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正交异性表层的夹层圆柱扁壳的非线性稳定性

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摘　　要

本文采用大挠度理论，对正交各向异性表层的夹层扁壳进行了几何非线性的稳定性分析，求出了屈曲载荷并研究了后屈曲特性。

一、引　　言

扁柱壳段的稳定性分析，是工程中的重要课题。各向同性扁圆柱壳，在轴向压力作用下的线性与非线性稳定性分析，已在弹性稳定性著作中（如[1]、[2]）给出。文[3]用数值分析方法研究了各向同性扁柱壳在横向载荷作用下的非线性稳定性。

夹层圆柱扁壳是航天航空工程中的重要构件，它的稳定性一直受到重视。文[4]研究了各向同性表层的夹层扁柱壳在轴压下的稳定性。文[5]研究了复合材料多层圆柱扁壳在轴向力作用下的后屈曲特点。

复合材料的夹层圆柱壳具有很高的结构效能，是航空和航天结构中所常用的结构形式，通常都将夹层板壳的表层铺设成正交各向异性的，表层可以折合为均匀正交各向异性的材料来作分析计算，这样处理要简单一些。本文具有这种应用背景。

本文给出了具有正交各向异性表层的夹层圆柱扁壳在轴向压力作用下的几何非线性稳定性分析。本文采用了大挠度理论，导出了具有正交各向异性表层的夹层圆柱扁壳的总势能表达式，用Ritz法和稳定性的能量判据，求出了屈曲载荷（或应力），并且给出了后屈曲的载荷挠度关系。

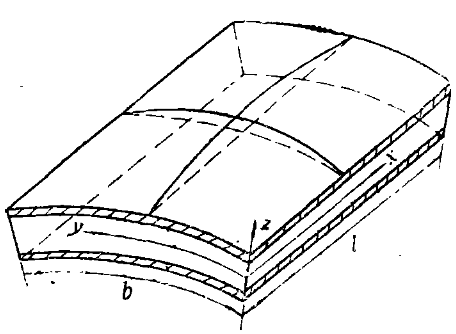

图1　夹层圆柱扁壳

本文假定：

〈i〉 上下表层为高强度的正交各向异性材料，材料的主方向平行于圆柱壳面的母线方向（x 方向，如图1所示）和环向（y方向）。表层按能抗弯的薄壳处理。

〈ii〉 夹心层由软而轻的各向同性材料构成。略去夹心层中沿x-y面内的应力，即 $\sigma_x=\sigma_y=\tau_{xy}=0$。在夹心层中，变形前垂直于中面的直线，变形后仍为直线，但一般不再垂直于中面。

〈iii〉 只考虑反对称挠曲变形，不考虑脱层问题，所以在夹心与表层中假定$\varepsilon_z=0$。

〈iv〉 在夹心与表层中，应力σ_z造成的应变可以略去。

取夹层扁壳的中面为 x-y 面，坐标系如图 1 所示。夹心层厚为h，上下表层厚度均为t，夹层扁壳总厚度为 $h+2t$。

设夹层圆柱扁壳的边界均为简支边，即$x=0$，l 及$y=0$，b均为简支边。在$x=0$ 及$x=l$处，作用有均匀轴向压力N_{x0}。本文研究在这样的载荷和边界条件下的正交异性 表层夹层圆柱扁壳的非线性稳定性问题。

二、变 形 的 描 述

我们用大挠度理论来描述夹层圆柱扁壳的变形。$R_y=R$是环向的曲率半径，$R_x=R_{xy}=\infty$。

设夹层圆柱扁壳中面的位移为

$$\vec{u}(x,y)=[u(x,y),v(x,y),w(x,y)]$$

夹心层内平行于中面的各曲面，在x,y方向的位移$u(x,y,z)$，$v(x,y,z)$形成的在 x-z 面内及y-z面内的剪应变所引起的法线转角分别为

$$\psi_x(x,y),\ \psi_y(x,y)$$

由此不难得出夹层扁壳上表层的位移为

$$\left.\begin{aligned}u^+&=u-\frac{h+t}{2}\psi_x-\left(z-\frac{h+t}{2}\right)\frac{\partial w}{\partial x}\\v^+&=v-\frac{h+t}{2}\psi_y-\left(z-\frac{h+t}{2}\right)\frac{\partial w}{\partial y}\\w^+&=w\end{aligned}\right\}\tag{2.1}$$

式中u，v表示中面位移$u(x,y)$，$v(x,y)$，以下都用此简化表示。下表层的位移为

$$\left.\begin{aligned}u^-&=u-\frac{h+t}{2}\psi_x-\left(z+\frac{h+t}{2}\right)\frac{\partial w}{\partial x}\\v^-&=v-\frac{h+t}{2}\psi_y-\left(z+\frac{h+t}{2}\right)\frac{\partial w}{\partial y}\\w^-&=w\end{aligned}\right\}\tag{2.2}$$

按照我们所用的假定，为保证在夹层扁壳中位移的连续性，应取夹心层位移为

$$\left.\begin{aligned}u^c&=u-z\left(\frac{h+t}{h}\psi_x-\frac{t}{h}\frac{\partial w}{\partial x}\right)\\v^c&=v-z\left(\frac{h+t}{h}\psi_y-\frac{t}{h}\frac{\partial w}{\partial y}\right)\\w^c&=w\end{aligned}\right\}\tag{2.3}$$

