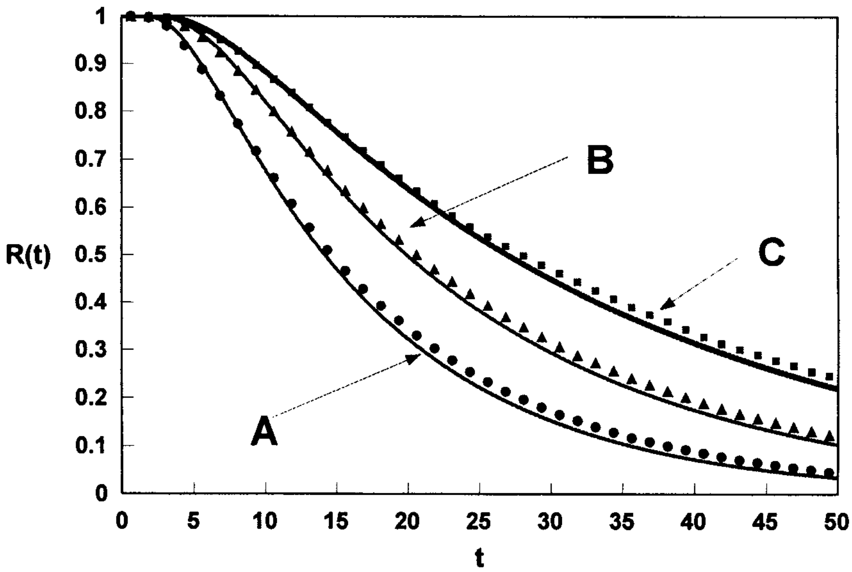


*Figure 6.* Reliability function of system (1) in primary external resonance $\Omega = 1.2$ and the other parameters are the same as those in Figure 3; — analytical result; ● ▲ ■ result from digital simulation.

$$+\frac{1}{2}(\beta_{11})_{i,j}\frac{R_{i,j+1}^{n+1} - 2R_{i,j}^{n+1} + R_{i,j-1}^{n+1}}{(\delta a_0)^2}$$
$$+\frac{1}{2}(\beta_{22})_{i,j}\frac{R_{i+1,j}^{n+(1/2)} - 2R_{i,j}^{n+(1/2)} + R_{i-1,j}^{n+(1/2)}}{(\delta \gamma_0)^2}. \tag{37}$$

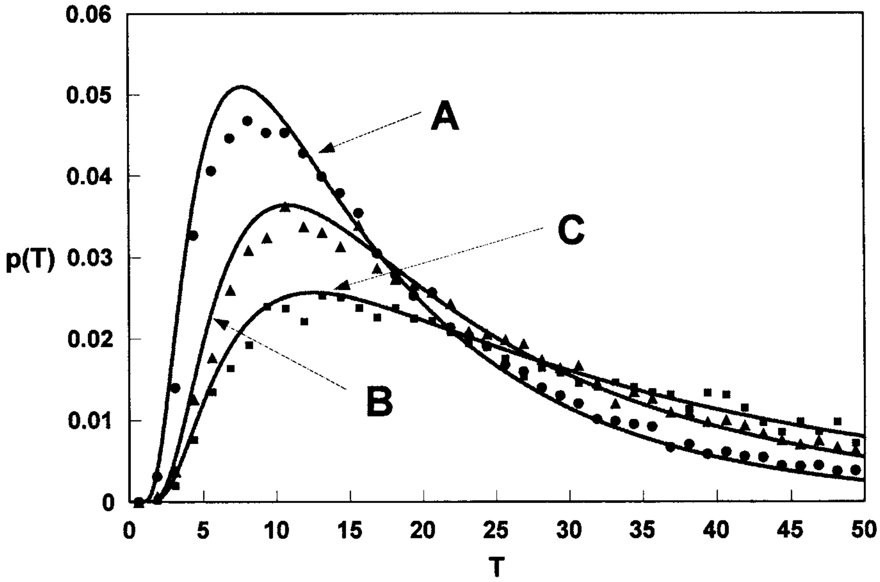


*Figure 7.* Probability density of first-passage time of system (1) in primary external resonance. The parameters are the same as those in Figure 6; — analytical result; ● ▲ ■ result from digital simulation.

