## BRIEF COMMUNICATIONS

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# Numerical estimate of the stability curve for the flow past a rotating cylinder

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It is demonstrated that the primary instability of the wake of a two-dimensional circular cylinder rotating with constant angular velocity can be qualitatively well described by the Landau equation. The coefficients of the Landau equation are determined by means of numerical simulations for the Navier–Stokes equations. The critical Reynolds numbers, which depend on the angular velocity of the cylinder, are evaluated correctly by linear regression. 
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The flow past a constantly rotating circular cylinder is characterized by the Reynolds number Re and the spin parameter $\dot{\alpha}$, which is defined as the ratio of the peripheral speed of the cylinder surface to the free stream velocity. The flow at low Reynolds numbers, concerning the primary instability of the wake, has been studied both numerically and experimentally.[1–3] The most recent studies include Hu *et al.*,[4] Kang *et al.*,[5] and Barnes.[6] Hu *et al.*[4] obtained the stability curve by low-dimensional Galerkin method. Kang *et al.*[5] estimated the stability curve by numerically solving the two-dimensional unsteady Navier–Stokes (NS) equations with a fully implicit fractional-step method in time and a second-order central difference scheme in space. Barnes[6] experimentally measured $\dot{\alpha}$ at which the periodical vortex shedding is suppressed. Note that Kang *et al.*[5] and Barnes[6] obtained not a definite stability curve but a stability "boundary" with width of $\Delta\dot{\alpha}=0.1$ due to the confinement of their approaches. Hu *et al.*[4] obtained a definite stability curve, but the finite dimensions of the Galerkin method limited the accuracy of the results. Naturally their results, which are shown in Fig. 2, scatter distinctly.

In this paper the stability curve in the regime $0\leqslant\dot{\alpha}\leqslant 1.5$ is determined by employing the Landau equation, which was formulated by Landau *et al.*[7] and has been widely used in the stability analysis of the steady flows, especially for the bluff body wakes. For instance, Provansal *et al.*,[8] Noack *et al.*,[9] and Schumm *et al.*[10] studied and described the transient regime above the oscillation threshold in the wake of a circular cylinder by the Landau equation. Albarède *et al.*[11] investigated the formation of oblique shedding and "chevron" patterns in cylinder wakes by the idealized model of a transverse Ginzburg–Landau equation. Schatz *et al.*[12] employed the Landau equation to analyze the onset of the primary stability of the plane channel flow with a streamwise-periodic array of cylinders and to determine the critical Reynolds number.

Below we briefly introduce the Landau equation. Consider the complex amplitude of an independent solution for the disturbed equation $A(t)=ce^{\gamma t}e^{-i\omega t}$, in which $c$, $\gamma$, and $\omega$ are real, and

$$\gamma=k(\mathrm{Re}-\mathrm{Re}_c)+O((\mathrm{Re}-\mathrm{Re}_c)^2)\quad\text{when}\quad \mathrm{Re}\to\mathrm{Re}_c. \tag{1}$$

$|A|$, i.e., the amplitude of the disturbance, approximately satisfies the Landau equation

$$d|A|/dt=\gamma|A|-g|A|^3/2. \tag{2}$$

According to Schatz *et al.*,[12] $g>0$ for the supercritical bifurcation of the bluff body wakes. It follows that the asymptotic amplitude of the disturbance $|A|_m=(2\gamma/g)^{1/2}$. Consequently,

$$|A|_m^2\cong\frac{2k}{g}(\mathrm{Re}-\mathrm{Re}_c)\quad\text{when}\quad \mathrm{Re}\to\mathrm{Re}_c+0. \tag{3}$$

It indicates that $|A|_m^2$ approximately linearly increases with Re. Hence $\mathrm{Re}_c$ can be estimated through the linear regression

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of the samples of Re and $|A|_m^2$ by the least-squares method. In the present work, the data needed for the fittings are obtained from the numerical simulations of the unsteady wake. Below the numerical method is outlined.