显然，由于$\varepsilon_z^c=0$，故有$w=w(x,y)$，各层的挠度相同。

扁柱壳大挠度分析的 Green 应变为

$$\left.\begin{aligned}\varepsilon_x^* &= \frac{\partial u^*}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\ \varepsilon_y^* &= \frac{\partial v^*}{\partial y}+\frac{w}{R}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\\ \gamma_{xy}^* &= \frac{\partial u^*}{\partial y}+\frac{\partial v^*}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\end{aligned}\right\}\tag{2.4}$$

考虑到薄壳的假定：$h+2t\ll R$，在薄壳结构的允许误差范围内，我们可以用中面的曲率半径代替整个夹层壳各层的曲率半径。将上下表层的位移(2.1)、(2.2)分别代入(2.4)，得到上表层的应变：

$$\left.\begin{aligned}\varepsilon_x^+ &= \frac{\partial u}{\partial x}-\frac{h+t}{2}\frac{\partial\psi_x}{\partial x}-\left(z-\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial x^2}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\ \varepsilon_y^+ &= \frac{\partial v}{\partial y}-\frac{h+t}{2}\frac{\partial\psi_y}{\partial y}-\left(z-\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial y^2}+\frac{w}{R}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\\ \gamma_{xy}^+ &= \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-\frac{h+t}{2}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)-2\left(z-\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\end{aligned}\right\}\tag{2.5}$$

下表层应变：

$$\left.\begin{aligned}\varepsilon_x^- &= \frac{\partial u}{\partial x}+\frac{h+t}{2}\frac{\partial\psi_x}{\partial x}-\left(z+\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial x^2}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\ \varepsilon_y^- &= \frac{\partial v}{\partial y}+\frac{h+t}{2}\frac{\partial\psi_y}{\partial y}-\left(z+\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial y^2}+\frac{w}{R}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\\ \gamma_{xy}^- &= \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{h+t}{2}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)-2\left(z+\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\end{aligned}\right\}\tag{2.6}$$

夹心层的剪应变，由平行于中面的各曲面的面内位移形成的剪应变及壳体挠曲形成的剪应变两部分构成：

$$\left.\begin{aligned}\gamma_{xz}^c &= \frac{\partial u^c}{\partial z}+\frac{\partial w}{\partial x}=\frac{h+t}{h}\left(\frac{\partial w}{\partial x}-\psi_x\right)\\ \gamma_{yz}^c &= \frac{\partial v^c}{\partial z}+\frac{\partial w}{\partial y}=\frac{h+t}{h}\left(\frac{\partial w}{\partial y}-\psi_y\right)\end{aligned}\right\}\tag{2.7}$$

从而

$$\psi_x=-\left(\gamma_{xz}^c\frac{h}{h+t}-\frac{\partial w}{\partial x}\right),\quad \psi_y=-\left(\gamma_{yz}^c\frac{h}{h+t}-\frac{\partial w}{\partial y}\right)\tag{2.8}$$

由此可知 ψ_x，ψ_y 的几何意义。

变形的描述，共用五个独立变元：u,v,w,ψ_x,ψ_y。

三、本 构 关 系

上下表层由同一种正交各向异性材料构成，它们的材料主方向平行于x,y轴。上下表层的应力应变关系为：

$$\left.\begin{aligned}\sigma_x^{\pm}&=\frac{E_x}{1-\nu_{xy}\nu_{yx}}\varepsilon_x^{\pm}+\frac{\nu_{xy}E_y}{1-\nu_{xy}\nu_{yx}}\varepsilon_y^{\pm}\\ \sigma_y^{\pm}&=\frac{\nu_{xy}E_y}{1-\nu_{xy}\nu_{yx}}\varepsilon_x^{\pm}+\frac{E_y}{1-\nu_{xy}\nu_{yx}}\varepsilon_y^{\pm}\\ \tau_{xy}^{\pm}&=G_{xy}\gamma_{xy}^{\pm}\end{aligned}\right\}\tag{3.1}$$

式中E_x,E_y为弹性模量，G_{xy}为剪切模量，ν_{xy}，ν_{yx}为Poisson比。

夹心的应力应变关系为

$$\tau_{xz}^c=G_c\gamma_{xz}^c,\ \tau_{yz}^c=G_c\gamma_{yz}^c\tag{3.2}$$

式中G_c为夹心层的剪切模量。

利用(3.1)和(3.2)，不难得出

$$\begin{aligned}N_x&=\frac{1}{2t}\left[\int_{h/2}^{h/2+t}\sigma_x^+dz+\int_{-(h/2+t)}^{-h/2}\sigma_x^-dz\right]\\&=\frac{E_x}{1-\nu_{xy}\nu_{yx}}\left\{\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+\nu_{xy}\frac{E_y}{E_x}\left[\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\frac{w}{R}\right]\right\}\end{aligned}\tag{3.3}$$

$$\begin{aligned}N_y&=\frac{1}{2t}\left[\int_{h/2}^{h/2+t}\sigma_y^+dz+\int_{-(h/2+t)}^{-h/2}\sigma_y^-dz\right]\\&=\frac{E_y}{1-\nu_{xy}\nu_{yx}}\left\{\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+\frac{w}{R}+\nu_{xy}\left[\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\right\}\end{aligned}\tag{3.4}$$

$$\begin{aligned}N_{xy}&=\frac{1}{2t}\left[\int_{h/2}^{h/2+t}\tau_{xy}^+dz+\int_{-(h/2+t)}^{-h/2}\tau_{xy}^-dz\right]\\&=G_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\end{aligned}\tag{3.5}$$