The associated boundary conditions are approximated as

$$R_{i,N}^{n+1} = 0, \quad R_{i,0}^{n+1} = R_{i,1}^{n+1}, \quad R_{0,j}^{n+1} = R_{M,j}^{n+1}, \quad R_{1,j}^{n+1} = R_{M+1,j}^{n+1}. \tag{38}$$

Equations (35) and (37) are of tri-diagonal form and they together with boundary conditions (36), (38) and initial condition $R_{i,j}^0 = 1$ can be solved efficiently by using the modified standard Thomas algorithm.

Let $\mu_{i,j}^1 = \mu_1(j\delta a_0, i\delta\gamma_0)$. The finite difference approximation to Equation (30) with drift and diffusion coefficients in Equation (11) is

$$(\alpha_1)_{i,j}\frac{\mu_{i,j+1}^1 - \mu_{i,j-1}^1}{2\delta a_0} + (\alpha_2)_{i,j}\frac{\mu_{i+1,j}^1 - \mu_{i-1,j}^1}{2\delta\gamma_0} + \frac{1}{2}(\beta_{11})_{i,j}\frac{\mu_{i,j+1}^1 - 2\mu_{i,j}^1 + \mu_{i,j-1}^1}{(\delta a_0)^2} + \frac{1}{2}(\beta_{22})_{i,j}\frac{\mu_{i+1,j}^1 - 2\mu_{i,j}^1 + \mu_{i-1,j}^1}{(\delta\gamma_0)^2} = -1. \tag{39}$$

The associated boundary conditions are approximated as

$$\mu_{i,N}^1 = 0, \quad \mu_{i,0}^1 = \mu_{i,1}^1, \quad \mu_{0,j}^1 = \mu_{M,j}^1, \quad \mu_{1,j}^1 = \mu_{M+1,j}^1. \tag{40}$$

Equation (39) and conditions (40) can be solved by using successive over-relaxation method.

Case 2 can be treated similarly. The only difference is that the drift and diffusion coefficients in Equation (15) rather than those in Equation (11) are used.

Some numerical results for the conditional reliability function, the conditional probability density and mean of the first-passage time for system (1) for both primary external resonance and primary parametric resonance are obtained and shown in Figures 3–14 with solid lines. To assess the validity and accuracy of the proposed procedure, corresponding results are obtained from digital simulation and shown in Figures 3–14 using symbols ▲, ●, ■. It is seen that

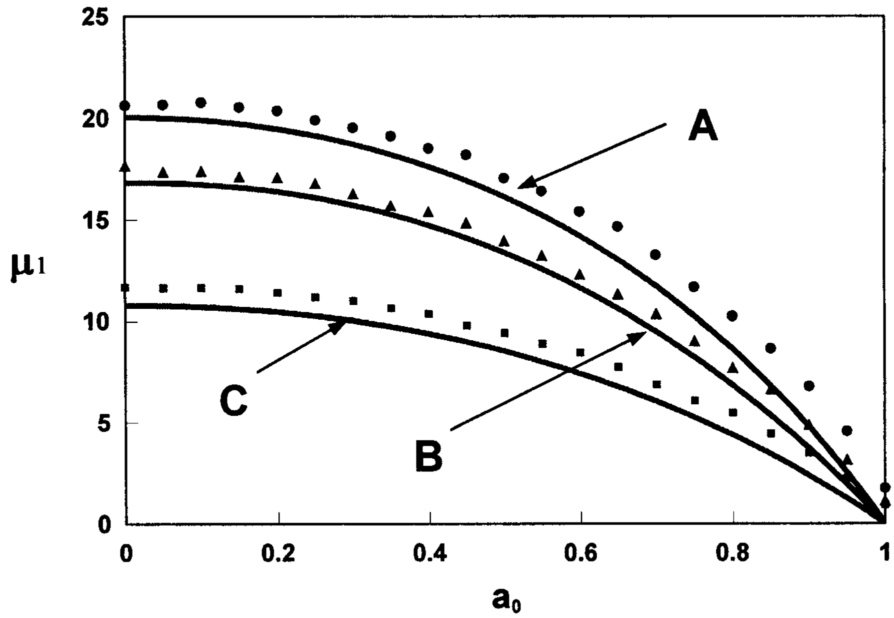


*Figure 8.* Mean first-passage time of system (1) in primary external resonance $\Omega = 1.4$ and the other parameters are the same as those in Figure 5.

