Applied Mathematics and Mechanics (English Edition), 2007, **28**(10):1299–1308

Dynamic equations for curved submerged floating tunnel *

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Abstract In virtue of reference Cartesian coordinates, geometrical relations of spatial curved structure are presented in orthogonal curvilinear coordinates. Dynamic equations for helical girder are derived by Hamilton principle. These equations indicate that four generalized displacements are coupled with each other. When spatial structure degenerates into planar curvilinear structure, two generalized displacements in two perpendicular planes are coupled with each other. Dynamic equations for arbitrary curvilinear structure may be obtained by the method used in this paper.

Key words submerged floating tunnel (SFT), dynamic equations, Hamilton principle, curved girder

Chinese Library Classification U459.9, TB122
2000 Mathematics Subject Classification 74K10, 70H25
Digital Object Identifier(DOI) 10.1007/s10483-007-1003-z

Introduction

Submerged floating tunnel (SFT), also called Archimedes bridge, is a potential traffic channel floating between water surface and water bed, which is a solid structure made of metals or reinforced concrete or mixed. In some water areas of environmental restriction or protection, it may be impossible to set up a traditional crossing. For such cases, SFT could be a promising solution. However, till now, there is not an SFT constructed in the world. This is a great challenge to engineering science community and has attracted preliminary investigations[1−6].

Due to local geographical and environmental conditions, a curved sect at the end of an SFT to connect the water bank sometimes is more feasible so as to decrease tunnel length and/or reduce tunnel slope. Double wall structure is often used in the conceptual design of SFT for which warp effect could be ignored. Thus the SFT structure may be regarded as a spatial girder without warping. Several research results are available on dynamic behavior of curved girder[7−12], which mostly are related to planar curved bridge structure, especially about planar circular structure. However, there are few reports related to dynamic behavior of spatial curved structure.

The dynamic behavior of SFT subjected to water wave and current is a major problem in the SFT realization. This paper considers the dynamic behavior of curved girder used in civil engineering especially in SFT structure. Based on reference Cartesian coordinates and orthogonal curvilinear coordinates, dynamic equations for helical girder are derived by means of Hamilton principle.

* Received Jan. 17, 2006; Revised Apr. 16, 2007
Project supported by the National Natural Science Foundation of China (No. 10532070)
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1 Hamilton principle[13]

Hamilton principle is written as

$$\int_{t_0}^{t_1} (\delta T + \delta \Pi)\, dt = 0. \tag{1}$$

Equation (1) means that for an actual movement, the total integral of $(\delta T + \delta \Pi)$ is zero in any time interval, where δT is the variation of dynamic energy and $\delta \Pi$ is the virtual work due to all effective forces. $\delta \Pi$ may be expressed as

$$\delta \Pi = -\delta V + \delta W,$$

where δV is the virtual work due to potential forces, and δW is the virtual work due to non-potential forces. Thus we have the general form of Hamilton principle:

$$\int_{t_0}^{t_1} (\delta T - \delta V + \delta W)\, dt = 0. \tag{2}$$

When the forces applied on the system are only potential forces, i.e., $\delta W = 0$, we have

$$\int_{t_0}^{t_1} (\delta T - \delta V)\, dt = 0.$$

The above equation of Hamilton principle may also be expressed by Lagrangian function of $L = T - V$, then

$$\int_{t_0}^{t_1} \delta L dt = 0.$$

For holonomic system, variation symbol and integral symbol are substitutable each other. So that

$$\int_{t_0}^{t_1} \delta L dt = \delta \int_{t_0}^{t_1} L dt.$$

By introducing the expression of Hamilton's action, $I = \int_{t_0}^{t_1} L dt$, thus Hamilton principle for holonomic system may be written as variation form:

$$\delta I = \delta \int_{t_0}^{t_1} L dt = 0. \tag{3}$$

Equation (3) suggests that for a holonomic system applied by potential forces, Hamilton's action of actual movement takes stagnation value at any time interval, compared with possible movement for the same boundary condition. In short, the variation of Hamilton's action is zero.

Hamilton principle involves only two dynamic quantities of dynamic energy and work (or potential energy), and there is no any restrictions for system geometry. For straight girder, the configuration function represents deflection. When the girder is spatially curvilinear, torsion angle and axial displacement need to be considered.

2 Geometrical relations of curvilinear girder

It is convenient to analyze the movement of spatial curvilinear girder by introducing reference Cartesian coordinates xyz, in which x and z axes are in horizontal plane, and y axis points vertically and downward. Analysis coordinates take the form of orthogonal curvilinear coordinates $\zeta\xi\eta$ with the origin overlapping that of reference coordinates, as shown in Fig. 1.

