

ON DISPERSION-CONTROLLED PRINCIPLES FOR NON-OSCILLATORY SHOCK-CAPTURING SCHEMES*

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ABSTRACT: The role of dispersions in the numerical solutions of hydrodynamic equation systems has been realized for long time. It is only during the last two decades that extensive studies on the dispersion-controlled dissipative (DCD) schemes were reported. The studies have demonstrated that this kind of the schemes is distinct from conventional dissipation-based schemes in which the dispersion term of the modified equation is not considered in scheme construction to avoid nonphysical oscillation occurring in shock wave simulations. The principle of the dispersion controlled aims at removing nonphysical oscillations by making use of dispersion characteristics instead of adding artificial viscosity to dissipate the oscillation as the conventional schemes do. Research progresses on the dispersion-controlled principles are reviewed in this paper, including the exploration of the role of dispersions in numerical simulations, the development of the dispersion-controlled principles, efforts devoted to high-order dispersion-controlled dissipative schemes, the extension to both the finite volume and the finite element methods, scheme verification and solution validation, and comments on several aspects of the schemes from author's viewpoint.

KEY WORDS: DCD schemes, stability conditions, numerical methods, CFD validation

1 INTRODUCTION

Computational simulation is now becoming a promising approach in solving full Reynolds averaged Navier-Stokes equations for geometrically realistic three-dimensional problems with supercomputers available. This technology enhances people's ability to highlight physics in fluid flows that are too complicated to be clearly visualized experimentally. However, there are still some major issues in CFD, which need further investigation. One of the major issues arises from the simulation of flowfields with shock waves, that is, the nonphysical oscillation occurring near shock waves or other kinds of discontinuities, such as the contact surface and the strong expansion waves. These kinds of discontinuities are the fundamental flow phenomena resulting in more complex fluid flows. Taking shock wave, a highly nonlinear phenomenon in aerodynamics, as an example,

we believe it is a unique source that may generate vortices and turbulence in fluid flows except for solid walls. On other hand, numerical theories on shock-capturing schemes have occupied an important position in the CFD development because of nature of the variable discontinuities across shock waves. The weak problem of fluid dynamics has attracted many scientists to dedicate to for more than three decades, and remains an interesting topic in the new century.

A number of stability conditions has been proposed as criteria for numerical scheme development. The relevant introduction can be found in the CFD text books and some of them are cited here as references. The discrete perturbation analysis was firstly proposed by Thom and Apelt in 1961^[1], with which scheme stability is examined by introducing a discrete perturbation at arbitrary mesh point and its effect is followed. Stability is indicated if the perturbation dies out as iteration procedure proceeds.

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In the von Neumann stability analysis^[2], a more widely applied method, the finite Fourier expansion of numerical solutions to a model equations is made. Whether each mode remains bounded or not is considered separately to determine scheme stability. In the Hirt stability theory^[3], all the terms of finite difference equations are expanded with Taylor series at the presently-calculated mesh point to develop continuum differential equations. The stability requires that the effective viscosity in the continuum equations must be proposed non-negative. The Warming stability condition^[4] was based on the same idea as applied in the Hirt stability theory, but it requires that the sum of all the coefficients of even-order derivatives in the modified equation must be positive. In the derivation of the modified equation, it is not necessarily assumed that numerical solutions satisfy the original partial differential equation for eliminating the time or the mixed time and space derivatives.

It is obvious that the above-mentioned stability criteria are proposed mainly based on dissipation principle, that is, the amplitude of numerical errors must be decreasing as computational iteration proceeds. Furthermore, the Total Variation Diminishing (TVD) condition, developed by Harten and Osher^[5], requires that the total variation in the next time step should be no more than that at the present time step. However, most schemes widely used in practice still exhibit nonphysical oscillation near shock waves in predicted solutions even though one or two foregoing stability criteria are satisfied. This circumstance has led people reluctantly to utilize artificial viscosity to obtain non-oscillatory solutions.

In Warming's numerical theories, a modified equation can be derived from a difference equation obtained by discretizing a partial differential equation with any numerical scheme. The modified equation with more higher order derivatives is different from the original partial differential equation. The two differential equations are equivalent only when these high order derivatives are negligible if time step and mesh size are infinitely small. Unfortunately, the requirement cannot be achieved in practice. Therefore, it is not surprise to see that the numerical solution is quite different from the exact solution of the original partial differential equation even if the same initial and boundary conditions are specified. For instance, there is no spurious oscillation in the exact solution, but the nonphysical oscillation may often occur in the numerical solution. And also, the dissipation effect may exist in the numerical solution, even

if the original partial differential equation is, actually, viscosity-free. As a matter of fact, they come from the contribution of the high-order terms in the modified equation. Nowadays, it is understood that the generation of spurious oscillation is closely related with dispersion terms in the modified equation due to phase shift errors, and has little connection to dissipation. The artificial viscosity does work to suppress the oscillations and capture shock waves as the Lax-Wendroff and the Beam-Warming schemes demonstrate, but ruins the resolution of numerical schemes in recognizing shock waves. Even though non-linear artificial viscosity could be introduced in schemes like ENO and TVD schemes, the over-dissipated problem was often reported because of the widely-varying intensity of various discontinuities in flowfields.

Effects of the dispersion terms on numerical solutions have been realized for a long time, but extensive studies on the dispersion-controlled dissipative scheme have just been carried out for only two decades. The work on scheme dispersion reduction was reported at first by Fromm as early as in 1968^[6]. Two difference schemes having opposite phase errors are linearly combined to eliminate dispersion or even achieve a zero average phase shift. Later on, Rusanov succeeded in minimizing dispersive errors in 1970 by proposing a third-order difference scheme, of which the third-order dispersion term vanishes^[7]. However, his numerical results still exhibited tiny overshoot near discontinuities. A significant advance on dispersion study was made by Warming and Hyett in 1974^[4]. The modified equation was derived by initially expanding each term of a difference scheme in a Taylor series, and then eliminating time derivatives higher than first-order, and mixed time and space derivatives. Contrary to common practice, the original partial differential equation has not been used to eliminate these derivatives. They declared that the modified equation represents the actual partial differential equation solved when a numerical solution is computed by solving a finite difference equation. A truncated version of the modified equation can be used to gain an insight into the nature of both dissipative and dispersive errors. The role of the third-order dispersion term in the modified equation was clearly demonstrated in a paper on dispersion control in scheme construction presented by Zhang in 1988^[9]. Then, he proposed NND scheme based on a criterion that the sign of the coefficient of the third-order derivative must be changed when computation iteration goes across shock waves. He further proved that the scheme

constructed in this manner satisfies the entropy increase condition. Along this line, a further study has been carried out theoretically by Jiang in 1993^[10] and was summarized for publishing later in 1995^[11]. In his work, general dispersion conditions, based phrase shift error analysis, were proposed for construction of non-oscillatory shock capturing schemes. It is found that the conditions implemented with the Warming necessary stability condition can be taken as a sufficient stability criterion for the non-oscillatory shock-capturing schemes, and the dispersion conditions are reduced to Zhang's criteria if the first-order approximation is accepted. Fu et al. analyzed nonphysical oscillations from a viewpoint of wavelet group velocity and proposed some schemes by combining fast or slow schemes^[12]. The fast schemes have the leading phrase shift error and the slow ones have the lagging one. The dispersion conditions were extended to the third and fourth-order schemes by He and Zhang^[13], and Li et al.^[14], respectively. Reviewing these research progresses, Zhuang and Zhang et al. have recommended the dispersion conditions as a principle in the construction of high order non-oscillatory shock-capturing schemes^[15], the principle based on dispersion control. Extension of the dispersion-controlled principle to the Finite Volume Method (FVM) on unstructured grid was reported by Zhang and Zhang^[16], and the relevant scheme of the Finite Element Method (FEM) was proposed by Wu and Cai^[17]. Extensive verification, validation and applications were reported by Takayama and Jiang^[18], Jiang and Takayama^[19], Zhuang^[20], Huang et al.^[21], Ye et al.^[22], Li and Zhang^[23], Jiang et al.^[24] and Jiang (2003)^[25]. From their work, efficiency and accuracy of the dispersion-controlled dissipative schemes have been well demonstrated.