四、总　势　能

为了运用能量准则研究稳定性，下面导出具有正交各向异性表层的夹层圆柱扁壳的总势能表达式。

由(2.5)、(3.3)、(3.4)、(3.5)，对于上表层可得

$$\left.\begin{aligned}\varepsilon_x^+&=\frac{1}{E_x}\left\{N_x-\nu_{xy}N_y-E_x\left[\frac{h+t}{2}\cdot\frac{\partial\psi_x}{\partial x}+\left(z-\frac{h+t}{2}\right)\frac{\partial^2w}{\partial x^2}\right]\right\}\\ \varepsilon_y^+&=\frac{1}{E_y}\left\{N_y-\nu_{yx}N_x-E_y\left[\frac{h+t}{2}\cdot\frac{\partial\psi_y}{\partial y}+\left(z-\frac{h+t}{2}\right)\frac{\partial^2w}{\partial y^2}\right]\right\}\\ \gamma_{xy}^+&=\frac{N_{xy}}{G_{xy}}-\frac{h+t}{2}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)-2\left(z-\frac{h+t}{2}\right)\frac{\partial^2w}{\partial x\partial y}\end{aligned}\right\}\tag{4.1}$$

类似地，对于下表层

$$\left.\begin{aligned}
\varepsilon_x^- &= \frac{1}{E_x}\left\{N_x-\nu_{xy}N_y+E_x\left[\frac{h+t}{2}\cdot\frac{\partial\psi_x}{\partial x}-\left(z+\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial x^2}\right]\right\}\\
\varepsilon_y^- &= \frac{1}{E_y}\left\{N_y-\nu_{yx}N_x+E_y\left[\frac{h+t}{2}\cdot\frac{\partial\psi_y}{\partial y}-\left(z+\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial y^2}\right]\right\}\\
\gamma_{xy}^- &= \frac{N_{xy}}{G_{xy}}+\frac{h+t}{2}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)-2\left(z+\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial x\partial y}
\end{aligned}\right\} \tag{4.2}$$

于是，上下表层的应变能为

$$\begin{aligned}
U_s=&\frac{1}{2}\iiint_{h/2}^{h/2+t}\left[\frac{E_x}{1-\nu_{xy}\nu_{yx}}(\varepsilon_x^+)^2+\frac{2\nu_{xy}E_y}{1-\nu_{xy}\nu_{yx}}\varepsilon_x^+\varepsilon_y^++\frac{E_y}{1-\nu_{xy}\nu_{yx}}(\varepsilon_y^+)^2\right.\\
&\left.+G_{xy}(\gamma_{xy}^+)^2\right]dxdydz+\frac{1}{2}\iiint_{-(h/2+t)}^{-h/2}\left[\frac{E_x}{1-\nu_{xy}\nu_{yx}}(\varepsilon_x^-)^2+\frac{2\nu_{xy}E_y}{1-\nu_{xy}\nu_{yx}}\varepsilon_x^-\varepsilon_y^-\right.\\
&\left.+\frac{E_y}{1-\nu_{xy}\nu_{yx}}(\varepsilon_y^-)^2+G_{xy}(\gamma_{xy}^-)^2\right]dxdydz\\
=&t\iint\left\{\frac{1}{1-\nu_{xy}\nu_{yx}}\left[\frac{(N_x-\nu_{xy}N_y)^2}{E_x}+\frac{2\nu_{xy}(N_x-\nu_{xy}N_y)(N_y-\nu_{yx}N_x)}{E_x}\right.\right.\\
&\left.+\frac{(N_y-\nu_{yx}N_x)^2}{E_y}\right]+\frac{N_{xy}^2}{G_{xy}}+\frac{(h+t)^2}{4(1-\nu_{xy}\nu_{yx})}\left[E_x\left(\frac{\partial\psi_x}{\partial x}\right)^2\right.\\
&\left.+2\nu_{xy}E_y\frac{\partial\psi_x}{\partial x}\frac{\partial\psi_y}{\partial y}+E_y\left(\frac{\partial\psi_y}{\partial y}\right)^2\right]+\frac{(h+t)^2}{4}G_{xy}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)^2\\
&+\frac{t^2}{12(1-\nu_{xy}\nu_{yx})}\left[E_x\left(\frac{\partial^2 w}{\partial x^2}\right)^2+2\nu_{xy}E_y\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right.\\
&\left.\left.+E_y\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right]+\frac{t^2}{12}G_{xy}\left(2\frac{\partial^2 w}{\partial x\partial y}\right)^2\right\}dxdy
\end{aligned} \tag{4.3}$$