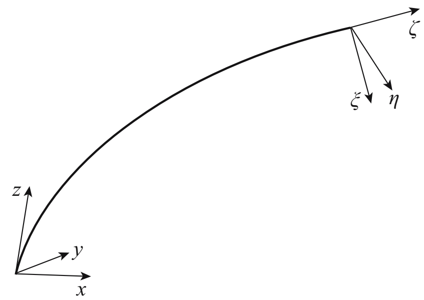

Fig. 1 Reference coordinates and curvilinear coordinates

The curvilinear structure is of spiral shape. The parametric equations in reference coordinates are

$$\begin{cases} x = R - R\cos t, \\ y = -kt, \quad k > 0, \\ z = R\sin t, \end{cases} \tag{4}$$

where t is a parameter and k is screw-pitch.

The relation between curvature radius ρ_1 and deflective radius ρ_2 of the curve is

$$\rho_1 = \frac{R^2 + k^2}{R}, \quad \rho_2 = \frac{R^2 + k^2}{k}. \tag{5}$$

The direction of ζ axis in curvilinear coordinates is the tangent of the curve, i.e.,

$$\boldsymbol{\zeta} = \frac{1}{\sqrt{R^2 + k^2}}(R\sin t\boldsymbol{i} - k\boldsymbol{j} + R\cos t\boldsymbol{k}). \tag{6}$$

The direction of ξ axis in curvilinear coordinates is the principal normal of the curve, i.e.,

$$\boldsymbol{\xi} = \cos t\boldsymbol{i} - \sin t\boldsymbol{k}. \tag{7}$$

The direction of η axis is perpendicular to both axes of ζ and ξ, and satisfies right hand principle, i.e.,

$$\boldsymbol{\eta} = \boldsymbol{\zeta} \times \boldsymbol{\xi} = \frac{1}{\sqrt{R^2 + k^2}}(k\sin t\boldsymbol{i} + R\boldsymbol{j} + k\cos t\boldsymbol{k}). \tag{8}$$

Suppose that the left end of the girder be located at the origin of coordinates and that it extends along the spiral line as shown in Fig. 1. Letting the parameter t equal t_1 at the vicinity of left end, the axis directions in curvilinear coordinates are

$$\begin{cases} \boldsymbol{\zeta}_1 = \dfrac{1}{\sqrt{R^2 + k^2}}\left(R\sin t_1\boldsymbol{i} - k\boldsymbol{j} + R\cos t_1\boldsymbol{k}\right), \\ \boldsymbol{\xi}_1 = \cos t_1\boldsymbol{i} - \sin t_1\boldsymbol{j}, \\ \boldsymbol{\eta}_1 = \dfrac{1}{\sqrt{R^2 + k^2}}\left(k\sin t_1\boldsymbol{i} + R\boldsymbol{j} + k\cos t_1\boldsymbol{k}\right). \end{cases} \tag{9}$$

For the increment of the parameter dt, the axis directions for an arbitrary small portion corresponding to $t + dt$ in curvilinear coordinates are

$$\begin{cases} \boldsymbol{\zeta}_2 = \dfrac{1}{\sqrt{R^2+k^2}}\left(R\sin(t_1+dt)\,\boldsymbol{i} - k\boldsymbol{j} + R\cos(t_1+dt)\,\boldsymbol{k}\right), \\ \boldsymbol{\xi}_2 = \cos(t_1+dt)\,\boldsymbol{i} - \sin(t_1+dt)\,\boldsymbol{k}, \\ \boldsymbol{\eta}_2 = \dfrac{1}{\sqrt{R^2+k^2}}\left(k\sin(t_1+dt)\,\boldsymbol{i} + R\boldsymbol{j} + k\cos(t_1+dt)\,\boldsymbol{k}\right). \end{cases} \tag{10}$$

So the orientation correlations of the axes at the two ends for the small portion in curvilinear coordinates are as follows:

$$\begin{cases} \cos(\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) = \dfrac{R^2\cos dt + k^2}{R^2+k^2}, & \cos(\boldsymbol{\zeta}_1,\boldsymbol{\xi}_2) = -\dfrac{R\sin dt}{\sqrt{R^2+k^2}}, & \cos(\boldsymbol{\zeta}_1,\boldsymbol{\eta}_2) = \dfrac{kR(\cos dt - 1)}{R^2+k^2}, \\ \cos(\boldsymbol{\xi}_1,\boldsymbol{\zeta}_2) = \dfrac{R\sin dt}{\sqrt{R^2+k^2}}, & \cos(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) = \cos dt, & \cos(\boldsymbol{\xi}_1,\boldsymbol{\eta}_2) = \dfrac{k\sin dt}{\sqrt{R^2+k^2}}, \\ \cos(\boldsymbol{\eta}_1,\boldsymbol{\zeta}_2) = \dfrac{kR(\cos dt - 1)}{R^2+k^2}, & \cos(\boldsymbol{\eta}_1,\boldsymbol{\xi}_2) = -\dfrac{k\sin dt}{\sqrt{R^2+k^2}}, & \cos(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2) = \dfrac{k^2\cos dt + R^2}{R^2+k^2}. \end{cases} \tag{11}$$