In this paper, research progresses on the dispersion-controlled principle for non-oscillatory shock-capturing schemes are reviewed, including the exploration of the role of dispersions in numerical simulations, the development of the dispersion-controlled principles, efforts devoted to high-order dispersion-controlled dissipative schemes, the extension to both FVM and FEM, scheme verification and solution validation, and comments on several aspects of the schemes from author's viewpoint.

2 WARMING'S MODIFIED EQUATION

The modified equation was proposed by Warming and Hyett in 1974 to analyze stability and accuracy of finite difference schemes^[4]. In development of

the dispersion-controlled principle, this equation occupies a very important position. From more general gas dynamic equations, its derivation is described here for completeness.

To begin with, the one-dimensional Euler equation is adopted as a model equation, given by

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = 0 \quad (1)$$

which can be rewritten as

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{U}}{\partial x} = 0 \quad (2)$$

here $\mathbf{A} = \frac{\partial \mathbf{F}(\mathbf{U})}{\partial \mathbf{U}}$ denotes Jacobian Matrix, \mathbf{U} unknown variable vector, and \mathbf{F} flux vector. According to the local linearization assumption, \mathbf{A} is taken as a constant matrix at any given grid point for analysis, therefore, it can be written as $\mathbf{A} = \mathbf{S}^{-1} \mathbf{\Lambda} \mathbf{S}$, where \mathbf{S} denotes \mathbf{A} 's right eigenvector matrix and $\mathbf{\Lambda}$ an diagonal matrix consisting of matrix \mathbf{A} 's eigenvalues. Let $\mathbf{W} = \mathbf{S}\mathbf{U}$, Eq.(2) can be rewritten as

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{\Lambda} \frac{\partial \mathbf{W}}{\partial x} = 0 \quad (3)$$

Considering only one equation in Eq.(3) for simplicity, we can write it in the form of a simple wave equation by letting $w^l = u$ and $c = \lambda^l$ to follow after Warming's notation, where ($l = 1, \dots, 3$).

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0 \quad (4)$$

It is easy to verify that

$$u(x, t) = e^{ik(x-ct)} \quad \text{for } c > 0 \quad (5)$$

is one of the exact solutions of Eq.(4) with the initial condition of $u(x) = e^{ikx}$ in the case of $c > 0$. The solution describes a simple wave with a wave number of k propagating at speed of c from left to right. If the initial condition given contains shock waves, the solution is composed of a series of the simple waves with various wave numbers. All the waves propagate at the same speed in the same direction with different amplitudes.

Partial differential Eq.(4) must be discretized with a suitable numerical scheme in computation to obtain an algebraic equation or a finite difference equation. A general form of the finite difference equation at grid point j in time step $n+1$ can be written as

$$u_j^{n+1} = f(u_{j+l}^n) \quad l = \pm 1, \dots, L \quad (6)$$

where parameter L varies depending on the numerical scheme used for discretization. Assuming that the

solution of the finite difference equation does not necessarily satisfy the corresponding partial differential equation, another partial differential equation can be derived by expanding all the terms in Eq.(6) with Taylor series at grid point j in time step n , given in the form of

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = \sum_{i=2}^{\infty} \mu_i \frac{\partial^i u}{\partial x^i} \quad (7)$$

where μ_i denotes the coefficient of the i -th order derivative. This is the equation that has been proposed and referred to as the modified equation of the finite difference equation by Warming and Hyett in 1974^[4]. Comparing it with Eq.(4), one can see that two equations are equivalent only if all the terms in the right of Eq.(7) approach to zero. It is impossible because of the limited mesh size and the time step possibly to take in computation. The numerical result of the finite difference equation, actually, is the solution of the modified equation instead of the original partial differential equation. It is not surprised to see that the numerical solution behaves different from what we expect according to the original partial differential equation, for example, smeared shock waves due to numerical viscosity, nonphysical oscillations near discontinuities and solution instability. Such discrepancies also vary depending on numerical schemes adopted, actually, on the variations of all the right terms in Eq.(7). From a viewpoint of the modified equation, the discrepancies, especially for the nonphysical oscillations, are not induced by numerical errors or perturbations possibly introduced in computational procedure, but represent the intrinsic characteristics of the modified equation. If the contribution of all the right terms in Eq.(7) to numerical solutions can be clearly classified, we may will benefit from this understanding in the construction of numerical schemes.

3 NND SCHEME

The significant progress in the application of dispersion-controlled principle to numerical scheme construction was achieved by Zhang in 1988^[9]. He proposed a stability criteria for his NND scheme, which reads

$$\begin{aligned} \mu_3 > 0 & \quad \text{behind shock waves} \\ \mu_3 < 0 & \quad \text{ahead of shock waves} \end{aligned} \quad (8)$$

To elucidate physical natures of the criteria, Zhang had carried out a theoretical analysis based on the second law of thermodynamics. By adding the third-order derivative to one-dimensional Navier-Stokes

equations to achieve a high-order analogy of the modified equation, an entropy equation is derived as follows

$$\rho T \frac{DS}{Dt} = \frac{4}{3} \mu_2 \left(\frac{\partial u}{\partial x} \right)^2 + 3 \mu_3 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} \quad (9)$$

where ρ denotes density, T temperature, DS/Dt the mass derivative of entropy. Assuming that a shock wave propagates from left to right, the following relations should be satisfied ahead of or behind the shock wave, respectively:

ahead of shock waves

$$\begin{aligned} \frac{\partial u}{\partial x} &< 0 \\ \frac{\partial^2 u}{\partial x^2} &< 0 \end{aligned} \quad (10a)$$

behind shock waves

$$\begin{aligned} \frac{\partial u}{\partial x} &< 0 \\ \frac{\partial^2 u}{\partial x^2} &> 0 \end{aligned} \quad (10b)$$

Viscosity μ_2 is zero for the Euler equation, the entropy condition cannot be satisfied if the sign of μ_3 is kept unchanged in the entire computational domain. In other words, this sign has to be varied according to Eq.(10) when calculating shock waves to simulate a physically realistic process of entropy increase. Even in solving the Navier-Stokes equation, the above-discussed problem may arise because the viscosity in aerodynamic problems is usually too small to make Eq.(9) satisfy the entropy increase principal.

According to the criteria, Zhang proposed his Non-oscillatory and Non-free parameter Dissipation (NND) difference scheme given by^[9]

$$\left(\frac{\partial U}{\partial t} \right)_j^n = -\frac{1}{\Delta x} (\mathbf{H}_{j+\frac{1}{2}}^n - \mathbf{H}_{j-\frac{1}{2}}^n) \quad (11)$$

with

$$\mathbf{H}_{j+\frac{1}{2}}^n = \mathbf{F}_{j+\frac{1}{2}L}^+ + \mathbf{F}_{j+\frac{1}{2}R}^- \quad (12)$$

where

$$\mathbf{F}_{j+\frac{1}{2}L}^+ = \mathbf{F}_j^+ + \frac{1}{2} \text{minimod}(\Delta \mathbf{F}_{j-\frac{1}{2}}^+, \Delta \mathbf{F}_{j+\frac{1}{2}}^+) \quad (13)$$

$$\mathbf{F}_{j+\frac{1}{2}R}^- = \mathbf{F}_{j+1}^- - \frac{1}{2} \text{minimod}(\Delta \mathbf{F}_{j+\frac{1}{2}}^-, \Delta \mathbf{F}_{j+\frac{3}{2}}^-)$$

$$\Delta \mathbf{F}_{j+\frac{1}{2}}^\pm = \mathbf{F}_{j+1}^\pm - \mathbf{F}_j^\pm \quad (14)$$

$$\mathbf{F}^\pm = \mathbf{A}^\pm \mathbf{U} \quad (15)$$

where the minimod function is implemented according to the shock wave properties expressed with Eq.(10),

and acts automatically as a shock wave identifier, and flux splitting is carried out with the Steger and Warming method^[8]. Actually, the NND scheme is a combination of the central difference and the second-order upwind scheme: the central difference being of zero phase shift error and the upwind scheme of the leading one. The two schemes are applied ahead of or behind shock waves, respectively, to realize the dispersion-controlled principle.