At low Reynolds numbers the flow past a rotating cylinder may be assumed to be two-dimensional. So the two-dimensional NS equations for the incompressible fluid

$$\partial \mathbf{V}/\partial t = -\nabla p + \nabla^2 \mathbf{V}/\mathrm{Re} + \mathbf{N}(\mathbf{V}), \tag{4}$$

$$\nabla \cdot \mathbf{V} = 0, \tag{5}$$

are considered, where $\mathbf{N}(\mathbf{V}) = -\mathbf{V}\cdot\nabla\mathbf{V}$ represents the nonlinear convection operator. The solving procedure is time-split into the substeps as

$$\left(\mathbf{V}^{n+1/3} - \sum_{q=0}^{J-1} \alpha_q \mathbf{V}^{n-q}\right) \Big/ \Delta t = \sum_{q=0}^{J-1} \beta_q \mathbf{N}(\mathbf{V}^{n-q}), \tag{6}$$

$$\nabla^2 p^{n+1} = \nabla \cdot (\mathbf{V}^{n+1/3}/\Delta t), \tag{7}$$

$$(\mathbf{V}^{n+2/3} - \mathbf{V}^{n+1/3})/\Delta t = -\nabla p^{n+1}, \tag{8}$$

$$(\gamma_0 \mathbf{V}^{n+1} - \mathbf{V}^{n+2/3})/\Delta t = \nabla^2 \mathbf{V}^{n+1}/\mathrm{Re}, \tag{9}$$

where the superscripts denote the discrete time step. $\mathbf{V}^{n+1/3}$ and $\mathbf{V}^{n+2/3}$ are intermediate values of the velocity. The coefficients $J$, $\alpha_q$, $\beta_q$, and $\gamma_0$, as well as the details of the splitting algorithm, can be found in Karniadakis *et al.*[13] The spectral element method is employed to solve Eqs. (7) and (9). The detailed implementation can be found in Patera,[14] Korczak *et al.*,[15] and Xiong *et al.*[16] Xiong *et al.*[16] also validated the mesh in the simulations of the flow past a cylinder.

The NS equations are solved as described above at a series of Reynolds numbers beyond $\mathrm{Re}_c$. When the flow reaches the asymptotic periodic state, the amplitudes of the velocities and the pressure at specific locations in the flow field are extracted and play the role of $|A|_m$ in Eq. (3). Note that the Reynolds numbers chosen as the samples for the linear regression should be close enough to $\mathrm{Re}_c$ so that the Landau equation is valid. On the other side, the samples selected for the linear regression should not be too close to each other, otherwise the error of the linear regression will increase. In practice the above two aspects should be balanced. For instance, the flow at Re=52, 55, 60, 70, and 80 are calculated for $\dot{\alpha}=0$ in the present work.

In practice the obtained $\mathrm{Re}_c$ depends on the quantity being extracted as well as on the extraction location, which means that the accuracy for the Landau model to describe the instability dynamics varies with the location. Due to the spatial evolution of the wake, the velocity and pressure signals at the location downstream distinctly deviate from the trigonometric wave in shape. To be consistent with the assumption of $A(t) = ce^{\gamma t}e^{-i\omega t}$, the extraction location is therefore chosen in the vicinity of the cylinder, e.g., the points $P_1(-0.486, 0.728)$ and $P_2(0.726, 0.486)$, where the shedding vortices emerge from the instability. Note that the Landau equation is substantially an extrapolation tool. Thus it would be useful to check how linear the plot of $|A|_m^2$ vs Re is. Consequently the correlation coefficient of the linear regression is used to validate the Landau model and to pick out the proper quantity and location further. The more the correlation coefficient approaches 1, the more accurately the Landau model describes the dynamics at the specific location and consequently the more accurate the evaluated $\mathrm{Re}_c$ is.

