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SHORT-AND RESONANT-RANGE INTERACTIONS BETWEEN SCALES IN TURBULENCE AND THEIR APPLICATIONS *

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Abstract: *Interactions between different scales in turbulence were studied starting from the incompressible Navier-Stokes equations. The integral and differential formulae of the short-range viscous stresses, which express the short-range interactions between contiguous scales in turbulence, were given. A concept of the resonant-range interactions between extreme contiguous scales was introduced and the differential formula of the resonant-range viscous stresses was obtained. The short- and resonant-range viscous stresses were applied to deduce the large-eddy simulation (LES) equations as well as the multiscale equations, which are approximately closed and do not contain any empirical constants or relations. The properties and advantages of using the multiscale equations to compute turbulent flows were discussed. The short-range character of the interactions between the scales in turbulence means that the multiscale simulation is a very valuable technique for the calculation of turbulent flows. A few numerical examples were also given.*

Key words: turbulence; interacting scale; eddy viscosity; short-range viscous stress; resonant-range viscous stress; multiscale equation

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Introduction

Turbulent flow contains a wide range of time- and length-scales. The nonlinear interactions between different scales play a key role in the evolution of the turbulent flow. In the theoretical researches and engineering calculations for the turbulent flows, the concept of eddy-viscosity, which was presented and extended by J. Boussinesq, G. Taylor and L. Prandtl, has been widely used in the last hundred years[1,2]. In the eddy-viscosity theory, the virtual turbulent-eddy motions are analogized to gas molecular motions, therefore, the eddy-viscosity should be originated from the interactions between widely separated scales in turbulence. However, it is generally believed that the dominant interactions are between contiguous, rather than widely separated, scales[3]. This subject has been studied in many papers. For example, the "direct

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interaction" theories presented by R. Kraichnan confirm that at sufficiently large Reynolds number the energy transfer is mainly amongst traids for which three Fourier components satisfy $k \approx k' \approx k - k'$ [4]. Some new results for nonlinear interactions between the scales had been acquired recently through the analysis of direct numerical simulation databases for the incompressible channel flow by J. Domaradzki *et al.* [5,6]. Their main results are as follows: i) The nonlinear dynamics of the resolved modes with wave number $k < k_1$ are governed by their interactions with a limited range of modes with wave number not exceeding $2k_1$ and much smaller scales have a negligible effect on the resolved ones; ii) The nonlinear dynamics of the modes with wave number ranging from k_1 to $2k_1$ are largely determined by their interactions with the resolved scales with wave numbers $k < k_1$. Both of the large-small scales equations for turbulence calculation developed by Gao-Zhuang[7~9] and the multiscale method in turbulence presented by T. Hughes *et al.* [10] are built undoubtedly on the basis of the interactions between the scales (wave numbers) are mainly those between contiguous scales (wave numbers).

The nonlinear interactions between the scales in turbulence are further analyzed starting from the incompressible Navier-Stokes equations. The dominant interactions were proved to be short-range ones between contiguous scales. A concept of resonant-range interactions between extreme contiguous scales was introduced. Three integral and differential formulae of the short- and resonant-range viscous stresses were given. With these formulae a new large eddy simulation (LES) equations and the multiscale equations governing both the motions of large eddies and the fluctuation motions of contiguous small scales relating to the large ones were presented.

1 Short-Range Interactions Between the Scales in Turbulence

The short-range interactions between contiguous scales in turbulence can be vividly expressed as: the turbulent viscous stresses of the scale-range with scales $\Delta x < \Delta x_c, \Delta y < \Delta y_c$ and $\Delta z < \Delta z_c$ (for short, with scales $\Delta x < \Delta x_c$, the same below) acting on the large scale-range with scales $\Delta x > \Delta x_c$ are mainly supplied by a limited range of scales lying between Δx_f and Δx_c, where $\Delta x_f < \Delta x_c$ (see Fig. 1). The procedure of proving the proposition of the short-range interactions is as follows: taking the space averaged system of the incompressible Navier-Stokes (NS) equations, deducing the differential formulae of turbulent viscous stresses, and then the proposition of short-range interactions may be proved and both the integral and differential formulae of short-range viscous stresses can be deduced; introducing a concept of resonant-range interactions and giving the differential formula of resonant-range viscous stresses.

The incompressible Navier-Stokes(NS) equations are

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{1}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 u_i}{\partial x_j \partial x_j}, \tag{2}$$

where $Re = \dfrac{U_0 L}{\nu}$, u_i, x_i, t and p are made dimensionless with reference to the free-stream velocity U_0, the boundary characteristic length $L, L/U_0$ and ρU_0^2, respectively. The space averaged (or call the box-filtered) system of the incompressible NS Eqs. (1) and (2) can be written as