4 DISPERSION CONDITIONS

Motivated by the NND scheme, Jiang has carried out a systematic study on the dispersion-controlled principle since 1993^[10] and published his dispersion conditions for non-oscillatory shock-capturing schemes in 1995^[11]. The work is based on the analysis on the modified equation and begins with the simple wave equation expressed with Eq.(4). Assuming that the solution $u(x, t)$ has a series of form and following after the exact solution of Eq.(5), one has

$$u(x, t) = \sum_m e^{\alpha_m t} e^{ik_m x} \quad (16)$$

where k_m is the wave number and real, but α_m may be complex. Because the equation is linear, superposition can be used, therefore, we can examine behaviors of only a single term of the series.

$$u_m(x, t) = e^{\alpha_m t} e^{ik_m x} \quad (17)$$

Substitute Eq.(17) into Eq.(7) yields

$$\alpha_m + ik_m c = \sum_{n=1}^{\infty} (-1)^n k_m^{2n} \mu_{2n} + i \sum_{n=1}^{\infty} (-1)^n k_m^{2n+1} \mu_{2n+1} \quad (18)$$

Because k_m is real, α_m can be re-written as

$$\alpha_m = \alpha_r + i\alpha_i \quad (19)$$

where

$$\alpha_r = \sum_{n=1}^{\infty} (-1)^n k_m^{2n} \mu_{2n} \quad (20)$$

and

$$\alpha_i = -k_m \left[c - \sum_{n=1}^{\infty} (-1)^n k_m^{2n} \mu_{2n+1} \right] \quad (21)$$

The real part, α_r , contains all the coefficients of the even-order derivatives and the imagery part, α_i , includes all the odd-order ones. Substituting Eqs.(19)~(21) into Eq.(17) yields

$$u_m(x, t) = \exp \left\{ \left[\sum_{n=1}^{\infty} (-1)^n k_m^{2n} \mu_{2n} \right] t \right\}.$$

$$\exp \left\{ ik \left\{ x - \left[c - \sum_{n=1}^{\infty} (-1)^n k_m^{2n} \mu_{2n+1} \right] t \right\} \right\} \quad (22)$$

The first term of the solution represents the wave amplitude evolution resulting from all the dissipation terms in the modified equation, and the second one denotes the wave propagating speed that varies depending on all the dispersion terms. If the terms in the right of Eq.(7) approaches to zero, the solution reduces to the exact solution of Eq.(5). This requirement can imply that

$$\sum_{n=1}^{\infty} (-1)^n k_m^{2n} \mu_{2n} = 0 \quad (23a)$$

and

$$\sum_{n=1}^{\infty} (-1)^n k_m^{2n} \mu_{2n+1} = 0 \quad (23b)$$

These two conditions are impossible to satisfy because the scheme being of infinitely high order of accuracy is not available in practice. In computation, numerical solutions must be bounded so that numerical schemes have to be dissipative. To ensure a scheme being dissipative, the requirement could be satisfied if the sum of all the dissipation coefficients is negative. The condition reads

$$\sum_{n=1}^{\infty} (-1)^n k_m^{2n} \mu_{2n} < 0 \quad (24)$$

To obtain a criteria more applicable according to Eq.(24), Warming and Hyett proposed their heuristic (necessary) condition which can be expressed as

$$\sum_{n=1}^{\infty} (-1)^{(n-1)} \mu_{2n} > 0 \quad (25)$$

The Warming's scheme^[8] can be demonstrated to satisfy the necessary condition, but nonphysical oscillations still occur near shock waves. Therefore, artificial viscosity has to be introduced into the scheme to suppress the oscillations. From Eq.(22), it is obvious that the higher frequency wave will decay more rapidly due to its bigger dissipation coefficient and the role of the artificial viscosity is easy to understand.

After examining dispersion terms in the modified equation, Jiang^[10] pointed out that the non-physical oscillation is resulting from phase shift errors represented by the dispersion terms but it is also impossible to remove all the terms because these arise from the intended analogy of the convective flow phenomenon in fluid dynamics. It may be easy to explain

if one examine a shock solution in spectrum space since the solution of a shock wave can be considered to be composed of a series of the simple waves of different wave numbers with help of the Fourier transformation. The transformation can be demonstrated with Fig.1, where a periodic square wave is decom-

posed into a series of the simple sine waves. The contribution of the first three low frequency sine waves to the periodic square wave is presented in Figs.1(b)~1(d), respectively. For these waves, the higher the wave number is, the lower the amplitude becomes.

From the exact solution of Eq.(4), it is known that these simple sine waves must propagate at the same speed so that the discontinuity in the rectangular wave could be kept sharp and moves at a given speed. In case of the wave speed changing upon their wave numbers due to phase shift, the leading or lagging errors as demonstrated in examining the modified equation, these simple waves will behave ahead of or behind the discontinuity, which leads to the so-called nonphysical oscillation. The oscillation is, actually, not numerical errors arising from computation, but the reflection of nature of the modified equation associated with numerical schemes in use.

To verify the idea, Jiang et al.^[11] examined both the Lax-Wendroff and the Beam-Warming schemes by solving Eq.(4) in the domain $(-1, 1)$ with initial and boundary conditions for $u(x, t)$ as given below

$$u(x, 0) = \begin{cases} 1 & x \in [-1, -0.5] \\ 0 & x \in (-0.5, 1] \end{cases} \quad (26)$$

The modified equation of the difference equation obtained with the Lax-Wendroff scheme is

$$u_t + cu_x = -\frac{1}{6}c(\Delta x)^2(1 - \nu^2)u_{xxx} - \frac{1}{8}c(\Delta x)^3\nu(1 - \nu^2)u_{xxxx} + \dots \quad (27)$$

where $\nu = (c\Delta t)/(\Delta x)$, and Δt and Δx are time step and mesh size, respectively. By applying the Beam-Warming upwind scheme^[8] to Eq.(4), the modified equation can be expressed as

$$u_t + cu_x = \frac{1}{6}c(\Delta x)^2(1 - \nu)(2 - \nu)u_{xxx} - \frac{1}{8\Delta t}c(\Delta x)^4\nu(1 - \nu)^2(2 - \nu)u_{xxxx} + \dots \quad (28)$$

If $\nu < 1$, the coefficient of the third-order derivative is negative in Eq.(27), but positive in Eq.(28). The coefficient of the fourth-order derivative is negative in both the equations, and therefore, satisfies the Warming's stability condition. Furthermore, two third-order dispersion coefficient is equal to each other if ν is taken to be 0.5, but the fourth-order dissipation coefficients are different. The numerical results obtained with two schemes are plotted in Fig.2. Fig-

