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多层圆柱壳在轴压下稳定问题的数学弹性力学解

戴滂陵　王震鸣

（中国科学院力学研究所）

提要　本文给出了稳定问题的 Новожилов [1] 平衡方程和边界条件的曲线坐标形式，用数学弹性力学的方法分析了两端简支的各向同性多层圆柱壳在轴压下的稳定问题，给出了可求解临界载荷的超越方程．用数值计算方法算得了临界载荷，并与夹层壳理论算得的结果作了比较．

一、引　言

由于数学上的困难，用数学弹性力学方法严格求解板、壳稳定问题的文献并不多见．近代各种近似计算方法的出现和应用，迫切地希望得到一些问题（那怕是一些简单问题）的精确解，以验证这些方法的正确性及其计算结果的精度．这种趋势使得人们重新认真地考虑一些可能得到的严格解问题．在文献〔1〕中，Новожилов给出了可求解稳定问题的平衡方程和边界条件，但没有给出具体问题的解．文献〔2〕严格按照Новожилов 的平衡方程和边界条件求解了圆柱壳在轴压下的轴对称失稳等若干问题．用级数展开的方法求得了相应于经典理论临界载荷公式的修正项．文献〔3〕求得了四边简支均匀受压矩形板临界载荷的精确解．文献〔4〕探讨了两端简支的夹层圆柱壳在轴压下轴对称失稳问题的严格解（其中，表层按经典壳理论处理，夹芯按数学弹性力学方法处理），用数值计算方法算得了临界载荷．本文将 Новожилов[1] 的稳定问题平衡方程（（v. 13）式）和边界条件（（v. 30）式）改写为张量方程，使之具有一般性．然后在圆柱坐标形式下，用数学弹性力学方法分析了两端简支的各向同性多层圆柱壳在轴压下的稳定问题，得到了可求解各种临界载荷的超越方程，用数值计算方法可算得所需的结果．对于三层壳的情况，本文给出了在轴压下轴对称失稳时的算例，并与夹层壳理论算得的结果作了比较．

二、曲线坐标形式的Новожилов平衡方程

Новожилов 在文献〔1〕中给出了直角坐标形式的稳定问题平衡方程及其边界条件．为了运用方便，首先将其改写为张量方程．采用和文献〔5〕相同的记号，利用哑标求和约定，通过升、降指标和变普通偏导数为协变导数等措施，可得到用张量形式写出的稳定问题平衡方程：

$$(\sigma^{ij}+\sigma_0{}^{jk}u^i|_{;k})_{;j}=0 \qquad (i=1,\ 2,\ 3) \tag{2.1}$$

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相应的边界条件（不受静水压力作用）为：

$$(\sigma^{ij}+\sigma_0^{kj}u^i|_k)n_j=0 \qquad (i=1,\ 2,\ 3) \tag{2.2}$$

其中，n_j 为物体变形前界面单位法向矢量的协变分量.

将(2.1)式和(2.2)式的协变导数写开得到：

$$\left[\sigma^{ij}+\sigma_0^{kj}\left(\frac{\partial u^i}{\partial x^k}+u^m\Gamma^i_{mk}\right)\right]_{,j}+\left[\sigma^{nj}+\sigma_0^{kj}\left(\frac{\partial u^n}{\partial x^k}+u^m\Gamma^n_{mk}\right)\right]\Gamma^i_{nj}$$
$$+\left[\sigma^{in}+\sigma_0^{nk}\left(\frac{\partial u^i}{\partial x^k}+u^m\Gamma^i_{mk}\right)\right]\Gamma^j_{nj}=0 \qquad (i=1,\ 2,\ 3) \tag{2.3}$$

和

$$\left[\sigma^{ij}+\sigma_0^{kj}\left(\frac{\partial u^i}{\partial x^k}+u^m\Gamma^i_{mk}\right)\right]n_j=0 \qquad (i=1,\ 2,\ 3) \tag{2.4}$$

由(2.3)式和(2.4)式可写出各种曲线坐标形式的稳定问题平衡方程和相应的边界条件.

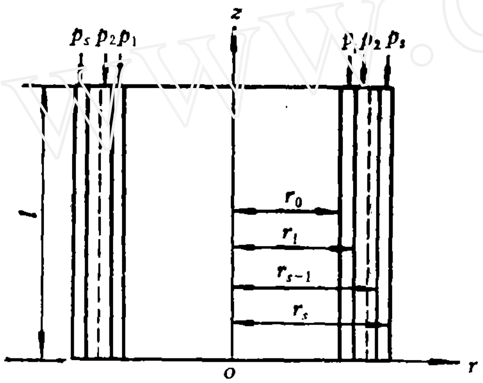
采用圆柱坐标，取 $x^1=r,x^2=\theta,x^3=z$ 从而 $\Gamma^1_{22}=-r,\Gamma^2_{12}=\Gamma^2_{21}=\dfrac{1}{r}$（其余为零），这时 $n_r=1$，$n_\theta=n_z=0$，代入(2.3)，(2.4)式可得到圆柱坐标形式的Новожилов平衡方程：