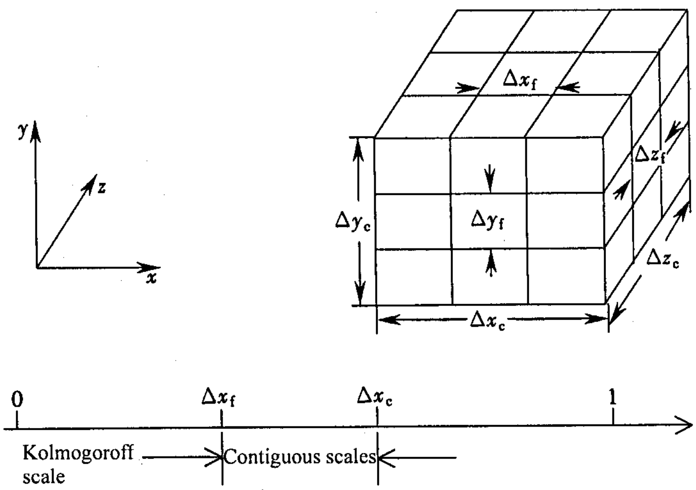

Fig.1 A sketch of interactions of between the scales in turbulence (the scales are made dimensionless with reference to the boundary characteristic length)

$$\frac{\partial U_{ci}}{\partial x_i} = 0, \tag{3}$$

$$\frac{\partial U_{ci}}{\partial t} + U_{cj}\frac{\partial U_{ci}}{\partial x_j} = -\frac{\partial p_c}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 U_{ci}}{\partial x_j \partial x_j} - F_{ci}(u_i, U_{ci}), \tag{4}$$

where

$$(U_{ci}, p_c) = \frac{1}{V_c}\int_{V_c}(u_i, p)\mathrm{d}v, \qquad V_c = \Delta x_c \Delta y_c \Delta z_c, \tag{5a}$$

$$F_{ci}(u_i, U_{ci}) = \frac{1}{V_c}\int_{V_c}(u_j - U_{cj})\frac{\partial}{\partial x_j}(u_i - U_{ci})\mathrm{d}v. \tag{5b}$$

Here U_{ci}, p_c and $F_{ci}(u_i, U_{ci})$ are respectively the i-component of the space averaged velocity, the space averaged pressure and the i-component of viscous stresses of the small-scale-range with scales $\Delta x < \Delta x_c$ acting on the large one with scales $\Delta x > \Delta x_c$.

Suppose the solutions u_i of the NS Eqs. (1) and (2) be continuity and differential, the derivatives of u_i exist. Taking the center of the small volume element $V_c = \Delta x_c \Delta y_c \Delta z_c$ as the original point of Cartesian coordinates, owing to $\Delta x_c \ll 1, \Delta y_c \ll 1, \Delta z_c \ll 1$ then we have

$$U_{cj} = \frac{1}{V_c}\int u_j \mathrm{d}v = \frac{1}{\Delta x_c \Delta y_c \Delta z_c}\int_{-\Delta x_c/2}^{\Delta x_c/2}\int_{-\Delta y_c/2}^{\Delta y_c/2}\int_{-\Delta z_c/2}^{\Delta z_c/2}\left[u_{j0} + \left(\frac{\partial u_j}{\partial x}\right)_0 x + \left(\frac{\partial u_j}{\partial y}\right)_0 y + \right.$$

$$\left.\left(\frac{\partial u_j}{\partial z}\right)_0 z + \frac{1}{2}\left(\frac{\partial^2 u_j}{\partial x^2}\right)_0 x^2 + \frac{1}{2}\left(\frac{\partial^2 u_j}{\partial y^2}\right)_0 y^2 + \frac{1}{2}\left(\frac{\partial^2 u_j}{\partial z^2}\right)_0 z^2 + \cdots\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z =$$

$$u_{j0} + \frac{1}{24}\left[\left(\frac{\partial^2 u_j}{\partial x^2}\right)_0 \Delta x_c^2 + \left(\frac{\partial^2 u_j}{\partial y^2}\right)_0 \Delta y_c^2 + \left(\frac{\partial^2 u_j}{\partial z^2}\right)_0 \Delta z_c^2\right] + O(\Delta x_c^4, \Delta y_c^4, \cdots), \tag{6a}$$

$$\frac{\partial}{\partial x_j}(U_{cj}) = \left(\frac{\partial u_i}{\partial x_j}\right)_0 + \frac{1}{24}\left[\left(\frac{\partial^3 u_i}{\partial x_j \partial x^2}\right)_0 \Delta x_c^2 + \right.$$

$$\left.\left(\frac{\partial^3 u_i}{\partial x_j \partial y^2}\right)_0 \Delta y_c^2 + \left(\frac{\partial^3 u_i}{\partial x_j \partial z^2}\right)_0 \Delta z_c^2\right] + O(\Delta x_c^4 \cdots). \tag{6b}$$