由(2.7)和(3.2)，可得夹心层的应变能为

$$\begin{aligned}
U_c&=\frac{1}{2G_c}\iiint_{-h/2}^{h/2}[(\tau_{xz}^c)^2+(\tau_{yz}^c)^2]dxdydz\\
&=\frac{C}{2}\iint\left[\left(\frac{\partial w}{\partial x}-\psi_x\right)^2+\left(\frac{\partial w}{\partial y}-\psi_y\right)^2\right]dxdy
\end{aligned} \tag{4.4}$$

其中，$C=G_c(h+t)^2/h$。

考虑到略去夹心层中的应力σ_x，σ_y，τ_{xy}，则外力势能为

$$U_L=N_{x0}\iint\left[\frac{1}{E_x}(N_x-\nu_{xy}N_y)-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]dxdy \tag{4.5}$$

综上所述，我们导出了具有正交各向异性表层的夹层圆柱扁壳的总势能为

$$J=U_s+U_c+U_L$$

五、解　　法

把夹层圆柱扁壳的总势能J，表达为以u，v，w，ψ_x，ψ_y为独立变元的泛函，则有

$$J=\frac{1}{2}\iiint_{h/2}^{h/2+t}\left\{\frac{E_x}{1-\nu_{xy}\nu_{yx}}\left[\frac{\partial u}{\partial x}-\frac{h+t}{2}\frac{\partial\psi_x}{\partial x}-\left(z-\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial x^2}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^2\right.$$

$$+\frac{2\nu_{xy}E_y}{1-\nu_{xy}\nu_{yx}}\left[\frac{\partial u}{\partial x}-\frac{h+t}{2}\frac{\partial\psi_x}{\partial x}-\left(z-\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial x^2}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\left[\frac{\partial v}{\partial y}-\frac{h+t}{2}\frac{\partial\psi_y}{\partial y}\right.$$

$$\left.-\left(z-\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial y^2}+\frac{w}{R}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]+\frac{E_y}{1-\nu_{xy}\nu_{yx}}\left[\frac{\partial v}{\partial y}\right.$$

$$\left.-\frac{h+t}{2}\frac{\partial\psi_y}{\partial y}-\left(z-\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial y^2}+\frac{w}{R}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]^2$$

$$+G_{xy}\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}-\frac{h+t}{2}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)-2\left(z-\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial x\partial y}\right.$$

$$\left.\left.+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]^2\right\}dxdydz+\frac{1}{2}\iiint_{-(h/2+t)}^{-h/2}\left\{\frac{E_x}{1-\nu_{xy}\nu_{yx}}\left[\frac{\partial u}{\partial x}+\frac{h+t}{2}\frac{\partial\psi_x}{\partial x}\right.\right.$$

$$\left.-\left(z+\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial x^2}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]^2+\frac{2\nu_{xy}E_y}{1-\nu_{xy}\nu_{yx}}\left[\frac{\partial u}{\partial x}+\frac{h+t}{2}\frac{\partial\psi_x}{\partial x}\right.$$

$$\left.-\left(z+\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial x^2}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]\left[\frac{\partial v}{\partial y}+\frac{h+t}{2}\frac{\partial\psi_y}{\partial y}-\left(z+\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial y^2}\right.$$

$$\left.+\frac{w}{R}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]+\frac{E_y}{1-\nu_{xy}\nu_{yx}}\left[\frac{\partial v}{\partial y}+\frac{h+t}{2}\frac{\partial\psi_y}{\partial y}-\left(z+\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial y^2}\right.$$

$$\left.+\frac{w}{R}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]^2+G_{xy}\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{h+t}{2}\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right)-2\left(z\right.\right.$$

$$\left.\left.\left.+\frac{h+t}{2}\right)\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]^2\right\}dxdydz+\frac{C}{2}\iint\left[\left(\frac{\partial w}{\partial x}-\psi_x\right)^2\right.$$

$$\left.+\left(\frac{\partial w}{\partial y}-\psi_y\right)^2\right]dxdy+N_{x0}\iint\frac{\partial u}{\partial x}dxdy \tag{5.1}$$

以u，v，w，ψ_x，ψ_y为自变函数，考虑总势能的一阶变分

$$\delta J=0$$

注意到独立变分δu，δv，δw，$\delta\psi_x$，$\delta\psi_y$的任意性，可以导出平衡方程组：

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0 \tag{5.2}$$

$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0 \tag{5.3}$$

$$\frac{\partial}{\partial x}\left(N_x\frac{\partial w}{\partial x}+N_{xy}\frac{\partial w}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w}{\partial x}+N_y\frac{\partial w}{\partial y}\right)$$

$$+\frac{C}{2t}\left(\nabla^2 w-\frac{\partial\psi_x}{\partial x}-\frac{\partial\psi_y}{\partial y}\right)-\frac{t^2}{12(1-\nu_{xy}\nu_{yx})}\left[E_x\frac{\partial^4 w}{\partial x^4}+2\nu_{xy}E_y\frac{\partial^4 w}{\partial x^2\partial y^2}\right.$$

$$\left.+E_y\frac{\partial^4 w}{\partial y^4}+4(1-\nu_{xy}\nu_{yx})G_{xy}\frac{\partial^4 w}{\partial x^2\partial y^2}\right]=0 \tag{5.4}$$