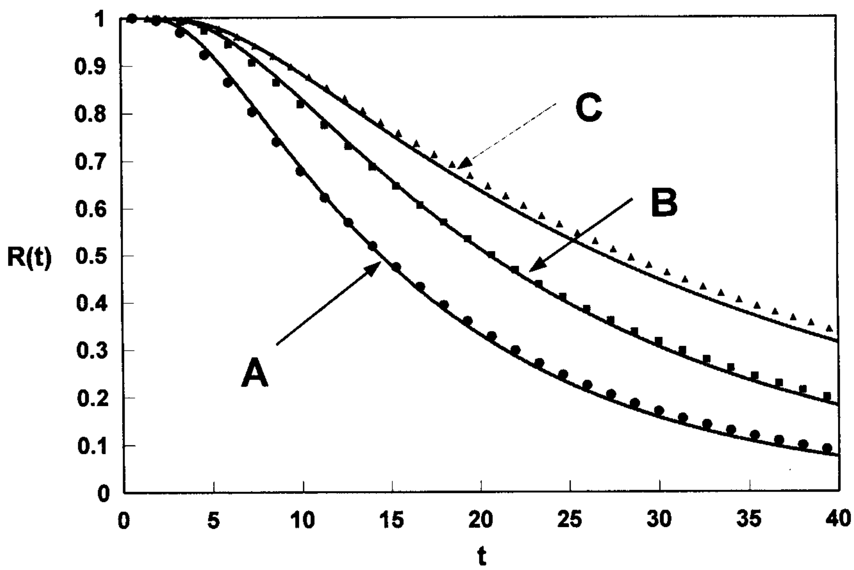


*Figure 9.* Reliability function of system (1) in primary parametric resonance. $\omega = 1.0$, $\Omega = 2.0$, $\alpha = 0.6$, $E = 0.01$, $D_2 = 0.04$, $a_0 = 0.24$, $\gamma_0 = 1.257$. (A) $D_1 = 0.03$, $\beta = 0.01$; (B) $D_1 = 0.02$, $\beta = 0.01$; (C) $D_1 = 0.02$, $\beta = 0.05$; — analytical result; ● ▲ ■ result from digital simulation.

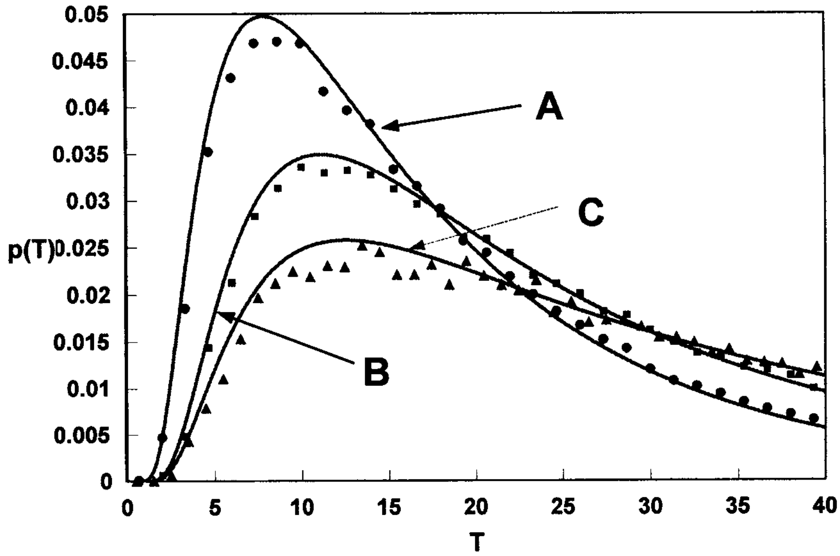


*Figure 10.* Probability density of first-passage time of system (1) in primary parametric resonance. The parameters are the same as those of Figure 9; — analytical result; ● ▲ ■ result from digital simulation.