There are four independent generalized displacements for curved girder, including three displacements u_ζ, u_ξ, and u_η along curvilinear axes and the torsion angle ϕ_ζ with respect to ζ axis. There exist following geometrical relations:

$$\begin{cases} d\zeta = \sqrt{R^2+k^2}\,dt, \\ \phi_\xi = -\dfrac{du_\eta}{d\zeta}, \\ \phi_\eta = \dfrac{du_\xi}{d\zeta}, \end{cases} \tag{12}$$

where ϕ_ξ is the torsion angle with respect to ξ axis and ϕ_η is the torsion angle with respect to η axis.

The spatial curvilinear axial strain ε_ζ is

$$\begin{aligned} \varepsilon_\zeta &= \lim_{dt\to 0} \frac{(u_\zeta + du_\zeta)\cos(\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) + (u_\xi + du_\xi)\cos(\boldsymbol{\zeta}_1,\boldsymbol{\xi}_2) + (u_\eta + du_\eta)\cos(\boldsymbol{\zeta}_1,\boldsymbol{\eta}_2) - u_\zeta}{d\zeta} \\ &= \frac{\partial u_\zeta}{\partial \zeta} - \frac{u_\xi}{\rho_1} - \frac{u_\eta}{\rho_2}. \end{aligned} \tag{13}$$

With the help of Fig. 2, we may describe the meaning of ε_ζ. First the projection of the displacements at the right end to the tangent of the left end subtracts the tangential displacement at the left end. Then the difference is divided by the length of small curve portion to give the value of ε_ζ.

The expression of original curvature for the curved girder with respect to ξ axis is $-k/(R^2+k^2)$. The increment of the deflection curvature with respect to ξ axis due to the displacement in η direction is

$$\chi'_\xi = \frac{1}{\rho_2 - u_\eta} - \frac{1}{\rho_2} \approx \frac{u_\eta}{\rho_2^2}. \tag{14}$$

The expression of original principal curvature for curved girder with respect to η axis is $R/(R^2+k^2)$. The increment of curvature with respect to η axis due to the displacement in ξ direction is

$$\chi'_\eta = \frac{1}{\rho_1 - u_\xi} - \frac{1}{\rho_1} \approx \frac{u_\xi}{\rho_1^2}. \tag{15}$$

Deformation curvatures of curved girder χ_ξ and χ_η are defined as the increment of the curvature due to deformation. In the calculation of χ_ξ, the curvature increments χ'_ξ and χ'_η and the increment induced by the girder rotation should be considered.

Referring to Fig. 3, deformation curvature χ_ξ with respect to ξ axis is

$$\begin{aligned}\chi_\xi &= \frac{u_\eta}{\rho_2^2} + \lim_{dt\to 0}\frac{(\phi_\xi + d\phi_\xi)\cos(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) + (\phi_\zeta + d\phi_\zeta)\cos(\boldsymbol{\xi}_1,\boldsymbol{\zeta}_2) + (\phi_\eta + d\phi_\eta)\cos(\boldsymbol{\xi}_1,\boldsymbol{\eta}_2) - \phi_\xi}{d\zeta} \\ &= \frac{u_\eta}{\rho_2^2} - \frac{\partial^2 u_\eta}{\partial\zeta^2} + \frac{\phi_\zeta}{\rho_1}.\end{aligned} \tag{16}$$

Similarly, deformation curvature χ_η with respect η to axis is

$$\begin{aligned}\chi_\eta &= \frac{u_\xi}{\rho_1^2} + \lim_{dt\to 0}\frac{(\phi_\eta + d\phi_\eta)\cos(\boldsymbol{\eta}_1,\boldsymbol{\eta}_2) + (\phi_\zeta + d\phi_\zeta)\cos(\boldsymbol{\eta}_1,\boldsymbol{\zeta}_2) + (\phi_\xi + d\phi_\xi)\cos(\boldsymbol{\eta}_1,\boldsymbol{\xi}_2) - \phi_\eta}{d\zeta} \\ &= \frac{u_\xi}{\rho_1^2} + \frac{\partial^2 u_\xi}{\partial\zeta^2} + \frac{\phi_\zeta}{\rho_2}.\end{aligned} \tag{17}$$