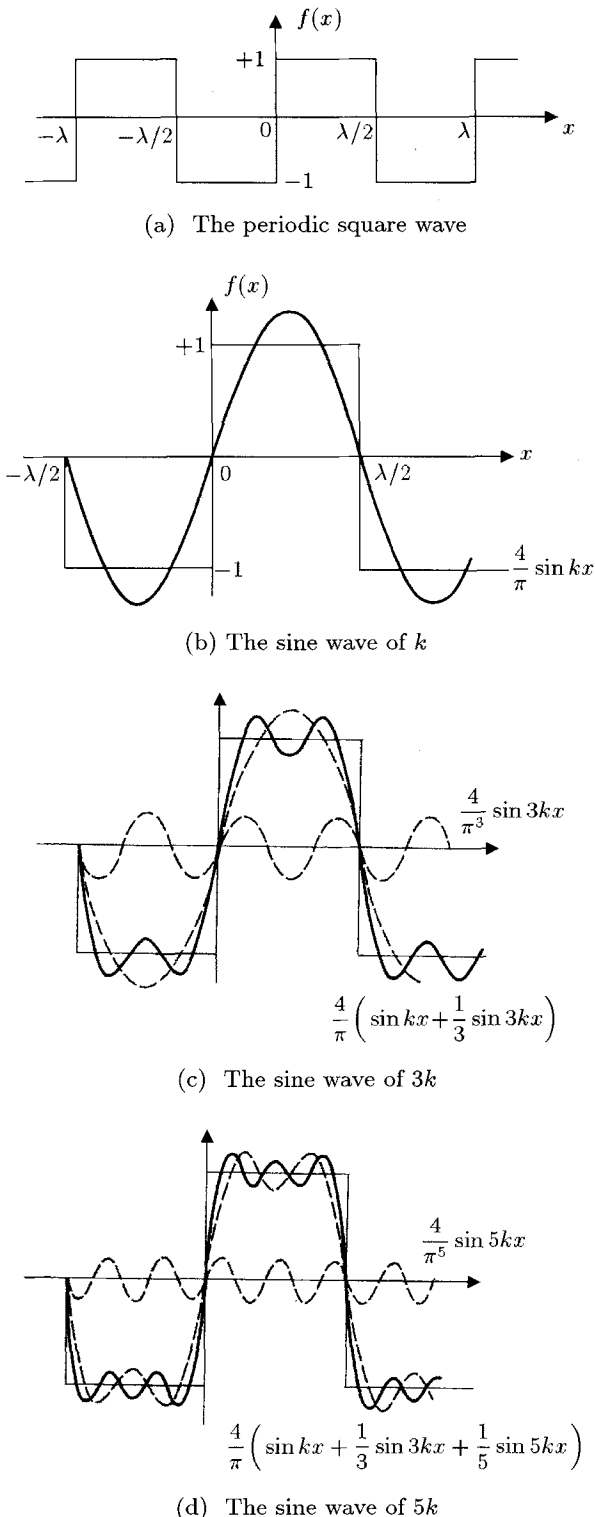
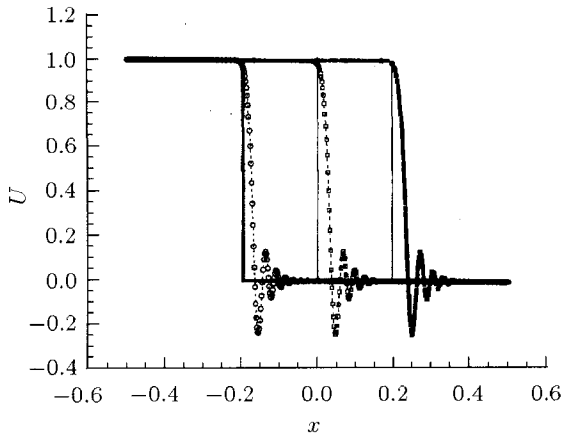
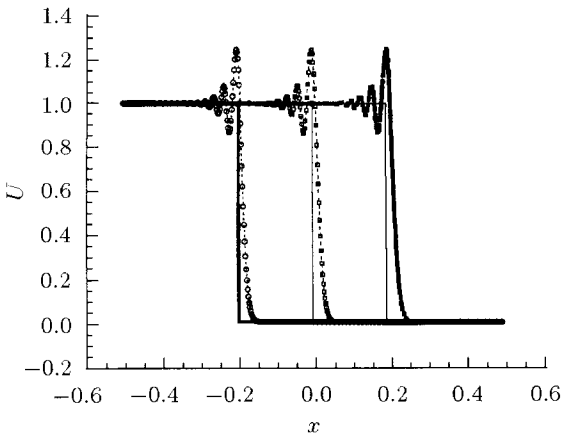


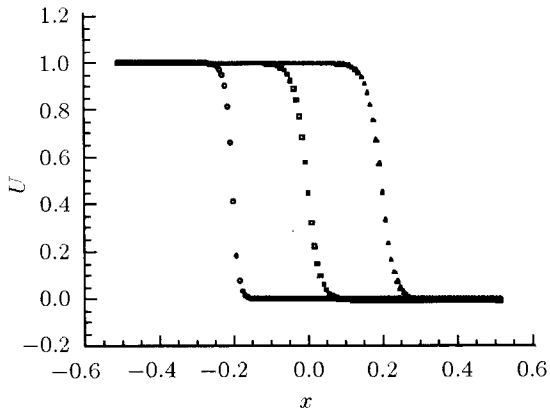
Fig.1 Synthesis of a periodic square wave, the contribution of the first three low frequency waves



(a) Beam-Warming scheme



(b) Lax-Wendroff scheme



(c) Dispersion-Controlled Scheme

Fig.2 Dispersion behaviors of three scheme in solving the scale wave equation for shock wave propagation^[11]

ure 2(a) shows that the positive dispersion does produce pre-shock oscillations due to the leading phrase error of the Beam-Warming scheme, and Fig.2(b) shows that the negative dispersion leads to post-shock

oscillations due to the lagging one of the Lax-Wendroff scheme. And also, the oscillations are of the same amplitude and the same frequency. This may imply that the third-order dispersion plays an important role in generating the nonphysical oscillation and the dissipation has little effect on it in the test case. Figure 2(c) shows the result obtained by applying the Beam-Warming scheme behind the shock wave and the Lax-Wendroff scheme ahead of the shock. It is amazing that the oscillation totally disappeared without any need of the artificial viscosity.

Based on the above analysis and the numerical experiments, Jiang proposed his dispersion conditions for non-oscillatory shock-capturing scheme^[10,11]. The conditions read, behind shock waves

$$\sum_{n=1}^{\infty} (-1)^{n+1} k_m^{2n+1} \mu_{2n+1} > 0 \quad (29a)$$

ahead of shock waves

$$\sum_{n=1}^{\infty} (-1)^{n+1} k_m^{2n+1} \mu_{2n+1} < 0 \quad (29b)$$

The primary concept having been casted into the conditions is to force high frequency waves to concentrate at a shock wave, in which the high frequency waves should be located, by actively changing the sign of phase shift errors of numerical schemes when computation proceeds across a shock wave. The dispersion conditions implemented with the Warming's stability condition were evaluated as the sufficient conditions for non-oscillatory shock-capturing schemes.

If the first-order approximation is applied to the dispersion conditions, the conditions for second-order schemes are reduced to the stability criteria proposed by Zhang in 1988. It reads

$$\begin{aligned} \mu_3 > 0 & \quad \text{behind shock waves} \\ \mu_3 < 0 & \quad \text{ahead of shock waves} \\ \mu_4 < 0 & \quad \text{in the whole region} \end{aligned} \quad (30)$$

5 DISPERSION-CONTROLLED SCHEMES

In order to verify the dispersion conditions, Jiang et al. had proposed a second-order scheme referred to as the Dispersion-Controlled Scheme (DCS)^[11]. The scheme is a combination of the Lax-Wendroff and Beam-Warming schemes with the minimod limiter and can be also expressed with Eqs.(11) and (12), and its numerical flux is expressed as

$$\begin{aligned}
 F_{j+\frac{1}{2}L}^+ &= F_j^+ + \frac{1}{2}\Phi_A^+ \\
 \text{minimod}(\Delta F_{j-\frac{1}{2}}^+, \Delta F_{j+\frac{1}{2}}^+) & \\
 F_{j+\frac{1}{2}R}^- &= F_{j+1}^- - \frac{1}{2}\Phi_A^- \\
 \text{minimod}(\Delta F_{j+\frac{1}{2}}^-, \Delta F_{j+\frac{3}{2}}^-) &
 \end{aligned}
 \quad (31)$$

and

$$\Phi_A^\pm = I \pm \beta \Lambda_A^\pm \quad (32)$$

where I is a unit matrix, $\beta = \Delta t / \Delta x$, and Λ_A is a matrix consisting of the eigenvalues of matrix A . In these equations, the $(\cdot)^+$ or $(\cdot)^-$ superscript denotes flux vector splitting according to Steger and Warming^[26].