Using the formulae (6a) and (6b) we deduce

$$F_{ci}(u_i, U_{ci}) = \frac{1}{\Delta x_c \Delta y_c \Delta z_c} \int_{-\Delta x_c/2}^{\Delta x_c/2} \int_{-\Delta y_c/2}^{\Delta y_c/2} \int_{-\Delta z_c/2}^{\Delta z_c/2} \Bigg\{ \Bigg[u_{j0} + \left(\frac{\partial u_j}{\partial x}\right)_0 x + \left(\frac{\partial u_j}{\partial y}\right)_0 y + \left(\frac{\partial u_j}{\partial z}\right)_0 z + \cdots - u_{j0} - \frac{1}{24}\left(\frac{\partial^2 u_j}{\partial x^2}\right)_0 \Delta x_c^2 - \cdots \Bigg] \times \Bigg[\left(\frac{\partial^2 u_i}{\partial x}\right)_0 + \left(\frac{\partial^2 u_i}{\partial x_j \partial x}\right)_0 x + \left(\frac{\partial^2 u_i}{\partial x_j \partial y}\right)_0 y + \left(\frac{\partial^2 u_i}{\partial x_j \partial z}\right)_0 z + \frac{1}{2}\left(\frac{\partial^3 u_i}{\partial x_j \partial x^2}\right)_0 x^2 + \cdots - \left(\frac{\partial u_i}{\partial x}\right)_0 \frac{1}{24}\left(\frac{\partial^3 u_i}{\partial x_j \partial x^2}\right)_0 \Delta x_c^2 - \frac{1}{24}\left(\frac{\partial^3 u_i}{\partial x_j \partial y^2}\right)_0 \Delta y_c^2 \Bigg] - \cdots \Bigg\} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{1}{12}\Bigg[\left(\frac{\partial u_j}{\partial x}\right)\left(\frac{\partial^2 u_i}{\partial x_j \partial x}\right)\Delta x_c^2 + \left(\frac{\partial u_j}{\partial y}\right)\left(\frac{\partial^2 u_i}{\partial x_j \partial y}\right)\Delta y_c^2 + \left(\frac{\partial u_j}{\partial z}\right)\left(\frac{\partial^2 u_i}{\partial x_j \partial z}\right)\Delta z_c^2 \Bigg] + O(\Delta x_c^4, \Delta y_c^4, \cdots). \tag{7}$$

Inference The viscous stresses of the small-scale range with scales $\Delta x < \Delta x_f$ acting on the large-one with the scales $\Delta x > \Delta x_f$ is (see Fig.1)

$$F_{fi}(u_i, U_{fi}) = \frac{1}{V_f}\int_{V_f} (u_j - U_{fj}) \frac{\partial}{\partial x_j}(u_i - U_{fi})\mathrm{d}v = \frac{1}{12}\left(\frac{\partial u_j}{\partial x}\frac{\partial^2 u_i}{\partial x_j \partial x}\Delta x_f^2 + \frac{\partial u_j}{\partial y}\frac{\partial^2 u_i}{\partial x_j \partial y}\Delta y_f^2 + \frac{\partial u_j}{\partial z}\frac{\partial^2 u_i}{\partial x_j \partial z}\Delta z_f^2 \right) + O(\Delta x_f^4, \Delta x_f^2 \Delta y_f^2, \cdots), \tag{8}$$

where

$$\begin{cases} (U_{fi}, p_f) = \dfrac{1}{V_f}\displaystyle\int_{V_f} (u_j, p)\mathrm{d}v, \\ V_f = \Delta x_f \Delta y_f \Delta z_f \qquad (\Delta x_f < \Delta x_c,\ \Delta y_f < \Delta y_c,\ \Delta z_f < \Delta z_c). \end{cases} \tag{8a}$$

Suppose, not losing generality, the side-length of the small volume elements V_c and V_f satisfy $\Delta x_f/\Delta x_c = \Delta y_f/\Delta y_c = \Delta z_f/\Delta z_c$ (see Fig.1), then we deduce from the formulae (7) and (8)

$$F_{fi}(u_i, U_{fi}) = \frac{\Delta x_f^2}{\Delta x_c^2} F_{ci}(u_i, U_{ci}) + O(\Delta x_c^4, \cdots). \tag{9}$$

From the formula (9) we deduce that the viscous stress of the scale-range with scales $\Delta x < \Delta x_f$ acting on the scale-range with scales $\Delta x > \Delta x_f$ is only $\Delta x_f^2/\Delta x_c^2$ that of the scale range with scales $\Delta x < \Delta x_c$ acting on the scale range with scales of $\Delta x > \Delta x_c$ (see Table 1). Imagine that even if the viscous stress $F_{fi}(u_i, U_{fi})$ exerts totally on the scale-range with scales $\Delta x > \Delta x_c$, the viscous stress $F_{ci}(u_i, U_{ci})$ is by $(1 - \Delta x_f^2/\Delta x_c^2) F_{ci}(u_i, U_{ci})$ larger than $F_{fi}(u_i, U_{fi})$. From physical consideration we know that the short-range stress $(1 - \Delta x_f^2/\Delta x_c^2) F_{ci}(u_i, U_{ci})$ can be supplied only by the scales ranging from Δx_f to Δx_c, which prove that the viscous stresses of the small scale range with scales $\Delta x < \Delta x_c$ acting on the large one with scales $\Delta x > \Delta x_c$ is supplied mainly by the scale range near Δx_c, precisely speaking, by the scale range lying between Δx_f and Δx_c. The state above is physical connotation of the short-range viscous stresses.

2 Integral and Differential Formulae of the Short-Range Viscous Stresses

From the integral formulae (5b) and (8) for $F_{ci}(u_i, U_{ci})$ and $F_{fi}(u_i, U_{fi})$ we may infer that the integral formula of the viscous stresses of the scales ranging from Δx_f and Δx_c acting on the scale-range with scales $\Delta x > \Delta x_c$, i.e., the integral formula of the short-range viscous stresses should be as follows:

$$F_{cfi}(U_{fi}, U_{ci}) = \frac{1}{V_c}\int_{V_c}(U_{fj} - U_{cj})\frac{\partial}{\partial x_j}(U_{fi} - U_{ci})\mathrm{d}v, \tag{10}$$

where U_{ci} and U_{fi} are given in the formulae (5a) and (8a), respectively.