Torsion curvature τ is

$$\begin{aligned}\tau &= \lim_{dt\to 0}\frac{(\phi_\zeta + d\phi_\zeta)\cos(\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) + (\phi_\xi + d\phi_\xi)\cos(\boldsymbol{\zeta}_1,\boldsymbol{\xi}_2) + (\phi_\eta + d\phi_\eta)\cos(\boldsymbol{\zeta}_1,\boldsymbol{\eta}_2) - \phi_\zeta}{d\zeta} \\ &= \frac{\partial\phi_\zeta}{\partial\zeta} + \frac{1}{\rho_1}\frac{\partial u_\eta}{\partial\zeta} + \frac{1}{\rho_2}\frac{\partial u_\xi}{\partial\zeta}.\end{aligned} \tag{18}$$

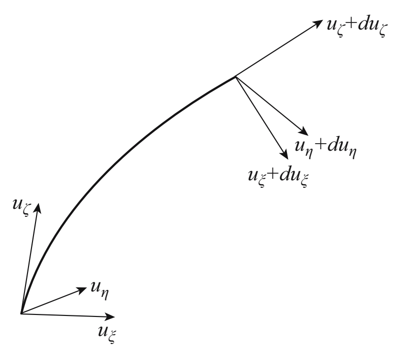

Fig. 2 A sect of girder schematically showing the displacements and their increments

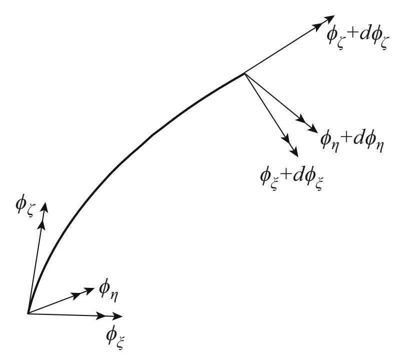

Fig. 3 Schematic showing ϕ_ζ, ϕ_ξ, ϕ_η and their increments

3 Dynamic equations of curved girders

The system energy of curved girder consists of system dynamic energy, elastic strain energy and work done by external forces. Dynamic energies due to rotation with respect to ξ and η axes are

$$\int_s \frac{1}{2}\rho\left(\eta\dot{\phi}_\xi\right)^2 dA = \frac{1}{2}\rho\dot{\phi}_\xi^2\int_s \eta^2 dA = \frac{1}{2}\rho I_\xi\dot{\phi}_\xi^2 = \frac{1}{2}\rho I_\xi \dot{u}_\eta'^2,$$
$$\int_s \frac{1}{2}\rho\left(\xi\dot{\phi}_\eta\right)^2 dA = \frac{1}{2}\rho\dot{\phi}_\eta^2\int_s \xi^2 dA = \frac{1}{2}\rho I_\eta\dot{\phi}_\eta^2 = \frac{1}{2}\rho I_\eta \dot{u}_\xi'^2.$$

Hence the dynamic energy of the structure is

$$T=\frac{1}{2}\rho\int_0^l\left[A\left(\dot{u}_\xi^2(\zeta)+\dot{u}_\eta^2(\zeta)+\dot{u}_\zeta^2(\zeta)+a^2\dot{\phi_\zeta}^2\right)+I_\xi\dot{u}_\eta'^2+I_\eta\dot{u}_\xi'^2\right]d\zeta, \tag{19}$$

where a is the radius of gyration of girder cross-section, A is the cross-section area, ρ is the density of structure, the head dot indicates the derivative with respect to time, and the use of prime denotes the deferential with respect to ζ.

Strain energy is

$$V=\int_0^l\frac{1}{2}\left(EI_\xi\chi_\xi^2+EI_\eta\chi_\eta^2+GI_d\tau^2+EA\varepsilon_\zeta^2\right)d\zeta, \tag{20}$$

where E is elastic modulus, I_ξ and I_η are the moments of inertia with respect to ξ and η, respectively, and I_d is the torsion moment.

Work done by external forces is

$$W=\int_0^l\left(f_\xi u_\xi+f_\eta u_\eta+f_\zeta u_\zeta+m_\xi\phi_\xi+m_\eta\phi_\eta+m_\zeta\phi_\zeta\right)d\zeta, \tag{21}$$

where f_ξ, f_η, f_ζ are forces in directions ξ,η,ζ, respectively, and m_ξ, m_η, m_ζ are the moments with respect to the unit axis of ξ,η,ζ.

It is convenient to perform the variation analysis of dynamic energy T, strain energy V and work W separately.