A meaningful numerical result of the shock tube problem, as shown in Fig.3, was reported by Jiang in 1993^[10]. For the classical problem, it is well known that the shock wave and the contact surface propagate down-stream, but the expansion waves travel up-

stream. Figure 3(a) shows the results calculated with the DCS scheme, with which the extension of the dispersion-controlled principle to nonlinear equations was well demonstrated. Figure 3(b) shows the result obtained with the Beam-Warming scheme, from which nonphysical oscillations are observable at all the types of discontinuities. According to the above analysis on the Beam-Warming scheme, it is known that the scheme has the leading phrase error, and its numerical solution must have oscillations in front of discontinuities. The conclusion was well verified with Fig.3(b), where oscillations observable ahead of the shock wave, the contact surface and the expansion waves. This indicates that the dispersion conditions are applicable not only to shock waves, but also to other discontinuities whatever has high gradient in its flow variables. Moreover, the dispersion conditions drawn from the analysis based on the linearized partial differential equation work well for non-linear one-dimensional Euler equations.

The analysis on phrase shift errors was also reported by Fu et al. in 1996 with the group velocity of wavelets^[12], and the effect of phrase shift errors on the solution of the partial differential equation was discussed from another viewpoint. Some schemes were tested as combinations of the fast schemes (with leading phrase error) and the slow schemes (with lagging phrase error). In the 7th International Symposium of Computational Fluid Dynamics, Zhuang and Zhang et al. (1997) had reviewed CFD research progresses in China, and the dispersion conditions were re-derived and evaluated as the principle for the construction of non-oscillatory shock-capturing schemes^[15].

The above discussion indicates that the schemes based on the dispersion conditions are different from conventional shock-capturing schemes in their principles of stability requirements. Therefore, it is recommended to define the scheme satisfying both the dispersion conditions and the Warming's necessary condition as the Dispersion-Controlled Dissipative (DCD) scheme to distinguish it from the conventional dissipative schemes. This is helpful not only for understanding the scheme, but also for improving it by considering its nature in future.

In order to solve partial differential equations with numerical methods by using computers, high-order errors, as expressed by the modified equations, cannot be avoided. However, if the high-order errors can be utilized to remove nonphysical oscillation, this must be better than adding more extra terms to dissipate it. This is because that there may be many

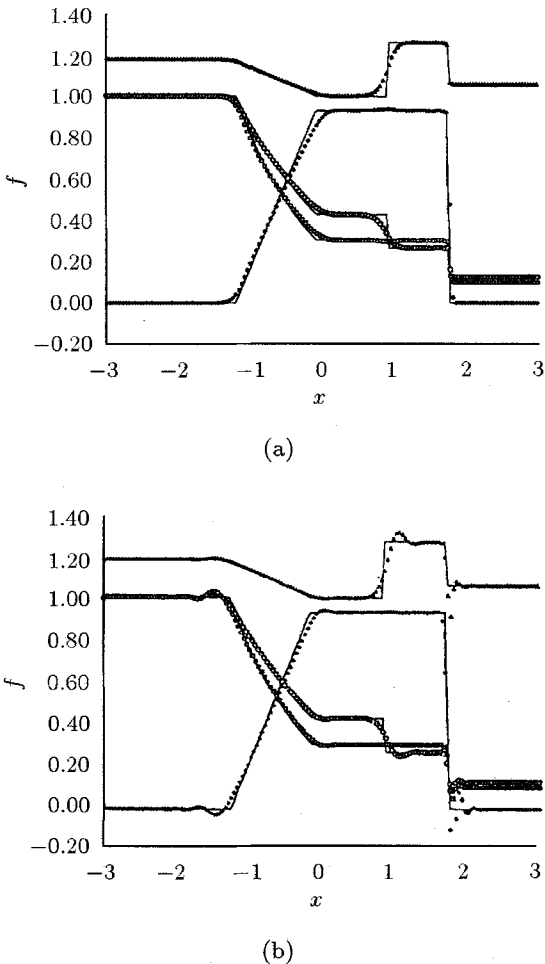


Fig.3 Solutions of the shock tube problem solved with (a) the DCS scheme and (b) the Beam-Warming scheme^[10]

shock waves and contact surfaces in complex flow-fields, and their intensity varies dramatically here and there. The artificial dissipation required must vary according to oscillation amplitudes that depend on the discontinuity intensity so that optimum effects could be achieved. Unfortunately, it is too difficult to be realized, therefore, the over or under-dissipated problems could not be avoided. Moreover, the free parameters for controlling artificial dissipation have no physical meaning and their selections induce something artificial in numerical solutions. In conclusion, the dispersion-controlled principal has been demonstrated to be an active way in which non-oscillatory shock-capturing schemes can be constructed, which indicates a new direction for the research of CFD theories. It has been proved that the dispersion-controlled dissipative schemes satisfy the entropy increase principal as the TVD scheme does. This does not mean the DCD schemes are the same as the TVD schemes, but indicates that all the well-designed schemes must satisfy some more fundamental physical laws like the

entropy increase principal.

6 DEVELOPMENT OF HIGHER ORDER DCD SCHEME

6.1 The Third-order DCD Scheme

To develop third-order non-oscillatory shock-capturing schemes based on the dispersion-controlled principal, the criteria for stability resulting from the second-order approximation of the dispersion conditions can be given as

$$\begin{aligned} \mu_5 < 0 & \quad \text{behind shock waves} \\ \mu_5 > 0 & \quad \text{ahead of shock waves} \\ \mu_4 < 0 & \quad \text{in the entire region} \end{aligned} \quad (33)$$

According to the criteria, He and Zhang proposed their third-order Essentially Non-oscillatory containing No free parameter (ENN) scheme^[13]. By following after Li's notation^[14], the numerical flux in the scheme can be written as

$$H_{j+\frac{1}{2}}^+ = F_j^+ + \frac{1}{2} \begin{cases} \Delta F_{j+\frac{1}{2}}^+ - \frac{1}{6} \text{minimod}(\Delta F_{j+\frac{3}{2}}^+ - \Delta F_{j+\frac{1}{2}}^+, \Delta F_{j+\frac{1}{2}}^+ - \Delta F_{j-\frac{1}{2}}^+) & \text{for } sw^+ < 0 \\ \Delta F_{j-\frac{1}{2}}^+ + \frac{1}{3} \text{minimod}(\Delta F_{j+\frac{1}{2}}^+ - \Delta F_{j-\frac{1}{2}}^+, \Delta F_{j-\frac{1}{2}}^+ - \Delta F_{j-\frac{3}{2}}^+) & \text{for } sw^+ \geq 0 \end{cases} \quad (34)$$

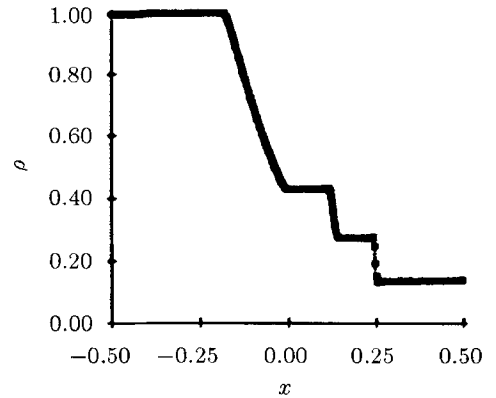
$$H_{j+\frac{1}{2}}^- = F_{j+1}^- - \frac{1}{2} \begin{cases} \Delta F_{j+\frac{3}{2}}^- + \frac{1}{3} \text{minimod}(\Delta F_{j+\frac{5}{2}}^- - \Delta F_{j+\frac{3}{2}}^-, \Delta F_{j+\frac{3}{2}}^- - \Delta F_{j+\frac{1}{2}}^-) & \text{for } sw^- < 0 \\ \Delta F_{j+\frac{1}{2}}^- - \frac{1}{6} \text{minimod}(\Delta F_{j+\frac{3}{2}}^- - \Delta F_{j+\frac{1}{2}}^-, \Delta F_{j+\frac{1}{2}}^- - \Delta F_{j-\frac{1}{2}}^-) & \text{for } sw^- \geq 0 \end{cases} \quad (35)$$

where, the switching limiter sw^\pm has the following definition

$$sw^+ = \Delta F_{j+\frac{1}{2}}^+ - \Delta F_{j-\frac{1}{2}}^+ \quad (36a)$$

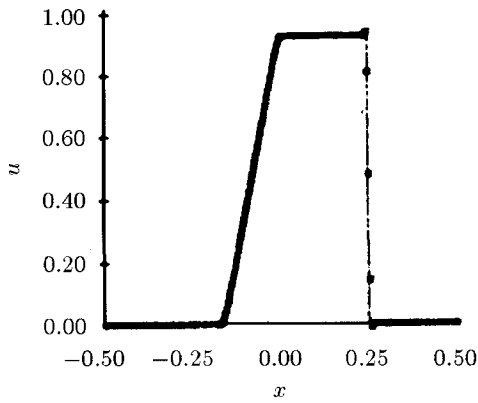
$$sw^- = \Delta F_{j+\frac{3}{2}}^- - \Delta F_{j+\frac{1}{2}}^- \quad (36b)$$

Numerical tests were carried out with the third-order scheme and satisfactory results of the shock tube problem were obtained with nice resolution of the contact surface that is often seared with second-order schemes, as shown in Fig.4. If carefully examining their numerical results, tiny oscillations are still observable near the vicinity of discontinuities.