$$E_x\frac{\partial^2\psi_x}{\partial x^2}+\nu_{xy}E_y\frac{\partial^2\psi_y}{\partial x\partial y}+(1-\nu_{xy}\nu_{yx})G_{xy}\left(\frac{\partial^2\psi_x}{\partial y^2}+\frac{\partial^2\psi_y}{\partial x\partial y}\right)+\kappa\left(\frac{\partial w}{\partial x}-\psi_x\right)=0 \tag{5.5}$$

$$\nu_{xy}E_y\frac{\partial^2\psi_x}{\partial x\partial y}+E_y\frac{\partial^2\psi_y}{\partial y^2}+(1-\nu_{xy}\nu_{yx})G_{xy}\left(\frac{\partial^2\psi_x}{\partial x\partial y}+\frac{\partial^2\psi_y}{\partial x^2}\right)+\kappa\left(\frac{\partial w}{\partial y}-\psi_y\right)=0 \tag{5.6}$$

其中

$$\kappa=\frac{2C(1-\nu_{xy}\nu_{yx})}{t(h+t)^2}=\frac{2G_c(1-\nu_{xy}\nu_{yx})}{ht}$$

并有相应的边界条件：

$x=0,l$ 时

$$\left.\begin{aligned}&N_x=-\frac{N_{x0}}{2t},\ N_{xy}=0\\&w=0,\quad \frac{\partial^2 w}{\partial x^2}+\nu_{yx}\frac{\partial^2 w}{\partial y^2}=0\\&E_x\frac{\partial\psi_x}{\partial x}+\nu_{xy}E_y\frac{\partial\psi_y}{\partial y}=0,\quad \frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}=0\end{aligned}\right\} \tag{5.7}$$

$y=0,b$ 时

$$\left.\begin{aligned}&N_{xy}=N_y=0\\&w=0,\quad \nu_{xy}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0\\&\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}=0,\quad \nu_{xy}\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y}=0\end{aligned}\right\} \tag{5.8}$$

引进应力函数 Φ：

$$N_x=\frac{\partial^2\Phi}{\partial y^2},\ N_y=\frac{\partial^2\Phi}{\partial x^2},\ N_{xy}=-\frac{\partial^2\Phi}{\partial x\partial y}$$

则平衡方程(5.2)、(5.3)自动满足。由(3.3)和(3.4)可以得到

$$\frac{\partial u}{\partial x}=\frac{1}{E_x}\frac{\partial^2\Phi}{\partial y^2}-\frac{\nu_{yx}}{E_y}\frac{\partial^2\Phi}{\partial x^2}-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

$$\frac{\partial v}{\partial y}=\frac{1}{E_y}\frac{\partial^2\Phi}{\partial x^2}-\frac{\nu_{xy}}{E_x}\frac{\partial^2\Phi}{\partial y^2}-\frac{w}{R}-\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2$$

再利用(3.5)式，可以导出协调方程：

$$\frac{1}{E_y}\frac{\partial^4\Phi}{\partial x^4}+\frac{1}{E_x}\frac{\partial^4\Phi}{\partial y^4}+\left(\frac{1}{G_{xy}}-2\frac{\nu_{xy}}{E_x}\right)\frac{\partial^4\Phi}{\partial x^2\partial y^2}=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\frac{1}{R}\frac{\partial^2 w}{\partial x^2} \tag{5.9}$$

我们采用Ritz法，用能量准则来分析具有正交各向异性表层的夹层圆柱扁壳的非线性稳定性。设挠度函数

$$w=f\sin\frac{m\pi}{l}x\sin\frac{n\pi}{b}y=f\sin\alpha x\sin\beta y \tag{5.10}$$

其中 f 是待定系数，$\alpha=m\pi/l$，$\beta=n\pi/b$。m,n 分别为 x,y 方向的失稳半波数。由方程(5.5)和(5.6)可以求出

$$\psi_x=Af\cos\alpha x\sin\beta y \tag{5.11}$$

$$\psi_y=Bf\sin\alpha x\cos\beta y \tag{5.12}$$

式中 $A=\Delta_1/\Delta_0$，$B=\Delta_2/\Delta_0$

$$\Delta_0=\kappa^2+\kappa[\alpha^2E_x+\beta^2E_y+(1-\nu_{xy}\nu_{yx})G_{xy}(\alpha^2+\beta^2)]$$
$$+\alpha^2\beta^2(1-\nu_{xy}\nu_{yx})(E_xE_y-2\nu_{xy}E_yG_{xy})+G_{xy}(1-\nu_{xy}\nu_{yx})(\alpha^4E_x+\beta^4E_y)$$
$$\Delta_1=\kappa\alpha[\kappa+\beta^2(1-\nu_{xy})E_y+(\alpha^2-\beta^2)(1-\nu_{xy}\nu_{yx})G_{xy}]$$
$$\Delta_2=\kappa\alpha[\kappa+\alpha^2(E_x-\nu_{xy}E_y)+(\beta^2-\alpha^2)(1-\nu_{xy}\nu_{yx})G_{xy}]$$

由协调方程(5.9)，可以求出应力函数Φ：

$$\Phi=C_1f\sin\alpha x\sin\beta y+C_2f^2\cos2\alpha x+C_3f^2\cos2\beta y-\frac{N_{x_0}}{4tC_1}y^2 \tag{5.13}$$