Proof of Eq.(10) $F_{ci}(u_i, U_{ci})$ is resolved into the following relation:

$$F_{ci}(u_i, U_{ci}) = \frac{1}{V_c}\int_{V_c}(u_j - U_{fj} + U_{fj} - U_{cj})\frac{\partial}{\partial x_j}(u_i - U_{fi} + U_{fi} - U_{ci})\mathrm{d}v =$$
$$\frac{1}{V_c}\int_{V_c}(u_j - U_{fj})\frac{\partial}{\partial x_j}(u_i - U_{fi})\mathrm{d}v + \frac{1}{V_c}\int_{V_c}(u_j - U_{fj})\frac{\partial}{\partial x_j}(U_{fi} - U_{ci})\mathrm{d}v +$$
$$\frac{1}{V_c}\int_{V_c}(U_{fj} - U_{cj})\frac{\partial}{\partial x_j}(u_i - U_{fi})\mathrm{d}v + F_{cfi}(U_{fi}, U_{ci}). \tag{11}$$

Not losing generality, we suppose that there are m^2 identical V_f within V_c and that both V_c and V_f are similar cubes. The surface of V_c is covered with surfaces of numerous V_f. For any arrangement of m^2 identical V_f within V_c, the equality of $V_c = m^2 V_f$ holds. S indicates the center of any V_f with V_c. Then the first term on the right hand side of the formula (11) can be transferred to

$$\frac{1}{V_c}\sum_s^{m^2}\int_{V_f}(u_j - U_{fi})\frac{\partial}{\partial x_j}(u_i - U_{fi})\mathrm{d}v = F_{fi}(u_i, U_{fi}) + O(\Delta x_f^2\Delta x_c^2). \tag{12a}$$

Next, we prove that the order-of-magnitude of the second term on the right hand side of the formula (11) is equal to $O(\Delta x_f^2\Delta x_c^2)$

$$\frac{1}{V_c}\int_{V_c}(u_j - U_{fj})\frac{\partial}{\partial x_j}(u_i - U_{ci})\mathrm{d}v =$$
$$\frac{1}{V_c}\sum_s^{m^2}\int_{V_f}\Big\{\Big[u_{js} + \left(\frac{\partial u_j}{\partial x}\right)_s(x - x_s) + \left(\frac{\partial u_j}{\partial y}\right)_s(y - y_s) +$$
$$\left(\frac{\partial u_j}{\partial z}\right)_s(z - z_s) + \frac{1}{2}\left(\frac{\partial^2 u_j}{\partial x^2}\right)_s(x - x_s)^2 + \cdots - u_{js} - \frac{1}{24}\left(\frac{\partial^2 u_j}{\partial x^2}\right)_s\Delta x_f^2 -$$
$$\frac{1}{24}\left(\frac{\partial^2 u_j}{\partial y^2}\right)_s\Delta y_f^2 - \cdots\Big]\times\Big[\left(\frac{\partial u_i}{\partial x_j}\right)_s + \frac{1}{24}\left(\frac{\partial^3 u_i}{\partial x_j\partial x^2}\right)_s\Delta x_f^2 + \frac{1}{24}\left(\frac{\partial^3 u_i}{\partial x_j\partial y^2}\right)_s\Delta y_f^2 + \cdots -$$
$$\left(\frac{\partial u_i}{\partial x_j}\right)_0 - \frac{1}{24}\left(\frac{\partial^3 u_i}{\partial x_j\partial x^2}\right)_0\Delta x_c^2 - \cdots\Big]\Big\}\mathrm{d}x\mathrm{d}y\mathrm{d}z =$$
$$\frac{V_f}{V_c}\sum_s^{m^2}\Big[-\frac{1}{192}\left(\frac{\partial^2 u_i}{\partial x^2}\right)_s\left(\frac{\partial^3 u_i}{\partial x_j\partial x^2}\right)_0\Delta x_f^2\Delta x_c^2 - \frac{1}{192}\left(\frac{\partial^2 u_i}{\partial x^2}\right)_s\left(\frac{\partial^3 u_i}{\partial x_j\partial y^2}\right)_0\Delta x_f^2\Delta y_c^2 - \cdots\Big] =$$
$$O(\Delta x_f^2\Delta x_c^2, \cdots). \tag{12b}$$

Through similar operation, one can prove that the third term on the right-hand side of the formula (11) is the same order of magnitude as the second term

$$\frac{1}{V_c}\int_{V_c}(U_{fj} - U_{cj})\frac{\partial}{\partial x_j}(u_i - U_{fi})\mathrm{d}v = O(\Delta x_f^2\Delta x_c^2, \cdots). \tag{12c}$$

Using the formulae (9) and (12a) ~ (12c), we obtain the following relation:

$$F_{ci}(u_i, U_{ci}) = F_{fi}(u_i, U_{fi}) + F_{cfi}(U_{fi}, U_{ci}) + O(\Delta x_f^2 \Delta x_c^2), \tag{13}$$

$$F_{cfi}(U_{fi}, U_{ci}) = \left(1 - \frac{\Delta x_f^2}{\Delta x_c^2}\right) F_{ci}(u_i, U_{ci}) + O(\Delta x_c^4, \cdots). \tag{14}$$