The variation of dynamic energy T is

$$\begin{aligned}\delta\int_{t_0}^{t_1}Tdt &=\delta\int_{t_0}^{t_1}\int_0^1\left(\frac{1}{2}m\left(\dot{u}_\xi^2+\dot{u}_\eta^2+\dot{u}_\zeta^2+a^2\dot{\phi_\zeta}^2\right)+\frac{1}{2}\rho I_\xi\dot{u}_\eta'^2+\frac{1}{2}\rho I_\eta\dot{u}_\xi'^2\right)d\zeta dt\\ &=\int_{t_0}^{t_1}\int_0^1\left(-\rho\left(A\ddot{u}_\xi-I_\eta\ddot{u}_\xi''\right)\delta u_\xi-\rho\left(A\ddot{u}_\eta-I_\xi\ddot{u}_\eta''\right)\delta u_\eta-\rho A\ddot{u}_\zeta\delta u_\zeta-\rho Aa^2\ddot{\phi}_\zeta\delta\phi_\zeta\right)d\zeta dt.\end{aligned} \tag{22}$$

The variation of strain energy V is

$$\begin{aligned}-\delta\int_{t_0}^{t_1}Vdt= &-\int_{t_0}^{t_1}\int_0^1\Bigg(\Bigg(EI_\xi\left(u_\eta^{(4)}-\frac{2u_\eta''}{\rho_2^2}+\frac{u_\eta}{\rho_2^4}-\frac{\phi_\zeta''}{\rho_1}+\frac{\phi_\zeta}{\rho_1\rho_2^2}\right)-\frac{GI_d}{\rho_1}\left(\phi_\zeta''+\frac{u_\eta''}{\rho_1}+\frac{u_\xi''}{\rho_2}\right)\\ &-\frac{EA}{\rho_2}\left(u_\zeta'-\frac{u_\xi}{\rho_1}-\frac{u_\eta}{\rho_2}\right)\Bigg)\delta u_\eta+\Bigg(EI_\eta\left(u_\xi^{(4)}+\frac{2u_\xi''}{\rho_1^2}+\frac{u_\xi}{\rho_1^4}+\frac{\phi_\zeta''}{\rho_2}+\frac{\phi_\zeta}{\rho_2\rho_1^2}\right)\\ &-\frac{GI_d}{\rho_2}\left(\phi_\zeta''+\frac{u_\eta''}{\rho_1}+\frac{u_\xi''}{\rho_2}\right)-\frac{EA}{\rho_1}\left(u_\zeta'-\frac{u_\xi}{\rho_1}-\frac{u_\eta}{\rho_2}\right)\Bigg)\delta u_\xi-EA\left(u_\zeta''-\frac{u_\xi'}{\rho_1}-\frac{u_\eta'}{\rho_2}\right)\delta u_\zeta\\ &+\left(\frac{EI_\xi}{\rho_1}\left(-u_\eta''+\frac{u_\eta}{\rho_2^2}+\frac{\phi_\zeta}{\rho_1}\right)+\frac{EI_\eta}{\rho_2}\left(u_\xi''+\frac{u_\xi}{\rho_1^2}+\frac{\phi_\zeta}{\rho_2}\right)-GI_d\left(\phi''+\frac{u_\eta''}{\rho_1}+\frac{u_\xi''}{\rho_2}\right)\right)\delta\phi_\zeta\Bigg)d\zeta dt.\end{aligned} \tag{23}$$

The variation of work done by external forces is

$$\delta\int_{t_0}^{t_1}Wdt=\int_{t_0}^{t_1}\int_0^1\left(f_\xi\delta u_\xi+f_\eta\delta u_\eta+f_\zeta\delta u_\zeta+m_\xi\delta\phi_\xi+m_\eta\delta\phi_\eta+m_\zeta\delta\phi_\zeta\right)d\zeta dt.$$

Considering the geometrical relation of Eq. (12), one may write the above equation as

$$\delta\int_{t_0}^{t_1}Wdt=\int_{t_0}^{t_1}\int_0^1\left(\left(f_\xi-m_\eta'\right)\delta u_\xi+\left(f_\eta+m_\xi'\right)\delta u_\eta+f_\zeta\delta u_\zeta+m_\zeta\delta\phi_\zeta\right)d\zeta dt. \tag{24}$$