其中

$$C_1=\frac{-\alpha^2/R}{\alpha^4/E_y+\beta^4/E_x+(1/G_{xy}-2\nu_{xy}/E_x)\alpha^2\beta^2}$$

$$C_2=\frac{E_y\beta^2}{32\alpha^2},\quad C_3=\frac{E_x\alpha^2}{32\beta^2}$$

由w,ψ_x,ψ_y,Φ的表达式(5.10)、(5.11)、(5.12)、(5.13)，可知以(4.3)、(4.4)、(4.5)式给出的总势能J可以表达为以f为独立变元的函数。

根据最小势能原理，总势能的一阶变分为零对应于夹层扁柱壳的平衡状态。

$$\delta J=\frac{\partial J}{\partial f}\delta f=0$$

由$\partial J/\partial f=0$，得到

$$2t\iint\Big\{\frac{1}{1-\nu_{xy}\nu_{yx}}\Big[\frac{1}{E_x}(C_1f(\nu_{xy}\alpha^2-\beta^2)\sin\alpha x\sin\beta y-4C_3\beta^2f^2\cos2\beta y$$
$$+4\nu_{xy}C_2\alpha^2f^2\cos2\alpha x)(C_1(\nu_{xy}\alpha^2-\beta^2)\sin\alpha x\sin\beta y-8C_3\beta^2f\cos2\beta y+8\nu_{xy}C_2f\cos2\alpha x)$$
$$+\frac{\nu_{xy}}{E_x}(C_1f(\nu_{xy}\alpha^2-\beta^2)\sin\alpha x\sin\beta y-4C_3\beta^2f^2\cos2\beta y+4\nu_{xy}C_2\alpha^2f^2\cos2\alpha x)(C_1(\nu_{yx}\beta^2$$
$$-\alpha^2)\cdot\sin\alpha x\sin\beta y-8C_2\alpha^2f\cos2\alpha x+8\nu_{yx}C_3\beta^2f\cos2\beta y)+\frac{\nu_{xy}}{E_x}(C_1f(\nu_{yx}\beta^2-\alpha^2)$$
$$\cdot\sin\alpha x\sin\beta y-4C_2\alpha^2f^2\cos2\alpha x+4\nu_{yx}C_3\beta^2f^2\cos2\beta y)(C_1(\nu_{xy}\alpha^2-\beta^2)\sin\alpha x\sin\beta y$$
$$-8C_3\beta^2f\cos2\beta y+8\nu_{xy}C_2\alpha^2f\cos2\alpha x)+\frac{1}{E_y}(C_1f(\nu_{yx}\beta^2-\alpha^2)\sin\alpha x\sin\beta y$$
$$-4C_2\alpha^2f^2\cos2\alpha x+4\nu_{yx}C_3\beta^2f^2\cos2\beta y)(C_1(\nu_{yx}\beta^2-\alpha^2)\sin\alpha x\sin\beta y-8C_2\alpha^2f\cos2\alpha x$$
$$+8\nu_{yx}C_3\beta^2f\cos2\beta y)\Big]+\frac{C_1^2}{G_{xy}}\alpha^2\beta^2\cos^2\alpha x\cos^2\beta y\Big\}dxdy+\frac{bl}{8}t(h+t)^2$$
$$\cdot\Big[\frac{1}{1-\nu_{xy}\nu_{yx}}(E_x\alpha^2A^2+2\nu_{xy}E_y\alpha\beta AB+E_y\beta^2B^2)+G_{xy}(\beta A+\alpha B)^2\Big]f$$
$$+\frac{bl}{24}t^3\Big[\frac{1}{1-\nu_{xy}\nu_{yx}}(E_x\alpha^4+2\nu_{xy}E_y\alpha^2\beta^2+E_y\beta^4)+4G_{xy}\alpha^2\beta^2\Big]f$$
$$+\frac{bl}{4}C[(\alpha-A)^2+(\beta-B)^2]f-N_{x_0}\cdot\frac{bl}{4}\alpha^2f=0 \tag{5.14}$$

此式可简记为

$$\frac{\partial J}{\partial f}=f\Big[N_{x_0}-\frac{1}{\alpha^2}(\kappa_0+\kappa_1f+\kappa_2f^2)\Big]=0 \tag{5.15}$$

所以，正交各向异性表层的夹层扁柱壳的平衡位置是

$$f=0 \tag{5.16}$$

$$N_{x_0}-\frac{1}{\alpha^2}(\kappa_0+\kappa_1 f+\kappa_2 f^2)=0 \tag{5.17}$$

$f=0$对应于夹层扁柱壳的初始平衡位置。曲线(5.16)和(5.17)的交点为

$$f=0,\ N_{x_0}-\kappa_0/\alpha^2=0. \tag{5.18}$$

总势能的二阶变分为

$$\delta^2 J=\frac{\partial^2 J}{\partial f^2}(\delta f)^2=N_{x_0}-\frac{1}{\alpha^2}(\kappa_0+2\kappa_1 f+3\kappa_2 f^2)$$

在交点(5.18)，$\delta^2 J=0$。由稳定性的能量判据[2]、[6]可知，交点(5.18)是临界点。曲线(5.17)描述了夹层圆柱扁壳的后曲屈特性(postbuckling behaviour)。