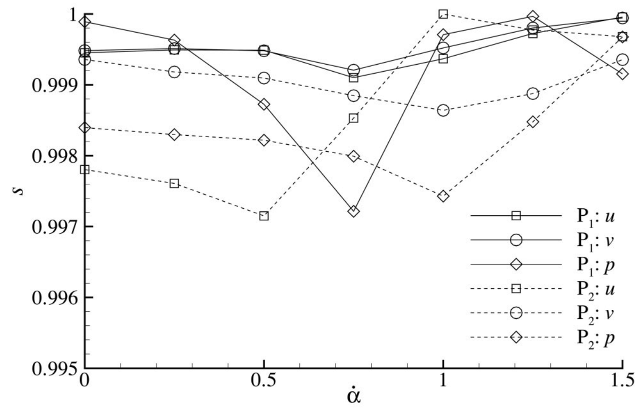


FIG. 1. The correlation coefficient of the linear fitting.

The correlation coefficients of the linear regression $s$ for the samples of Re and the squared amplitude of the velocities and the pressure at points $P_1$ and $P_2$ are shown in Fig. 1. The correlation coefficients derived from $v$ at point $P_1$ are generally closer to 1 with various angular velocities than that from the other quantities. Consequently the corresponding estimated $\mathrm{Re}_c$ are adopted. The determined stability curve is illustrated in Fig. 2, compared with the previous numerical and experimental results. The present results are essentially in agreement with those of the previous studies. When $\dot{\alpha} > 1$, the present stability curve lies between those of Kang *et al.*[5] and Barnes.[6] When $\dot{\alpha} < 1$, the critical Reynolds number given by the present work is higher than that of Hu *et al.*[4] In particular, when $\dot{\alpha}=0$, Hu *et al.*[4] found $\mathrm{Re}_c=45.6$; while the present study yields $\mathrm{Re}_c=47.5$, which is closer to the experimental results for the cylinder with large aspect ratio, for example, a value of 48 by Lee *et al.*[17] and 47.4 by Norberg.[18]

Meanwhile the investigations on the temporal evolution of the disturbance amplitude are carried out. The coefficients of the Landau equation are estimated with the method sug-

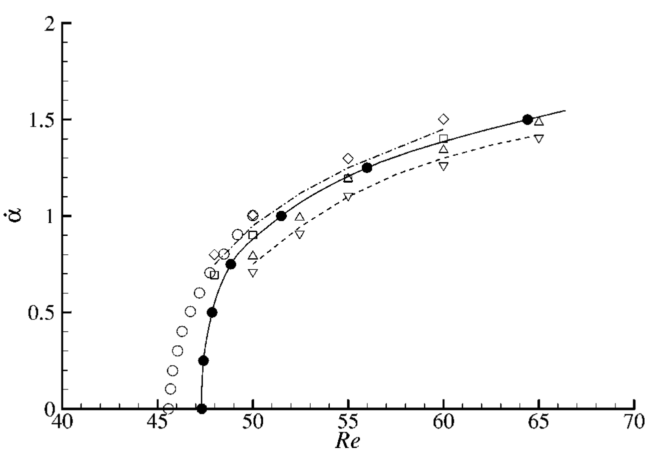


FIG. 2. The stability curve in the Re$-\dot{\alpha}$ plane. (○) The results of Hu *et al.* (Ref. 4); (□) and (◇) Kang *et al.* (Ref. 5); (▽) and (△) Barnes (Ref. 6); (●) the present paper. (□) and (▽) The periodic flow; (◇) and (△) the steady flow. The approximate stability boundary: dash and dot curve is from Kang *et al.* (Ref. 5); dashed curve from Barnes (Ref. 6). Solid curve is the present result, which is obtained by the Akima cubic spline.