In a word, based on the general form of Hamilton principle (Eq. (2)), one is able to write the dynamic equations of curved SFT without energy dissipation, such that

$$\begin{cases} \rho\left(A\ddot{u}_\xi - I_\eta \ddot{u}''_\xi\right) + EI_\eta\left(u_\xi^{(4)} + \dfrac{2u''_\xi}{\rho_1^2} + \dfrac{\phi''_\zeta}{\rho_2} + \dfrac{u_\xi}{\rho_1^4} + \dfrac{\phi_\zeta}{\rho_2\rho_1^2}\right) \\ \quad - \dfrac{GI_d}{\rho_2}\left(\phi''_\zeta + \dfrac{u''_\eta}{\rho_1} + \dfrac{u''_\xi}{\rho_2}\right) - \dfrac{EA}{\rho_1}\left(u'_\zeta - \dfrac{u_\xi}{\rho_1} - \dfrac{u_\eta}{\rho_2}\right) \\ = f_\xi - m'_\eta, \\ \rho\left(A\ddot{u}_\eta - I_\xi \ddot{u}''_\eta\right) + EI_\xi\left(u_\eta^{(4)} - \dfrac{2u''_\eta}{\rho_2^2} - \dfrac{\phi''_\zeta}{\rho_1} + \dfrac{u_\eta}{\rho_2^4} + \dfrac{\phi_\zeta}{\rho_1\rho_2^2}\right) \\ \quad - \dfrac{GI_d}{\rho_1}\left(\phi''_\zeta + \dfrac{u''_\eta}{\rho_1} + \dfrac{u''_\xi}{\rho_2}\right) - \dfrac{EA}{\rho_2}\left(u'_\zeta - \dfrac{u_\xi}{\rho_1} - \dfrac{u_\eta}{\rho_2}\right) \\ = f_\eta + m'_\xi, \\ \rho A\ddot{u}_\zeta - EA\left(u''_\zeta - \dfrac{u'_\xi}{\rho_1} - \dfrac{u'_\eta}{\rho_2}\right) = f_\zeta, \\ a^2\rho A\ddot{\phi}_\zeta + \dfrac{EI_\xi}{\rho_1}\left(-u''_\eta + \dfrac{u_\eta}{\rho_2^2} + \dfrac{\phi_\zeta}{\rho_1}\right) + \dfrac{EI_\eta}{\rho_2}\left(u''_\xi + \dfrac{u_\xi}{\rho_1^2} + \dfrac{\phi_\zeta}{\rho_2}\right) - GI_d\left(\phi''_\zeta + \dfrac{u''_\eta}{\rho_1} + \dfrac{u''_\xi}{\rho_2}\right) = m_\zeta. \end{cases} \tag{25}$$

When $k = 0$ and $\rho_2 \to \infty$, spatial curve degenerates into plane curve. So dynamic differential equations are

$$\begin{cases} \rho\left(A\ddot{u}_\xi - I_\eta \ddot{u}''_\xi\right) + EI_\eta\left(u_\xi^{(4)} + \dfrac{2u''_\xi}{\rho_1^2} + \dfrac{u_\xi}{\rho_1^4}\right) - \dfrac{EA}{\rho_1}\left(u'_\zeta - \dfrac{u_\xi}{\rho_1}\right) = f_\xi - m'_\eta, \\ \rho\left(A\ddot{u}_\eta - I_\xi \ddot{u}''_\eta\right) + EI_\xi\left(u_\eta^{(4)} - \dfrac{\phi''_\zeta}{\rho_1}\right) - \dfrac{GI_d}{\rho_1}\left(\phi''_\zeta + \dfrac{u''_\eta}{\rho_1}\right) = f_\eta + m'_\xi, \\ \rho A\ddot{u}_\zeta - EA\left(u''_\zeta - \dfrac{u'_\xi}{\rho_1}\right) = f_\zeta, \\ a^2\rho A\ddot{\phi}_\zeta + \dfrac{EI_\xi}{\rho_1}\left(-u''_\eta + \dfrac{\phi_\zeta}{\rho_1}\right) - GI_d\left(\phi''_\zeta + \dfrac{u''_\eta}{\rho_1}\right) = m_\zeta. \end{cases} \tag{26}$$

For the solution of Eqs. (25)and (26), it is necessary to convert the loading in the reference coordinate system into that in the curvilinear coordinate system. Thus one may solve them in the curvilinear coordinate system.

4 Intrinsic characteristics of curved structure vibration

The boundary conditions are given in terms of vibration function.