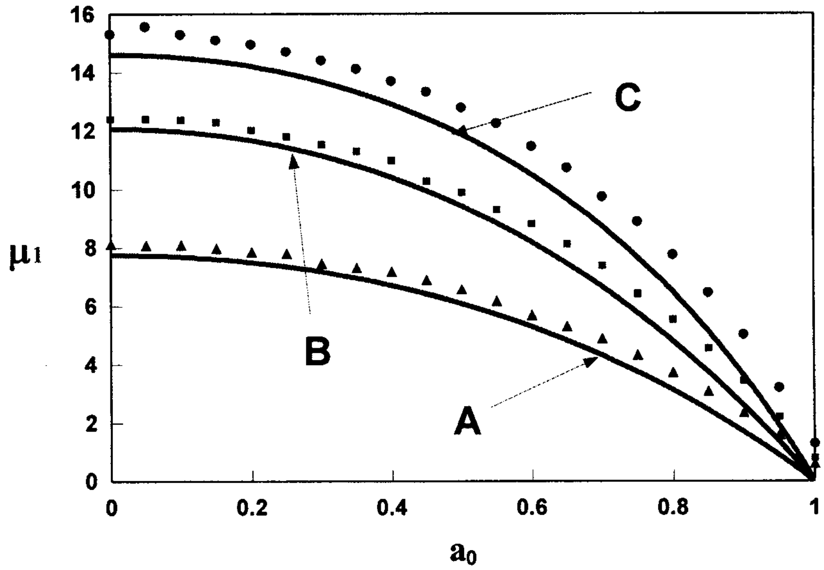


*Figure 11.* Mean first-passage time of system (1) in primary parametric resonance. $\omega = 1.0$, $\Omega = 2.0$, $\alpha = 0.6$, $E = 0.01$, $D_2 = 0.04$, $\gamma_0 = 3.771$. (A) $D_1 = 0.08$, $\beta = 0.01$; (B) $D_1 = 0.05$, $\beta = 0.01$; (C) $D_1 = 0.05$, $\beta = 0.07$; — analytical result; ● ▲ ■ result from digital simulation.

the analytical results agree well with simulation results and they are slightly conservative compared with simulation results. Figures 3, 6, 9, 12 show that the reliability function is a monotonously decreasing function of time while Figures 5, 8, 11, 14 show that the mean first-passage time is a monotonously decreasing function of initial amplitude. This observation is significant in the studying stochastic optimal control of the system with objectives of maximum reliability and maximum mean first passage time. It is also seen from Figures 3–14

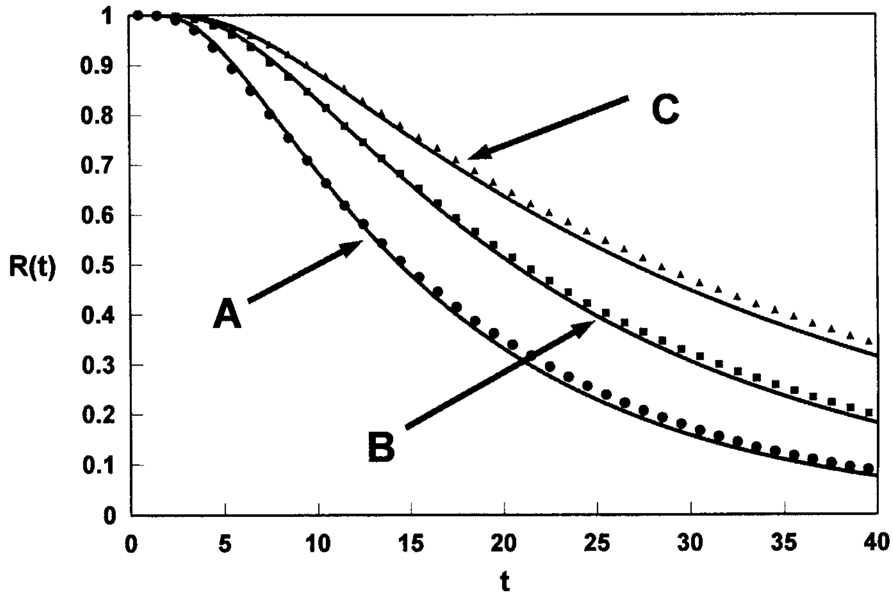


*Figure 12.* Reliability function of system (1) in primary parametric resonance $\Omega = 2.2$ and the other parameters are the same as those in Figure 9; — analytical result; ● ▲ ■ result from digital simulation.