The physical inference for the short-range interactions between scales given in Section 1 is confirmed by the formula (14), in which the $F_{ci}(u_i, U_{ci})$ expresses the viscous stress of whole small scale-range with scales $\Delta x < \Delta x_c$ acting on the large one with scales $\Delta x > \Delta x_c$, see Fig. 1. This is usually called the interactions between widely separated scales, i.e., the long-range viscous stresses[3,4]; $F_{cfi}(U_{fi}, U_{ci})$ expresses the viscous stress of the contiguous scales Δx ranging from Δx_f to $\Delta x_c (\Delta x_f < \Delta x < \Delta x_c)$, acting on the scale-range with scales $\Delta x > \Delta x_c$, i.e., the short range viscous stresses. The variation of the viscous stress ratio $|F_{cfi}| / |F_{ci}|$ with the scale-ratio $\Delta x_f / \Delta x_c$ is given in Table 1.

Table 1 Variation of the stress-ratio $|F_{cfi}| / |F_{ci}|$ and $|F_{ffi}| / |F_{cfi}|$ with the scale-ratio $\Delta x_f / \Delta x_c$

$\Delta x_f / \Delta x_c$	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
$\lvert F_{cfi} \rvert / \lvert F_{ci} \rvert$	0.36	0.51	0.64	0.75	0.84	0.91	0.96	0.99
$\lvert F_{ffi} \rvert / \lvert F_{cfi} \rvert$	/	/	/	0.33	0.19	0.10	0.04	0.01

From the data in Table 1 we know that the contiguous small scales, i.e., the short-range scales should range from Δx_c to $\frac{1}{2}\Delta x_c$ or $\frac{1}{3}\Delta x_c$, in both cases the contiguous scales offer 75% or 91% of the viscous stresses of the whole scale range with scales $\Delta x_c < \frac{1}{2}\Delta x_c$ or $\frac{1}{3}\Delta x_c$ acting on the large ones with scales $\Delta x > \Delta x_c$ and much smaller ones with scales $\Delta x_c < \frac{1}{2}\Delta x_c$ or $\frac{1}{3}\Delta x_c$ have a negligible effects on the large ones with scales $\Delta x > \Delta x_c$. The above conclusions are consistent with chief conclusions acquired through treating the direct numerical simulation (DNS) databases for the incompressible channel flow[5,6]. This chief conclusion is that the nonlinear dynamics of the resolved scales with wave numbers $k < k_1$ are governed almost exclusively by their interactions with a limited range of scales with wave numbers not exceeding $2k_1$ and much smaller scales have a negligible effect on the resolved ones. However, the formula (14) is not only suitable to the turbulent flow in the channels but also suitable to all turbulent flows.

As we know that the space average velocity-components U_{fi} and U_{ci} are continuity and differential, the derivatives of U_{fi} and U_{ci} exist. Therefore, starting from the integral formula (10) of the short range viscous stresses and through similar operation used in deducing the differential formula (7) of $F_{ci}(u_i, U_{ci})$, it is not difficult to obtain the following differential formula of the short range stresses:

$$F_{cfi}(U_{fi}, U_{ci}) = \frac{1}{12}\left(\frac{\partial U_{fj}}{\partial x}\frac{\partial^2 U_{fi}}{\partial x_j \partial x}\Delta x_c^2 + \frac{\partial U_{fj}}{\partial y}\frac{\partial^2 U_{fi}}{\partial x_j \partial y}\Delta y_c^2 + \frac{\partial U_{fj}}{\partial z}\frac{\partial^2 U_{fi}}{\partial x_j \partial z}\Delta z_c^2\right) + O(\Delta x_c^4, \cdots). \tag{15}$$

From the formula (15) we see that short-range viscous stress of the scale range lying between Δx_f and $\Delta x_c (\Delta x_f < \Delta x < \Delta x_c)$ acting on the large scales $\Delta x > \Delta x_c$ are nonlinear functions of the first- and second-order derivatives of the space average velocities on the fine grid (i.e. the scale Δx_f) and are proportional to the area of the course grid, i.e., the square of the scale Δx_c.

3 A Concept of Resonant Interactions and Resonant-Range Viscous Stresses

Consider the interactions between the scales being smaller than but near extremely Δx_c and the scales $\Delta x > \Delta x_c$, which are different from the short-range interactions. In order to distinguish them, we should introduce a concept of resonant range interactions, which express the interactions between the small scales being smaller than but near extremely Δx_c and the scales $\Delta x > \Delta x_c$. According to the definitions of the space averaged velocities we know that the U_{fi} tends to U_{ci} if the Δx_f tends to Δx_c, i.e., the V_f tends to V_c, see Fig. 1. Therefore, the differential formula of the resonant-range viscous stresses can be deduced directly from the formula (15) of the short-range ones.

$$F_{cci}(U_{ci}, U_{ci}) = \frac{1}{12}\left(\frac{\partial U_{cj}}{\partial x}\frac{\partial^2 U_{ci}}{\partial x_j \partial x}\Delta x_c^2 + \frac{\partial U_{cj}}{\partial y}\frac{\partial^2 U_{ci}}{\partial x_j \partial y}\Delta y_c^2 + \frac{\partial U_{cj}}{\partial z}\frac{\partial^2 U_{ci}}{\partial x_j \partial z}\Delta z_c^2\right) + O(\Delta x_c^4, \cdots). \tag{16}$$