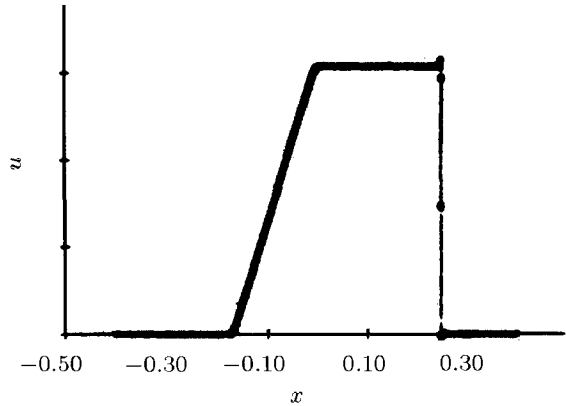
(a) Density

Fig.4 Solutions of the shock tube problem solved with the third-order DCD scheme^[13]



(b) Velocity

Fig.4 Solutions of the shock tube problem solved with the third-order DCD scheme^[15] (continued)



(b) Velocity

Fig.5 Solutions of the shock tube problem solved with the fourth-order DCD scheme^[15]

6.2 The Fourth-order DCD Scheme

For the fourth-order DCD scheme, the requirement for non-oscillations and stability can be proposed as follows

$$\begin{aligned}
 \mu_5 < 0 & \quad \text{behind shock waves} \\
 \mu_5 > 0 & \quad \text{ahead of shock waves} \\
 \mu_6 > 0 & \quad \text{in the entire region}
 \end{aligned}
 \tag{37}$$

Based on the requirements, Li et al.^[14] have developed a fourth-order scheme and more detailed information can be found in Refs.[9,15]. Even higher-order schemes are also possible to develop, but the schemes are usually too complex to be applied in practice.

Verification of the fourth-order DCD schemes was carried out, and the result of the shock tube problem was summarized by Zhuang et al. in his review lecture^[15], and is cited here in Fig.5. Figure 5 demon-

strated higher resolution of the discontinuities, especially for the contact surface. However, tiny oscillations are also observable near the shock wave. The reason for the problem is not clear and needs further investigation.

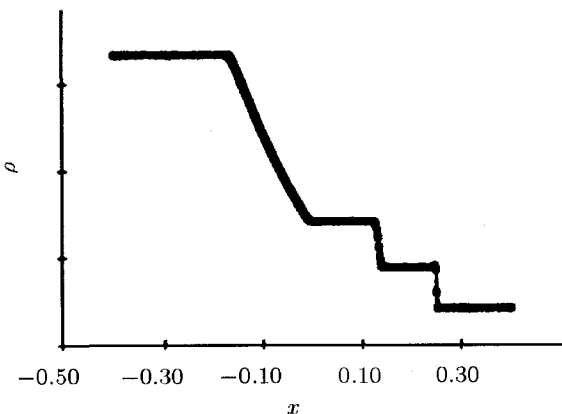
Both the second and third-order DCD schemes are dissipative because the schemes have the fourth-order dissipation term. The second DCD scheme demonstrates its non-oscillatory characteristic successfully, but the third-order one does not avoid tiny oscillations. Zhuang and Zhang et al. did not explain the reason for the observation, but recommended to use the entropy increase condition when calculation crossing shock waves^[15]. From author's viewpoint, the problem may be related to the scheme limiter. As is well known, the high-order schemes must be reduced to lower-order schemes in shock wave computation. It is expected that there should be no oscillations if the high-order schemes can be properly reduced to the second-order one. And also, the effort on pursuing high-order schemes in shock waves should be considered carefully because any order derivative in shock waves does not exist.

7 EXTENSION TO THE FVM AND FEM

7.1 The Finite Volume Scheme

The extension of the DCD schemes to the finite volume method was reported by Zhang et al. and numerical results on unstructured grid were presented in their work^[16]. In the extension, the two-dimensional Euler equations in Cartesian coordinates can be written as

$$\iint_{\Omega} \frac{\partial U}{\partial t} d\Omega + \oint (\mathbf{F}n_x + \mathbf{E}n_y) dl = 0 \tag{38}$$



(a) Density

where, \mathbf{F} and \mathbf{E} are flux vectors in x - and y -direction, respectively, Ω denotes the element cell for integration, and n_x and n_y are the components of the normal vector \mathbf{n} on boundaries in x - and y -directions. Integrating the second term in Eq.(38) on a triangle element yields

$$\oint (\mathbf{F}n_x + \mathbf{E}n_y)dl = \sum_{k=1}^3 (\mathbf{F}_k n_x + \mathbf{E}_k n_y)dl_k \quad (39)$$

To obtain a second-order scheme, the key fact is how to calculate flux vectors \mathbf{F} and \mathbf{E} on element boundaries. Taking it as a Riemann problem and considering the principal of the Gaudunov scheme, \mathbf{F}_k on the boundaries can be calculated with the linear interpolation.

$$\mathbf{F}_k = \mathbf{F}_L^+ + \mathbf{F}_R^- \quad (40)$$

$$\mathbf{F}_L^+ = \mathbf{F}_i^+ + \left(\frac{\partial \overline{\mathbf{F}}}{\partial x}\right)^+ \Delta x_{ik} + \left(\frac{\partial \overline{\mathbf{F}}}{\partial y}\right)^+ \Delta y_{ik} + \dots \quad (41a)$$

$$\mathbf{F}_R^- = \mathbf{F}_j^- + \left(\frac{\partial \overline{\mathbf{F}}}{\partial x}\right)^- \Delta x_{jk} + \left(\frac{\partial \overline{\mathbf{F}}}{\partial y}\right)^- \Delta y_{jk} + \dots \quad (41b)$$

where

$$\left(\frac{\partial \overline{\mathbf{F}}}{\partial x}\right)^+ = \text{minmod} \left[\left(\frac{\partial \mathbf{F}}{\partial x}\right)_i^+, \left(\frac{\partial \mathbf{F}}{\partial x}\right)_j^+ \right]$$

$$\left(\frac{\partial \overline{\mathbf{F}}}{\partial y}\right)^+ = \text{minmod} \left[\left(\frac{\partial \mathbf{F}}{\partial y}\right)_i^+, \left(\frac{\partial \mathbf{F}}{\partial y}\right)_j^+ \right]$$

$$\left(\frac{\partial \overline{\mathbf{F}}}{\partial x}\right)^- = \text{minmod} \left[\left(\frac{\partial \mathbf{F}}{\partial x}\right)_i^-, \left(\frac{\partial \mathbf{F}}{\partial x}\right)_j^- \right]$$

$$\left(\frac{\partial \overline{\mathbf{F}}}{\partial y}\right)^- = \text{minmod} \left[\left(\frac{\partial \mathbf{F}}{\partial y}\right)_i^-, \left(\frac{\partial \mathbf{F}}{\partial y}\right)_j^- \right]$$

where $\Delta x_{ik} = x_k - x_i$, $\Delta y_{ik} = y_k - y_i$, $\Delta x_{jk} = x_k - x_j$ and $\Delta y_{jk} = y_k - y_j$. The formula of flux vector \mathbf{E} can be also defined in the same way. The equation for calculation can be obtained by applying the definitions in Eq.(39) and completing the integral. Numerical tests showed the extension works well and shock waves can be captured with satisfactory accuracy.