(i) Fixed end: displacement and angular rotation equal zero, i.e.,

$$\boldsymbol{U}(x,t) = \boldsymbol{0}, \quad \frac{\partial \boldsymbol{U}(x,t)}{\partial \zeta} = \boldsymbol{0}, \tag{27}$$

where $\boldsymbol{U} = q(t)\boldsymbol{V}(\zeta) = q(t)(u_\xi, u_\eta, u_\zeta, \phi_\zeta)^{\mathrm{T}}$ and $x = 0$. Equation (27) can be expressed in terms of vibration function:

$$\boldsymbol{V}(x) = \boldsymbol{0}, \quad \frac{\partial \boldsymbol{V}(x)}{\partial \zeta} = \boldsymbol{0}. \tag{28}$$

(ii) Simply supported end: displacement and bending moment equal zero, i.e.,

$$\boldsymbol{U}(x,t) = \boldsymbol{0}, \quad \frac{\partial^2 \boldsymbol{U}(x,t)}{\partial \zeta^2} = \boldsymbol{0}. \tag{29}$$

It can be expressed in terms of vibration function:

$$\boldsymbol{V}(x)=\boldsymbol{0},\quad \frac{\partial^2\boldsymbol{V}(x)}{\partial\zeta^2}=\boldsymbol{0}. \tag{30}$$

(iii) Free end: bending moment and shear force equal zero, i.e.,

$$\frac{\partial^2\boldsymbol{U}(x,t)}{\partial\zeta^2}=\boldsymbol{0},\quad \frac{\partial^3\boldsymbol{U}(x,t)}{\partial\zeta^3}=\boldsymbol{0}. \tag{31}$$

It can be expressed in terms of vibration function:

$$\frac{\partial^2\boldsymbol{V}(x)}{\partial\zeta^2}=\boldsymbol{0},\quad \frac{\partial^3\boldsymbol{V}(x)}{\partial\zeta^3}=\boldsymbol{0}. \tag{32}$$

(iv) Ends with rotation resistance by spring matrix $\boldsymbol{k}_\alpha$ and with displacement restraint by spring matrix $\boldsymbol{k}_d$: bending moment $\boldsymbol{M}(x)$ is the product of multiplying $\boldsymbol{k}_\alpha$ and rotational angle, and shear force $\boldsymbol{Q}(x)$ is the product of multiplying $\boldsymbol{k}_d$ and end displacement, i.e.,

$$\boldsymbol{M}(x)=\boldsymbol{k}_\alpha^{\mathrm{T}}\cdot\frac{\partial\boldsymbol{U}(x,t)}{\partial\zeta},\quad \boldsymbol{Q}(x)=\boldsymbol{k}_d^{\mathrm{T}}\cdot\boldsymbol{U}(x,t). \tag{33}$$

(v) Ends with concentrated mass: bending moment is zero, and shear force equals inertial force, i.e.,

$$\boldsymbol{M}=\boldsymbol{0},\quad \boldsymbol{Q}=m\frac{\partial^2\boldsymbol{U}(x,t)}{\partial t^2}. \tag{34}$$

Here, the intrinsic characteristics of planar curved structure due to bending vibration and torsional vibration are analyzed by the use of second and fourth formulas in Eq. (25). Suppose that the boundary conditions be: simply supported at both ends and torsion of ends restricted, i.e.,

$$\boldsymbol{V}(0)=\boldsymbol{V}(l)=\boldsymbol{0},\quad \phi_\zeta(0)=\phi_\zeta(l)=0. \tag{35}$$

where l is the arc length of structure.

According to the boundary conditions, generalized displacements u_η and ϕ_ζ are

$$\begin{cases} u_\eta=\dfrac{2}{l}\displaystyle\sum_{j=1}^{\infty}U_\eta(j,t)\sin\frac{j\pi\zeta}{l},\\ \phi_\zeta=\dfrac{2}{l}\displaystyle\sum_{j=1}^{\infty}\Phi_\zeta(j,t)\sin\frac{j\pi\zeta}{l}, \end{cases} \tag{36}$$

where u_η and ϕ_ζ are Fourier-Sine transformations of U_η and Φ_ζ, i.e.,

$$\begin{cases} U_\eta=\displaystyle\int_0^l u_\eta(\zeta,t)\sin\frac{j\pi\zeta}{l}d\zeta,\\ \Phi_\zeta=\displaystyle\int_0^l \phi_\zeta(\zeta,t)\sin\frac{j\pi\zeta}{l}d\zeta, \end{cases} \tag{37}$$

where j is the number of vibration mode, with $j=1,2,\cdots,\infty$.