下面通过具体算例，求出临界点、临界载荷，并考察后屈曲特性。

六、数 值 结 果

设夹层扁柱壳的表层为正交各向异性的玻璃环氧(glass-epoxy)材料，夹心为软而轻的材料。我们选取

$$\frac{E_x}{E_y}=3.0,\quad \frac{G_{xy}}{E_y}=0.6,\quad \nu_{xy}=0.25,\quad \frac{G_c}{E_y}=10^{-3}$$

$$b=l,\ R=10l,\ h=8t,\ t=10^{-2}l$$

a) 我们考虑三种屈曲形态

〈i〉$m=n=1$。此时(5.17)式中的

$$\begin{aligned}\kappa_0=&C[(\alpha-A)^2+(\beta-B)^2]+\frac{t^3}{6}\Big[\frac{1}{1-\nu_{xy}\nu_{yx}}(E_x\alpha^4+2\nu_{xy}E_y\alpha^2\beta^2+E_y\beta^4)\\&+4G_{xy}\alpha^2\beta^2\Big]+\frac{t(h+t)^2}{2}\Big[\frac{1}{1-\nu_{xy}\nu_{yx}}(E_x\alpha^2A^2+2\nu_{xy}E_y\alpha\beta AB+E_y\beta^2B^2)\\&+G_{xy}(\beta A+\alpha B)^2\Big]+\frac{2t}{G_{xy}}C_1^2\alpha^2\beta^2+\frac{2t}{1-\nu_{xy}\nu_{yx}}\Big[\frac{C_1^2}{E_x}(\nu_{xy}\alpha^2-\beta^2)^2\\&+2\frac{\nu_{xy}}{E_x}C_1^2(\nu_{xy}\alpha^2-\beta^2)(\nu_{yx}\beta^2-\alpha^2)+\frac{C_1^2}{E_y}(\nu_{yx}\beta^2-\alpha^2)^2\Big]\end{aligned} \tag{6.1}$$

$$\begin{aligned}\kappa_1=&\frac{128tC_1}{\pi^2(1-\nu_{xy}\nu_{yx})}\Big\{\frac{1}{E_x}(\nu_{xy}\alpha^2-\beta^2)(C_3\beta^2-C_2\alpha^2\nu_{xy})+\frac{\nu_{xy}}{E_x}[C_2\alpha^2(2\nu_{xy}\alpha^2\\&-\beta^2-\nu_{xy}\nu_{yx}\beta^2)+C_3\beta^2(2\nu_{yx}\beta^2-\alpha^2-\nu_{xy}\nu_{yx}\alpha^2)]\\&+\frac{1}{E_y}(\nu_{yx}\beta^2-\alpha^2)(C_2\alpha^2-C_3\beta^2\nu_{yx})\Big\}\end{aligned} \tag{6.2}$$

$$\begin{aligned}\kappa_2=&\frac{128}{1-\nu_{xy}\nu_{yx}}\Big[\frac{1}{E_x}(C_2^2\alpha^4\nu_{xy}^2+C_3^2\beta^4)-\frac{2\nu_{xy}}{E_x}(C_2^2\alpha^4\nu_{xy}+C_3^2\beta^4\nu_{yx})\\&+\frac{1}{E_y}(C_2^2\alpha^4+C_3^2\beta^4\nu_{yx}^2)\Big]\end{aligned} \tag{6.3}$$

可以求得$A=0.307069/l$，$B=0.368532/l$，于是(5.17)式为

$$P=[2.03873+2.34710f/l+487.04546(f/l)^2]\times 10^{-3}$$

其中$P=\pi^2 N_{x_0}/E_y l$是无量纲轴向压力。

〈ii〉 $m=2$，$n=1$。此时κ_0同(6.1)式，$\kappa_1=0$，κ_2同(6.3)式。可求得$A=0.0991767/l$，$B=0.340784/l$，于是有

$$P=[1.50348+1501.29171(f/l)^2]\times10^{-3}$$

〈iii〉 $m=3$，$n=1$。此时κ_0同(6.1)式，κ_2同(6.3)式，

$$\kappa_1=\frac{128tC_1}{3\pi^2(1-\nu_{xy}\nu_{yx})}\Big\{\frac{1}{E_x}(\nu_{xy}\alpha^2-\beta^2)(C_3\beta^2-C_2\alpha^2\nu_{xy})+\frac{\nu_{xy}}{E_x}[C_2\alpha^2(2\nu_{xy}\alpha^2-\beta^2-\nu_{xy}\nu_{yx}\beta^2)+C_3\beta^2(2\nu_{yx}\beta^2-\alpha^2-\nu_{xy}\nu_{yx}\alpha^2)]+\frac{1}{E_y}(\nu_{yx}\beta^2-\alpha^2)(C_2\alpha^2-C_3\beta^2\nu_{yx})\Big\}$$