7.2 The Finite Element Scheme

The NND finite element scheme on unstructured grid was proposed by Wu and Cai by combining the central difference and upwind scheme^[17]. Difficulty for applying the dispersion-controlled principle to the

finite element method is how to achieve the upwind effect in numerical flux calculations in which the flux is represented by both the primary and the weighted functions. Wu and Cai, first, divided the NND scheme into two parts: the central difference part and the modified upwind one,

behind shock waves

$$\begin{aligned} \mathbf{U}_j^{n+1} = & \mathbf{U}_j - \frac{\Delta t}{2\Delta x} (\mathbf{F}_{j+1}^{+n} - \mathbf{F}_{j-1}^{+n}) + \\ & \frac{\Delta t}{2(\Delta x)^2} (\mathbf{F}_{j+1}^{+n} - 3\mathbf{F}_j^{+n} + 3\mathbf{F}_{j-1}^{+n} - \mathbf{F}_{j-2}^{+n}) \end{aligned} \quad (42a)$$

ahead of shock waves

$$\begin{aligned} \mathbf{U}_j^{n+1} = & \mathbf{U}_j - \frac{\Delta t}{2\Delta x} (\mathbf{F}_{j+1}^{-n} - \mathbf{F}_{j-1}^{-n}) + \\ & \frac{\Delta t}{2(\Delta x)^2} (\mathbf{F}_{j+2}^{-n} - 3\mathbf{F}_{j+1}^{-n} + 3\mathbf{F}_j^{-n} - \mathbf{F}_{j-1}^{-n}) \end{aligned} \quad (42b)$$

The second-order upwind scheme in the above equations will be reduced to the first-order one in shock wave computation. The NND finite element scheme can be written as,

behind shock waves

$$GE + \frac{\Delta t}{2} \int_{\Omega} (\Delta x_j)^2 \frac{\partial^3 \mathbf{F}_{ji}^{+n}}{\partial x_j^3} \mathbf{N}_k d\Omega \quad (43a)$$

ahead of shock waves

$$GE + \frac{\Delta t}{2} \int_{\Omega} (\Delta x_j)^2 \frac{\partial^3 \mathbf{F}_{ji}^{-n}}{\partial x_j^3} \mathbf{N}_k d\Omega \quad (43b)$$

with

$$GE = \sum_e \left(\int_{\Omega_e} \mathbf{N}_j \mathbf{N}_k d\Omega \right) d\mathbf{U}_j \quad (44)$$

where \mathbf{N}_k are the primary functions, \mathbf{N}_j are the weighted functions, and \mathbf{F}_{ji}^{\pm} is the upwind flux splitting. The central difference part, denoted by GE , is easy to be realized as done in the conventional finite element method, but the upwind part needs to be treated with a special care. To achieve the upwind effect, Wu and Cai suggested to specify an upwind coefficient for each element according to ratio of the element area to the total area of all the elements surrounding the presently-calculated mesh point. When one calculate \mathbf{F} at a given mesh point assuming linear interpolation, its element equation can be given as

$$\int_e G \mathbf{N}_j d\Omega = \int_e \frac{\partial \mathbf{F}}{\partial x} \mathbf{N}_j d\Omega \quad (45)$$

where $\mathbf{F} = \mathbf{F}_i \mathbf{N}_i$, $\mathbf{G} = \mathbf{G}_i \mathbf{N}_i$ and $\mathbf{G} = \partial \mathbf{F} / \partial x$. If one denotes $b_i = \partial \mathbf{G} / \partial x$, the equation can be written as

$$\begin{aligned} \mathbf{G}_i \int_e \mathbf{N}_i \mathbf{N}_j d\Omega &= \mathbf{F}_i \int_e \frac{\partial \mathbf{N}_i}{\partial x} \mathbf{N}_j d\Omega \\ &= b_i \mathbf{F}_i \int_e \mathbf{N}_j d\Omega \end{aligned} \quad (46)$$

For a given element, choosing different weight functions will induce controllable upwind effects. If the weighted functions are taken as the same as the primary functions, the central element scheme, as expressed by Eq.(44), can be achieved. Numerical experiments were carried out with the NND finite element scheme and predicted results indicated that the shock wave reflection agrees with the common understanding. It is important to be pointed out that the upwind difference expressions, adopted in Eq.(42), have to be calculated in a similar way as in the finite difference method.

8 SCHEME VALIDATION AND SOLUTION VERIFICATION

Dispersion-controlled dissipative schemes have been widely applied, especially in China, for more than ten years. Many numerical results have been published in journals or conference proceedings during the period, with which the reliability and accuracy of the schemes were well demonstrated. A number of selected results are presented here to show scheme verification and solution validation.

8.1 Shock Wave Reflection

For validation of two-dimensional numerical solutions of the Euler equations, the case with exact solution is not available. Therefore, the direct comparison with experiments is recommended for validation, from which confidence on CFD solutions could be gained. Such a comparison was reported by Takayama and Jiang and cited here in Fig.6 as reference^[18]. This is a shock wave propagating at Mach number of 2 and reflecting from a wedge with a 49° angle. The angle is close to the critical angle at which the regular reflection will transfer to the Mach reflection. Their report is a summary of a benchmark test called by Shock Waves Journal in 1995, and three sets of experimental images and more than ten numerical results, calculated with various schemes under the required computational conditions, were submitted. From the summary, it is observable that some schemes are failure to capture the short Mach stem correctly, and the

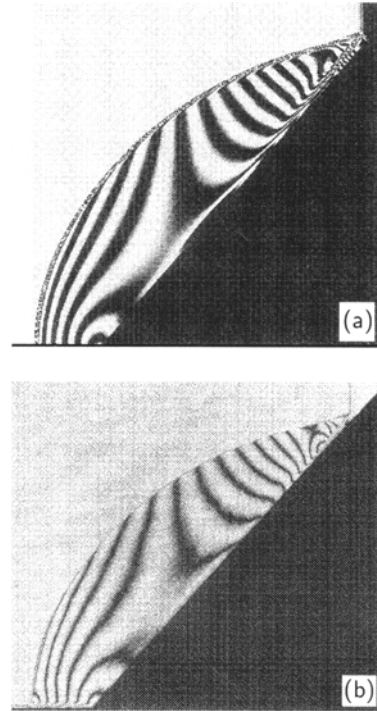


Fig.6 Shock wave reflecting over wedge of 49° angle for $M_i = 2.0$: (a) Numerical result; (b) Experimental interferogram^[19]

Mach stem even disappeared in a few of numerical results. If examining the numerical result in Fig.6(a), it is obvious that the Mach stem and the shock reflection simulated with the DCD scheme are in good agreement with the experimental image in Fig.6(b). The DCD scheme has demonstrated its superior performance.

8.2 Shock Wave Diffraction

The comparison of an axisymmetric case was provided by Jiang and Takayama in 1999^[19]. The case is a shock wave discharging from the open end of a shock tube at a Mach number of $M_i = 1.6$. The numerical result is presented in Fig.7(a) and the experimental interferogram in Fig.7(b). It can be seen from the comparison that the agreement between the numerical result and the experimental interferogram is excellent. This is not only because the number of fringes is identical but the distribution of the individual fringes matches well with each other with only minor exceptions. In fact, the largest deviation in fringe positions is less than half of the fringe distance. This case is more useful since there are many fringes that can be utilized in validation.