Suppose that the right sides of the second and the fourth formulas of Eq. (25) equal zero. According to the Fourier transformation and orthogonal property of vibration mode, we have

$$\begin{cases} \rho\left(A+I_\xi\left(\dfrac{j\pi}{l}\right)^2\right)\ddot{U}_\eta+\left(EI_\xi\left(\dfrac{j\pi}{l}\right)^4+\dfrac{GI_d}{R^2}\left(\dfrac{j\pi}{l}\right)^2\right)U_\eta+\left(\dfrac{EI_\xi}{R}\left(\dfrac{j\pi}{l}\right)^2+\dfrac{GI_d}{R}\left(\dfrac{j\pi}{l}\right)^2\right)\Phi_\zeta=0,\\ a^2\rho A\ddot{\Phi}+\left(\dfrac{EI_\xi}{R}\left(\dfrac{j\pi}{l}\right)^2+\dfrac{GI_d}{R}\left(\dfrac{j\pi}{l}\right)^2\right)U_\eta+\left(\dfrac{EI_\xi}{R^2}+GI_d\left(\dfrac{j\pi}{l}\right)^2\right)\Phi_\zeta=0. \end{cases} \tag{38}$$

Let

$$
\begin{cases}
\omega_\eta^2 = \left(EI_\xi \left(\dfrac{j\pi}{l}\right)^4 + \dfrac{GI_d}{R^2}\left(\dfrac{j\pi}{l}\right)^2 \right) \Big/ \left(\rho \left(A + I_\xi \left(\dfrac{j\pi}{l}\right)^2 \right) \right), \\
\omega_\phi^2 = \left(\dfrac{EI_\xi}{R^2} + GI_d \left(\dfrac{j\pi}{l}\right)^2 \right) \Big/ \left(a^2 \rho A \right), \\
\omega_{l\eta}^2 = \left(\dfrac{EI_\xi}{R} \left(\dfrac{j\pi}{l}\right)^2 + \dfrac{GI_d}{R}\left(\dfrac{j\pi}{l}\right)^2 \right) \Big/ \left(\rho \left(A + I_\xi \left(\dfrac{j\pi}{l}\right)^2 \right) \right), \\
\omega_{l\phi}^2 = \left(\dfrac{EI_\xi}{R} \left(\dfrac{j\pi}{l}\right)^2 + \dfrac{GI_d}{R}\left(\dfrac{j\pi}{l}\right)^2 \right) \Big/ \left(a^2 \rho A \right).
\end{cases}
\tag{39}
$$

Then Eq. (38) can be simplified as

$$
\begin{cases}
\ddot{U}_\eta + \omega_\eta^2 U_\eta + \omega_{l\eta}^2 \Phi_\zeta = 0, \\
\ddot{\Phi}_\zeta + \omega_{l\phi}^2 U_\eta + \omega_\phi^2 \Phi_\zeta = 0,
\end{cases}
\tag{40}
$$

where ω_η and ω_ϕ are the circular frequencies of bending vibration and torsion vibration for curved girder, respectively, and $\omega_{l\eta}$ and $\omega_{l\phi}$ are the coupling items due to curvature. When R tends to infinity, we have $\omega_{l\eta} = 0$ and $\omega_{l\phi} = 0$. If rotation energy is not considered, ω_η and ω_ϕ are frequencies of bending vibration and torsional vibration for straight beams, respectively.

In the design of potential SFT with thousands of meters in length, the length of one span module is of the order of magnitude of 100 meters. For the mode of low frequency vibration with $(j\pi/l)^2 \ll 1$, the value of rotational energy is small enough compared with dynamic energy and has little effect on frequencies. So low frequencies may be written as

$$
\begin{cases}
\omega_\eta^2 = \left(EI_\xi \left(\dfrac{j\pi}{l}\right)^4 + \dfrac{GI_d}{R^2}\left(\dfrac{j\pi}{l}\right)^2 \right) / m, \\
\omega_\phi^2 = \left(\dfrac{EI_\xi}{R^2} + GI_d \left(\dfrac{j\pi}{l}\right)^2 \right) / (a^2 m), \\
\omega_{l\eta}^2 = \left(\dfrac{EI_\xi}{R} \left(\dfrac{j\pi}{l}\right)^2 + \dfrac{GI_d}{R}\left(\dfrac{j\pi}{l}\right)^2 \right) / m, \\
\omega_{l\phi}^2 = \left(\dfrac{EI_\xi}{R} \left(\dfrac{j\pi}{l}\right)^2 + \dfrac{GI_d}{R}\left(\dfrac{j\pi}{l}\right)^2 \right) / (a^2 m).
\end{cases}
\tag{41}
$$

5 Conclusions

1) The paper gives dynamic equations of spatial curved SFT with Hamilton principle. In this system, four generalized displacements are coupled with each other. When spatial curvilinear structure degenerates into planar curvilinear structure, deflection is coupled with torsion angle and axial displacement is coupled with radial displacement. In other words, generalized displacements in a pair of perpendicular planes are coupled.

2) Dynamic equations of arbitrary curvilinear structures may be written by the present method, i.e., with a reference coordinate system.

3) When low frequencies of curved SFT are analyzed, rotational energy could be neglected.

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