Similarly, the differential formula of the resonant range viscous stresses for the scale Δx_f should be as follows:

$$F_{ffi}(U_{fi}, U_{fi}) = \frac{1}{12}\left(\frac{\partial U_{fj}}{\partial x}\frac{\partial^2 U_{fi}}{\partial x_j \partial x}\Delta x_f^2 + \frac{\partial U_{fj}}{\partial y}\frac{\partial^2 U_{fi}}{\partial x_j \partial y}\Delta y_f^2 + \frac{\partial U_{fj}}{\partial z}\frac{\partial^2 U_{fi}}{\partial x_j \partial z}\Delta z_f^2\right) + O(\Delta x_f^4, \cdots). \tag{17}$$

From the formulae (16) and (17), we see that the resonant-range viscous stresses of the scales $\Delta x < \Delta x_f$ (or Δx_c) acting on the scales $\Delta x > \Delta x_f$ (or Δx_c) are nonlinear functions of the firt- and second-order derivatives of the averaged velocities on the scale Δx_f (or Δx_c) and are proportional to the area of the grid, i.e., the square of the scale Δx_f (or Δx_c).

The resonant-range viscous stress $F_{ffi}(U_{fi}, U_{fi})$ is formally similar to the subgrid scale (SGS) model based on the Smagaorinsky hypothesis[1,2], but the concept causing them is different. The SGS model corresponding to $F_{ffi}(U_{fi}, U_{fi})$ is as follows[1,3]:

$$\frac{\partial}{\partial x_j}(\tau_{ij}), \ \tau_{ij} = (c\Delta)^2\left[\frac{\partial U_{fi}}{\partial x_j}\left(\frac{\partial U_{fi}}{\partial x_j} + \frac{\partial U_{fj}}{\partial x_i}\right)\right]^{1/2}\left(\frac{\partial U_{fi}}{\partial x_j} + \frac{\partial U_{fj}}{\partial x_i}\right), \tag{18a}$$

where $\frac{\partial}{\partial x_j}(\tau_{ij})$ is corresponding to $F_{ffi}(U_{fi}, U_{fi})$, c is dimensionless empirical constant, Δ indicates the filtered scale or grid scale and is chosen as

$$\Delta = (\Delta x_f \Delta y_f \Delta z_f)^{1/2} \text{ or } (\Delta x_f^2 + \Delta y_f^2 + \Delta z_f^2)^{1/2}. \tag{18b}$$

Comparing the resonant-range viscous stress with SGS model, the former does not include any empirical constant and relation and stems from the resonant-range interactions between extremely contiguous scales in turbulence; the SGS model includes empirical constant and relation and is based on classic eddy viscosity concept, i.e. the interactions between the scales are the long-range ones between widely separated scales in turbulence[1~4]. The SGS model is used to close approximately the large eddy simulation (LES) equations. Of course, the differential formula (17) of the resonant-range viscous stress can be used to induce new LES equations.

4 A New Large Eddy Simulation (LES) Equations

Using the resonant-range viscous stress tensor given in the formula (17), it is not difficult to construct the following space averaged (or box filtered) Navier-Stokes equations, i.e., a new large eddy simulation (LES) equations:

$$\frac{\partial U_{fi}}{\partial x_i} = 0 \qquad (i = 1,2,3), \tag{19a}$$

$$\frac{\partial U_{fi}}{\partial t} + U_{fj}\frac{\partial U_{fi}}{\partial x_j} = -\frac{\partial p_f}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 U_{fi}}{\partial x_j \partial x_j} - F_{ffi}(U_{fi}, U_{fi}) \qquad (i = 1,2,3), \tag{19b}$$

where U_{fi} and $F_{ffi}(U_{fi}, U_{fi})$ are given in the formulae (8a) and (17), respectively. According to the short-range character of interactions between the scales, the molecular viscous terms in Eq. (19b) ought to be neglected, reserving them is to match the boundary conditions at the solid surfaces. For the viscous fluid the condition of no slip on solid surfaces must be satisfied and the molecular viscous terms play an important role in the laminar sub-layer next to the wall. Similarly, the short-range character of interactions between the scales means that the resonant-range viscous stresses in LES Eq.(19b) act only on a local range with the scales $\Delta x > \Delta x_f$ but extremely near to Δx_f and do not act on the whole scale range with scales $\Delta x > \Delta x_f$, see Fig.1. Therefore, it would be best to adopt multiscale equations to compute turbulent flows, i.e., to divide prior the resolved scales into two or more than two scale-ranges and to solve simultaneously the multiscale equations governing the motions of these scale ranges. Only so the physical mechanism of the short range interactions between the scales can be detected perfectly.