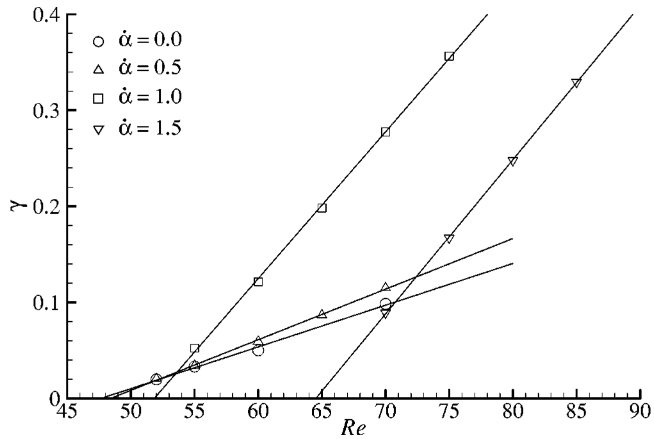


FIG. 3. The dependencies of the growth rate of the disturbance on the Reynolds number for various angular velocities.

gested by Schumm *et al.*[10] First the Landau equation is rewritten as

$$\frac{1}{|A|}\frac{d|A|}{dt} = \gamma - \frac{g}{2}|A|^2. \tag{10}$$

It indicates that the instantaneous growth rate $|A|^{-1}d|A|/dt$ is linear function of $|A|^2$. So the growth rate of the disturbance $\gamma$ can be evaluated by fitting Eq. (10) to the transient data obtained from the numerical simulations. The time derivatives of $|A|$ in Eq. (10) are evaluated by finite differences. The resulting dependencies of $\gamma$ on Re at various $\dot{\alpha}$ are illustrated in Fig. 3. For $\dot{\alpha}=0$, the estimated $\mathrm{Re}_c$ and the growth rate of $\gamma$ with Re, i.e., $k$, are

$$\mathrm{Re}_c = 47.64 \pm 0.61, \tag{11}$$

$$k = 0.004\,34 \pm 0.000\,21 \quad (\text{or equivalently} \quad k\,\mathrm{Re}_c = 0.207 \pm 0.013), \tag{12}$$

respectively, which essentially conform not only with the result $\mathrm{Re}_c=47.5$ obtained above from the asymptotic disturbance amplitudes but also with the value $\mathrm{Re}_c=46.7\pm0.3$, $k\,\mathrm{Re}_c=0.21\pm0.005$ by Schumm *et al.*,[10] and $k=0.003\,99$ (or equivalently $k\,\mathrm{Re}_c=0.215$) by Noack *et al.*[9] Figure 3 also indicates that both $\mathrm{Re}_c$ and $k$ increase with $\dot{\alpha}$. It is well known that the rotation of the cylinder can suppress the disturbance and therefore stabilize the wake, which counts for the increase of $\mathrm{Re}_c$ with $\dot{\alpha}$. At the same time, from the increase of $k$ (the growth rate of $\gamma$ with Re) with $\dot{\alpha}$ we speculate that the stabilizing effect of the rotation only acts in the vicinity of the critical Reynolds number. When Re is large enough, the effect of the rotation is to amplify the disturbance and destabilize the wake.

The stability curve can be approximately formulated as the dependence of the critical Reynolds number on the angular velocity. Due to symmetry (independence in the sense of rotation), the dependence is quadratic, so the polynomial form

$$\mathrm{Re}_c = b_0\dot{\alpha}^4 + b_1\dot{\alpha}^2 + b_2 \tag{13}$$

is adopted, where the values of $b_i$ ($i=0, 1, 2$) can be evaluated by least-squares fitting. Eventually we get

$$\mathrm{Re}_c = 2.917\dot{\alpha}^4 + 1.270\dot{\alpha}^2 + 47.65. \tag{14}$$

From Eq. (14) it is easy to obtain the critical rotation rate as a function of the Reynolds number. Here we will not write out the explicit expression.

In summary the combination of the Landau equation and the numerical simulations is successfully employed to investigate the primary instability of the flow past a constantly rotating circular cylinder. The numerical experiments are performed to determine oscillation amplitudes in the supercritical regime. Then these data are fitted to an amplitude function derived from the standard Landau equation, which leads to the critical Reynolds number of transition for the given rotation rate. In the fitting process, the correlation coefficient of the linear regression provides *a posteriori* estimation of the accuracy.

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