又求得$A=0.075849/l$，$B=0.090065/l$，于是有

$$P=[1.61515-0.09009f/l+3994.349889(f/l)^2]\times10^{-3}$$

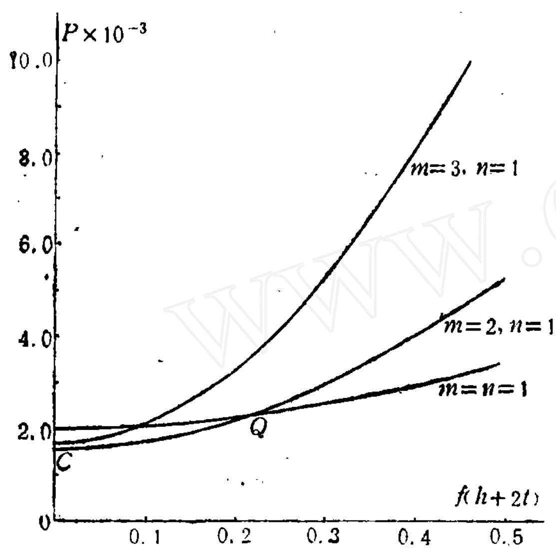

图 2

计算结果示于图2。图2中临界点C为$f=0$，$P=1.50348$，即临界载荷

$$(N_{x_0})_{cr}=1.50348E_yl/\pi^2$$

失稳时，最初的屈曲形态为$m=2$，$n=1$。当载荷达到$(N_{x_0})_Q=2.4E_yl/\pi^2$时，转而出现屈曲形态$m=n=1$。(见图3)。失稳后，轴向承载能力继续增大。

b) 对于屈曲形态$m=2$，$n=1$，分别取$G_c/E_y=10^{-3}$；5×10^{-3}；10^{-2}，并求出相应的后屈曲载荷挠度曲线：

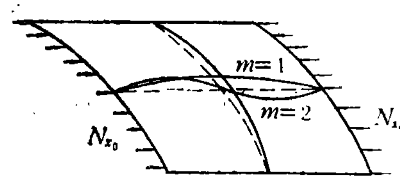

图 3

〈i〉 $G_c/E_y=10^{-3}$

$P=[1.50346+1501.29171(f/l)^2]\times10^{-3}$

〈ii〉 $G_c/E_y=5\times10^{-3}$

$P=[5.87929+1501.29171(f/l)^2]\times10^{-3}$

〈iii〉 $G_c/E_y=10^{-2}$

$P=[10.46797+1501.29171(f/l)^2]\times10^{-3}$

计算结果表明(图4)，当夹心层的剪切刚度C或剪切模量G_c增大时，临界载荷值提高，后屈曲轴向承载能力提高

c) 为研究曲率对后屈曲特性的影响，选取$h=3t$，$t=10^{-2}l$，$R=2l$；$4l$；$20l$，则无量纲曲率

$$\kappa_R=\frac{l^2}{R(h+2t)}=10;5;1$$

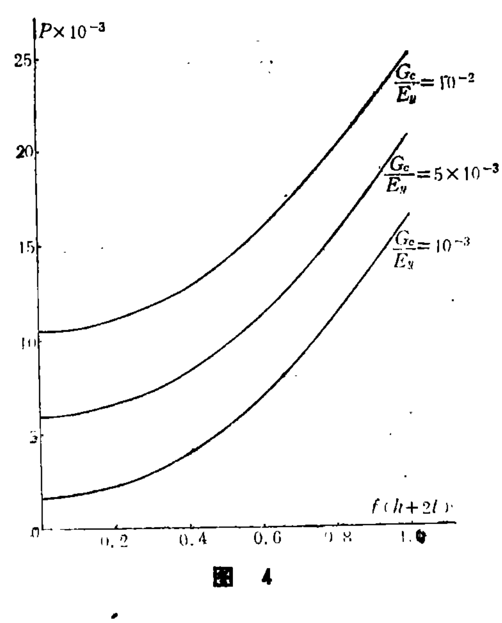

图 4

材料常数仍是$E_x/E_y=3.0$，$G_{xy}/E_y=0.6$，$G_c/E_y=10^{-3}$. 对于屈曲形态$m=n=1$，分别计算

〈i〉$\kappa_R=1$：$R=20l$，此时$A=0.331229/l$，$B=0.783532/l$，

$$P=[1.236449+0.187768f/l+487.045455(f/l)^2]\times10^{-3}$$

〈ii〉$\kappa_R=5$：$R=4l$，A,B值同上，

$$P=[1.240928+0.938840f/l+487.045455(f/l)^2]\times10^{-3}$$

〈iii〉$\kappa_R=10$：$R=2l$，A,B值同上，

$$P=[1.254924+1.87768f/l+487.045455(f/l)^2]\times10^{-3}$$

计算结果列于表1中. 由表可知，随着曲率增大，后屈曲轴向承载能力略有提高，临界载荷也略有提高.

表　1

κ_R \ $P\times10^{-3}$ \ f/l	0.000	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
1	1.23645	1.28703	1.43502	1.68042	2.02323	2.46345	3.00108	3.63612	4.36856	5.19842	6.12568
5	1.24093	1.29902	1.45452	1.70743	2.05775	2.50548	3.05062	3.69317	4.43313	5.27049	6.20527
10	1.25492	1.32241	1.48730	1.74960	2.10930	2.56642	3.12095	3.77288	4.52223	5.36898	6.31315

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Nonlinear Stability Analysis of a Sandwich Shallow Cylindrical Panel with Orthotropic Surfaces

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Abstract

In this paper, the large deflection theory is adopted to analyse the geometrical nonlinear stability of a sandwich shallow cylindrical panel with orthotropic surfaces. The critical point is determined and the postbuckling behaviour of the panel is studied.