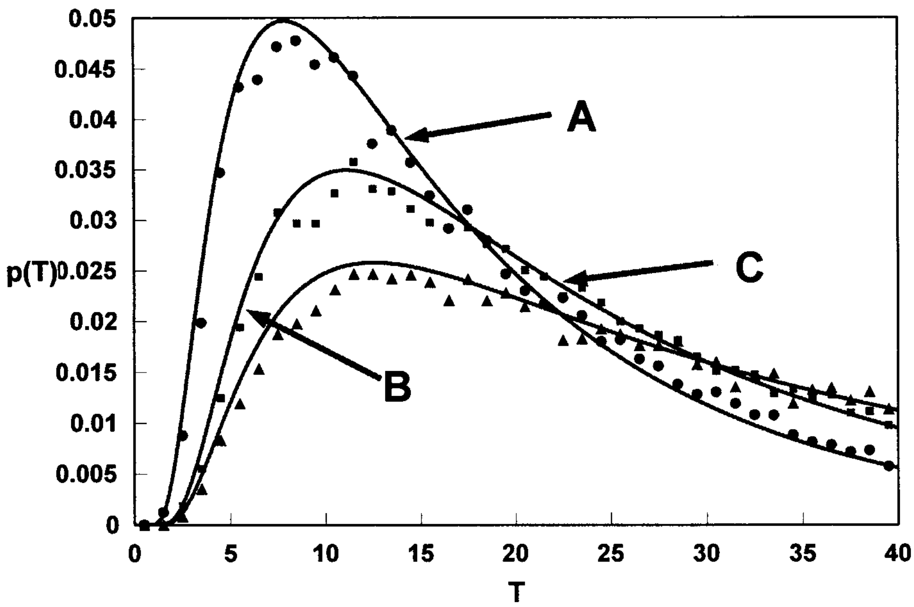


*Figure 13.* Probability density of first-passage time of system (1) in primary parametric resonance. The parameters are the same as those in Figure 12; — analytical result; ● ▲ ■ result from digital simulation.

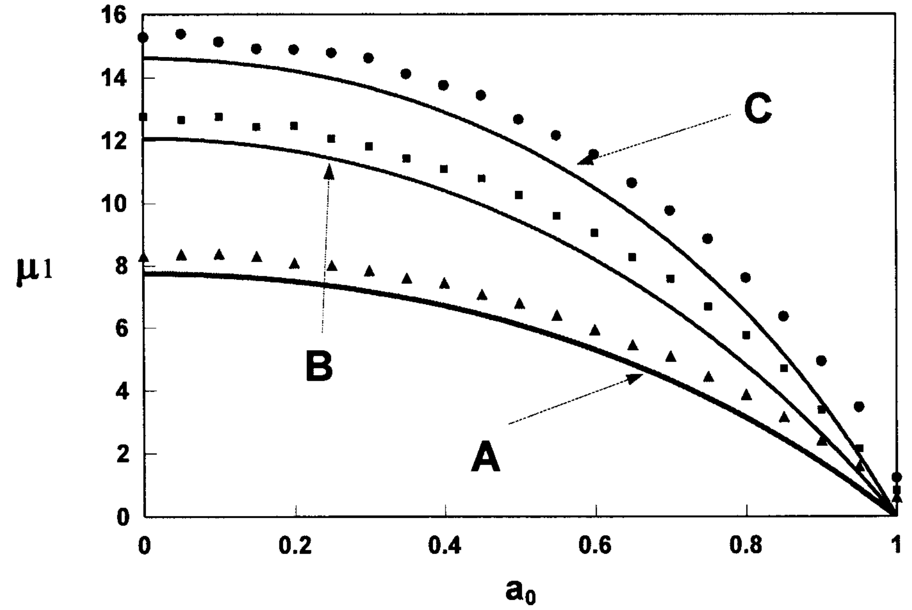


*Figure 14.* Mean first-passage time of system (1) in primary parametric resonance. $\Omega = 2.2$ and the other parameters are the same as those in Figure 11; — analytical result; ● ▲ ■ result from digital simulation.

that all the results depend strongly on the excitation intensity and damping while from the comparisons between Figures 3 and 6, 4 and 7, 5 and 8, 9 and 12, 10 and 13, 11 and 14 that all the results are not very sensitive to the change in detuning parameter.

## 4. Conclusions

In the present paper the first-passage time of Duffing oscillator under combined harmonic and white noise excitations has been investigated. The stochastic averaging method for strongly nonlinear oscillators under combined harmonic and white-noise excitations has been applied to reduce the equation of motion of the system to the averaged Itô equations for homogenous diffusion processes $\bar{A}(t)$ and $\bar{\Gamma}(t)$. The backward Kolmogorov equation for the conditional reliability function and the generalized Pontryagin equations for moments of first-passage time have been established from the averaged Itô equations. These equations have been solved numerically by using finite difference method and numerical results have been obtained for primary external and parametric resonance cases. Analytical results have been verified by using the results obtained from digital simulation. The results show that reliability function is a monotonously decreasing function of time and mean first-passage time is a monotonously decreasing function of initial amplitude. All the results depend strongly on the magnitudes of excitation intensity and damping while they are not very sensitive to the change in detuning parameter.

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