5 Multiscale Equations in Turbulence

Dividing the resolved scales into two or more than two scale ranges and utilizing the formulae of both short- and resonant-ranges viscous stresses, we can obtain multiscale equations governing turbulent flows. For example, if the resolved scales are divided into the small scales ranging from Δx_f to Δx_c and the large scale range lying between 1 and Δx_c, see Fig.1 the large scale (space averaged) equations governing the motion of large scale range is as follows:

$$\frac{\partial U_{ci}}{\partial x_c} = 0 \tag{20a}$$

$$(i = 1,2,3).$$

$$\frac{\partial U_{ci}}{\partial t} + U_{cj}\frac{\partial U_{ci}}{\partial x_j} = -\frac{\partial p_c}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 U_{ci}}{\partial x_j \partial x_j} - F_{cfi}(U_{fi}, U_{ci}) \tag{20b}$$

Substracting the large scale Eqs.(20a) ~ (20b) from the LES Eqs.(19a) ~ (19b), we obtain small scale equations governing the fluctuation motions of the small scale (fine-grid) averaged quantities relating to the large scale (coarse grid) averaged ones

$$\frac{\partial}{\partial x_i}(U_{fi} - U_{ci}) = 0, \tag{21a}$$

$$\frac{\partial}{\partial t}(U_{fi} - U_{ci}) + (U_{fj} - U_{cj})\frac{\partial}{\partial x_j}(U_{fi} - U_{ci}) =$$

$$-\frac{\partial}{\partial x_i}(p_f - p_c) - U_{cj}\frac{\partial}{\partial x_j}(U_{fi} - U_{ci}) - (U_{fj} - U_{cj})\frac{\partial U_{ci}}{\partial x_j} +$$

$$\frac{1}{Re}\frac{\partial^2}{\partial x_j \partial x_j}(U_{fi} - U_{ci}) + F_{cfi}(U_{fi}, U_{ci}) - F_{ffi}(U_{fi}, U_{fi}), \tag{21b}$$

where U_{ci} and U_{fi} are given in the formulae (5a) and (8a), respectively. Both the integral and differential formulae of the short range viscous stresses $F_{cfi}(U_{fi}, U_{ci})$ can be used and are given in the formulae (10) and (15), respectively. The resonant-range viscous $F_{ffi}(U_{fi}, U_{fi})$ is given in the formula (17). The $\Delta x_f, \Delta y_f$ and Δz_f (for short, the Δx_f) are consistent with the filtered scale in LES, and $\Delta x_c \approx (2 \sim 3)\Delta x_f$.

Obviously, the large-small scale (LSS) equations are closed approximately and do not contain any empirical constants or relations. The cause of retaining the molecular viscous stresses in the multiscale Eqs.(20) ~ (21) is the same as that in the LES Eqs.(19a) ~ (19b). The short-range viscous stresses in the large scale Eq.(20b) are supplied by the resolved small scale-range with scales Δx lying between Δx_f and $\Delta x_c (\Delta x_f < \Delta x < \Delta x_c)$. F_{ffi} in the small scale Eq.(21b) indicates the resonant-range viscous stress of the unresolved much smaller scales $\Delta x (\Delta x < \Delta x_f)$ acting on the resolved small scales $\Delta x (\Delta x_f < \Delta x < \Delta x_c)$. In the conditions of the coarse and fine grids are geometrically similar and satisfy $\Delta x_f/(\Delta x_c) = \Delta y_f/(\Delta y_c) = \Delta z_f/(\Delta z_c)$, we deduce from the formulae (9) and (14) that $F_{ffi}/F_{cfi} \approx \frac{\Delta x_f^2}{\Delta x_c^2}\left(1 - \frac{\Delta x_f^2}{\Delta x_c^2}\right)^{-1}$, see Table 1. Therefore, the viscous stresses acting on the resolved small scales $\Delta x (\Delta x_f < \Delta x < \Delta x_c)$ are mainly supplied by the resolved large scales $\Delta x (\Delta x > \Delta x_c)$ and much smaller unresolved scales $\Delta x (\Delta x < \Delta x_f)$ have secondary influences. The above states for the large small scales (LSS) Eqs.(20) ~ (21) are consistent with the conclusions given by treating DNS databases for channel turbulent flow[5,6]. A main conclusion of Refs.[5,6] is drawn in the illustration of Table 1, other main conclusion is that the dynamics of the subgrid scales with wave numbers ranging from k_1 to $2k_1$ are largely determined by their interactions with the resolved large scales with wave numbers $k < k_1$ and much higher wave numbers $k > 2k_1$ have secondary effects.

Some comparisons of the multiscale Eqs.(20) ~ (21) with the traditional LES Eq.(19) are as follows. In the former the unresolved scales act only on the small scale range of the resolved scales, and in the latter the unresolved scales act on whole range of the resolved scales; therefor, as to detecting the nonlinear interactions between the contiguous scales and their effects, the former gains dominance over the latter. On the other hand, the unresolved scales contain still a wide range of time- and length-scales, any method using a few parameters and formulae to describe the perfect effects of the unresolved scales certainly include undetermined and unknow factors. The effects of the above unknow factors are confined to the small scale range of the resolved scales in the multiscale method, however, these unknown factors do affect the whole range of the resolved scales in the traditional LES methods. The multiscale method can supply simultaneously data about the space average and fluctuation motions. There are obvious advantages of the multiscale method compared with the traditional LES method.

The momentum and energy transfer between the resolved scales are introduced in the multiscale method, which is, of course, able to describe well experimental phenomena and observation results, such as the large eddies breaking into small ones and the energy cascade etc. The data about the space averaged and fluctuation motions and the characters of transfer between the resolved scales given by the multiscale method can be obtained by the direct numerical simulation (DNS) for turbulent flow. For this end, it is necessary to divide afterwards the resolved scales into several scale-ranges and treating the DNS database. It is worthy of studying further consistency and difference between the results given by the multiscale method and treating the DNS databases. Such analysis for an incompressible planar shear flow is given in the next section.