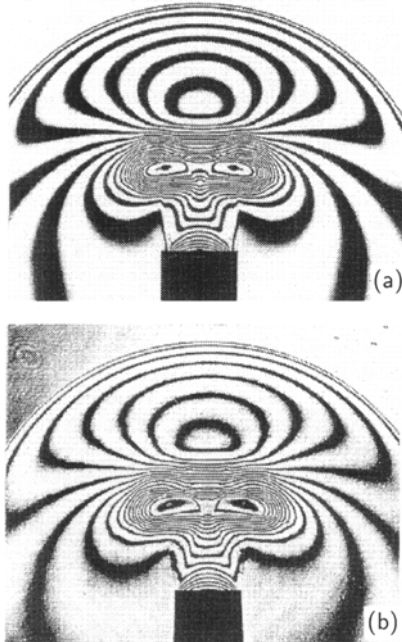


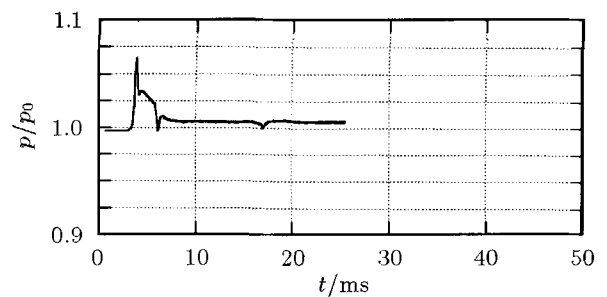
Fig.7 Shock wave diffracting at the open end of a shock tube: (a) Numerical result; (b) Experimental interferogram^[20]

8.3 High-speed Train/Tunnel Problem

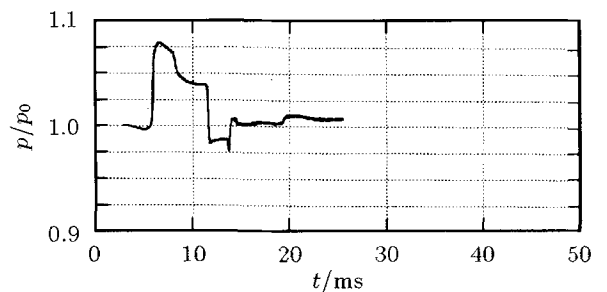
Another interesting result for axisymmetric cases was reported by Jiang et al. in 2002 and the train/tunnel problem was investigated in their work^[24]. Figures 8 and 9 show one pair of their results where numerical pressure variations recorded at three measurement stations were compared with the experimental data for a train speed of 360 km/h. The experiment was conducted in a scaled train/tunnel simulator in the Institute of Fluid Science, Tohoku University, Japan, and the numerical simulation was carried out by solving the Euler equations with a second-order DCD scheme in the computational domain similar to the scaled train/tunnel simulator. From the two figures, it is clearly observable that the pressure jump at the beginning is induced by the arrival of a weak shock wave due to the abrupt-entering of high-speed trains into the entrance of railway tunnels; the gradually decreasing pressure is related to the flow state transformation from the abrupt-entering state to the one created by an infinitely long train moving in a railway tunnel; the faster decreasing pressure is due to catching up of the expansion waves generated by the abrupt-entering of the tail of high-speed trains into the entrance of railway tunnels; the uniform pressure is considered to be created by an infinitely long train moving in a railway tunnel; finally, the sudden pressure decrease comes when the high-speed train passes

by the measurement stations. These important flow phenomena are reflected well both in the experimental data and the numerical results. Only notable discrepancy is the overpressure value that was over-estimated in the numerical simulation. It is believed that the discrepancy is due to viscosity and turbulence that are not considered in numerical simulations.

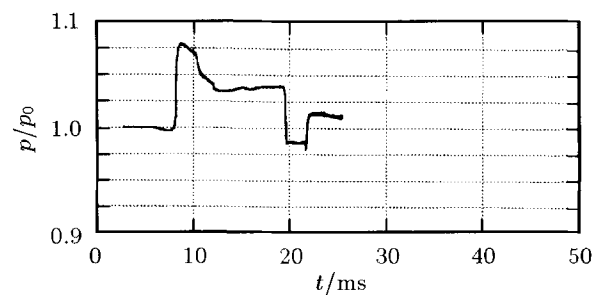
A lot of numerical simulations have been reported by many authors since the NND scheme was proposed, including three-dimensional problems^[19,20,25,27], chemically reacting gas flows^[21], magneto-hydrodynamic flowfield^[22], rarefield transition to continuum^[23], and others^[23,25]. These works will be not cited here due to length limitation of the



(a)



(b)



(c)

Fig.8 Numerical results showing pressure histories recorded at three measurement stations for a train speed of 360 km/h^[24]

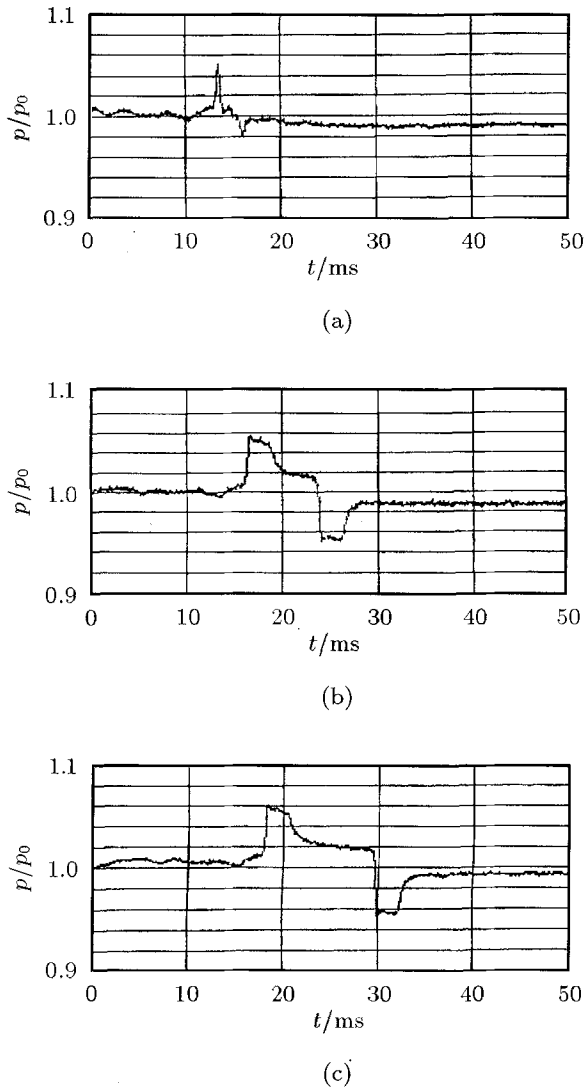


Fig.9 Experimental data showing pressure histories measured at three measurement stations for a train speed of 360 km/h^[24]

paper, however, reliability and accuracy of the dispersion-controlled dissipation schemes have been well demonstrated by these scheme verifications and solution validations.

The progress in dispersion-controlled dissipation schemes is remarkable, but there are still something important to be investigated in future. For example, the DCD scheme may result in the steepness of the flow phenomena being of high gradient because of its phrase shift nature. The effect may be negligible for most of the physical phenomena, but needs to be cleared. More attention should be paid to the development of high-order DCD schemes to meet requirements in simulation of high Reynolds number turbulent flows. Some promising progresses in the aspect have been achieved, but more work is still required for

further investigating into its mechanism in dispersion control. The extension of the dispersion-controlled principle to unstructured grid appears in progress and the proposed methods need to be improved for engineering applications.

9 CONCLUDING REMARKS

The dispersion condition implemented with Warming's stability condition works well in constructing the shock-capturing scheme with no oscillations, no free parameters and no need of artificial viscosity. It indicates that the two conditions are able to serve as fundamental principals for checking shock-capturing schemes. The DCD schemes based on the dispersion conditions are distinct from the conventional dissipation-based schemes, therefore, the difference should be emphasized to understand the schemes. The DCD schemes may be proved to have certain features similar to some well-known schemes, but this just implies that all the well-designed schemes have to satisfy certain physical principals. The DCD schemes have been well verified and validated during more than ten years' applications, from which confidence in the schemes to a certain degree has been gained. This brief review was prepared only from author's viewpoint, but it can serve as a useful reference to aid people in better understanding the DCD schemes and promote future relevant research.

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