In addition, the molecular viscous terms are linear ones, while the short- and resonant-range viscous terms are nonlinear ones. The energy ought not to accumulate at the smallest scale Δx_f of the resolved scales, hence the dissipation must be dominant at Δx_f, i.e. the sum of the last three terms on the right-hand side of Eq.(21b) must be larger than zero. If this requirement is not satisfied, we may use the multiscale equations with the scale-ranges being more than two, or increase properly the resonant-range viscous stresses, or use empirical subgrid scale (SGS) model instead of the resonant-range stresses.

6 Time Evolution of the Incompressible Planar Mixing Layer Flow

For the three-dimensional time evolution of an incompressible planar mixing layer flow we use the quasi-spectrum method to solve the multiscale Eqs.(20) ~ (21) and the unsteady Navier-Stokes equations[8,9].

The initial conditions of the incompressible planar mixing layer flow are as follows:

$$\begin{cases} (U_c, V_c, W_c) = (U_0 \tanh(2\bar{z}), 0, 0), \\ U_f - U_c = A_2 \sin\bar{x} \cos\bar{y} + A_3 \sin\bar{x} \cos\bar{y} \cos\bar{z}, \\ V_f - V_c = -A_2 \cos\bar{x} \sin\bar{y} - A_3 \cos\bar{x} \sin\bar{y} \cos\bar{z}, \\ W_f - W_c = 0, \ (\bar{x}, \bar{y}, \bar{z}) = \dfrac{1}{\delta_w^0}(x, y, z), \end{cases} \tag{22}$$

where $A_2, A_3 = 10^{-4}$ are the amplitudes of two- and three-dimensional initial disturbances, respectively. U_0 is the averaged velocity of two coming flows. The velocity, the time, the coordinate variables and the pressure are made dimensionless with reference to the $U_0, T_0 = \delta_w^0 / U_0, \delta_w^0$ and ρU_0^2, respectively. δ_w^0 is the initial thickness of vorticity. $Re = U_0 \delta_w^0 / \nu$, ν is the kinematic viscosity. In order to examine well the solutions of the large-small scale (LSS) equations and compare them with the solutions of the Navier-Stokes (NS) equations, the same grid system is used in solving of LSS and NS-equations. For instance for $Re = 10^2 \delta_w^2 / \nu$, the grid numbers are 32, 64 and 128 in the x-, y- and z-coordinate directions, respectively. Some typical results are as follows. The mean quantities of the velocity, the fluctuations and the shear viscous stress given by the LSS-solutions are consistent with those by treating database of the NS-solutions, see Fig. 2 and Fig. 3. Some new results are also given by the LSS-solutions, for examples, the time evolutions of both maximum fluctuation velocity and maximum shear viscous stress are obviously different from those of respective mean quantities, especially, both evolutions appear nearly in the meantime twice sudden-increases. The time of appearing sudden-increase is consistent with that of appearing inverse transfer of energy from the resolved small scales to the resolved large ones and also with that of rolling up spanwise- and rib-vortices. These facts mean that the momentum and energy exchanges between the resolved large and small scales are principal cause of evolutions of large scale motion. The sudden-increases of both maximum stress and maximum fluctuation are undoubtedly corresponding to the burst phenomena in the transition process. It should be mentioned that the maximum stress and maximum fluctuation are not acquired through the analysis of the database of the NS-solutions, see Fig.2 and Fig.3.

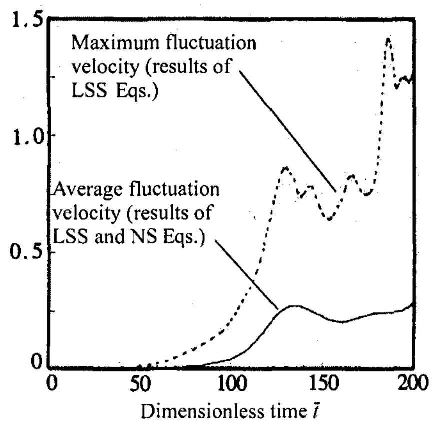

Fig.2 Maximum and average fluctuation velocities in an incompressible planar mixing flow

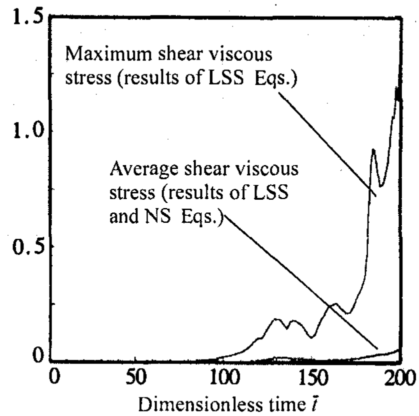

Fig.3 Maximum and average shear viscous stress in an incompressible planar mixing flow

7 Conclusion

Turbulent flow contains a wide range of time- and space-scales. The principal mechanism governing nonlinear dynamics of turbulence is the interactions between the scales, which is mainly short-range ones between the contiguous scales. Therefore, the calculations of turbulent flows ought to adopt the multiscale method. The short range interactions between the contiguous scales are studied, the formulae for the short- and resonant-range viscous stresses are obtained. These formulae are applied to deduce both the multiscale and LES equations, which are closed approximately and do not have any empirical constants and relations.

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