$$\left.\begin{aligned}
&\frac{\partial}{\partial r}\left[\sigma_r+\sigma_r^0\frac{\partial u_r}{\partial r}+\tau_{r\theta}^0\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}-u_\theta\right)+\tau_{rz}^0\frac{\partial u_r}{\partial z}\right]\\
&+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\tau_{r\theta}+\tau_{r\theta}^0\frac{\partial u_r}{\partial r}+\sigma_\theta^0\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}-u_\theta\right)+\tau_{\theta z}^0\frac{\partial u_r}{\partial z}\right]\\
&+\frac{\partial}{\partial z}\left[\tau_{rz}+\tau_{rz}^0\frac{\partial u_r}{\partial r}+\tau_{z\theta}^0\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}-u_\theta\right)+\sigma_z^0\frac{\partial u_r}{\partial z}\right]\\
&+\frac{1}{r}\left[\sigma_r+\sigma_r^0\frac{\partial u_r}{\partial r}+\tau_{r\theta}^0\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}-u_\theta\right)+\tau_{rz}^0\frac{\partial u_r}{\partial z}\right]\\
&-\frac{1}{r}\left[\sigma_\theta+\tau_{\theta r}^0\frac{\partial u_\theta}{\partial r}+\sigma_\theta^0\frac{1}{r}\left(\frac{\partial u_\theta}{\partial\theta}+u_r\right)+\tau_{\theta z}^0\frac{\partial u_\theta}{\partial z}\right]=0\\
&\frac{\partial}{\partial r}\left\{\frac{\tau_{\theta r}}{r}+\sigma_r^0\left[\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{u_\theta}{r^2}\right]+\frac{\tau_{r\theta}^0}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{u_\theta}{r}\right)+\frac{u_r}{r}\right]+\tau_{rz}^0\frac{\partial}{\partial z}\left(\frac{u_\theta}{r}\right)\right\}\\
&+\frac{1}{r}\frac{\partial}{\partial\theta}\left\{\frac{\sigma_\theta}{r}+\tau_{\theta r}^0\left[\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{u_\theta}{r^2}\right]+\frac{\sigma_\theta^0}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{u_\theta}{r}\right)+\frac{u_r}{r}\right]+\tau_{\theta z}^0\frac{\partial}{\partial z}\left(\frac{u_\theta}{r}\right)\right\}\\
&+\frac{\partial}{\partial z}\left\{\frac{\tau_{\theta z}}{r}+\tau_{rz}^0\left[\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{u_\theta}{r^2}\right]+\frac{\tau_{\theta z}^0}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{u_\theta}{r}\right)+\frac{u_r}{r}\right]+\sigma_z^0\frac{\partial}{\partial z}\left(\frac{u_\theta}{r}\right)\right\}\\
&+\frac{1}{r^2}\left\{3\tau_{r\theta}+\tau_{r\theta}^0\frac{\partial u_r}{\partial r}+\sigma_\theta^0\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}-u_\theta\right)+\tau_{\theta z}^0\frac{\partial u_r}{\partial z}\right\}\\
&+\frac{2}{r}\left\{\sigma_r^0\left[\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{u_\theta}{r^2}\right]+\tau_{r\theta}^0\frac{1}{r}\left[\frac{\partial}{\partial\theta}\left(\frac{u_\theta}{r}\right)+\frac{u_r}{r}\right]+\tau_{rz}^0\frac{\partial}{\partial z}\left(\frac{u_\theta}{r}\right)\right\}=0\\
&\frac{\partial}{\partial r}\left[\tau_{rz}+\sigma_r^0\frac{\partial z}{\partial r}+\tau_{r\theta}^0\frac{1}{r}\frac{\partial u_z}{\partial\theta}+\tau_{rz}^0\frac{\partial u_z}{\partial z}\right]\\
&+\frac{1}{r}\frac{\partial}{\partial\theta}\left[\tau_{\theta z}+\tau_{\theta r}^0\frac{\partial u_z}{\partial r}+\sigma_\theta^0\frac{1}{r}\frac{\partial u_z}{\partial\theta}+\tau_{\theta z}^0\frac{\partial u_z}{\partial z}\right]\\
&+\frac{\partial}{\partial z}\left[\sigma_z+\tau_{rz}^0\frac{\partial u_z}{\partial r}+\tau_{\theta z}^0\frac{1}{r}\frac{\partial u_z}{\partial\theta}+\sigma_z^0\frac{\partial u_z}{\partial z}\right]\\
&+\frac{1}{r}\left[\tau_{rz}+\sigma_r^0\frac{\partial u_z}{\partial r}+\tau_{r\theta}^0\frac{1}{r}\frac{\partial u_z}{\partial\theta}+\tau_{rz}^0\frac{\partial u_z}{\partial z}\right]=0
\end{aligned}\right\} \tag{2.5}$$

和边界条件：

$$\left.\begin{aligned}&\sigma_r+\sigma_r^0\frac{\partial u_r}{\partial r}+\tau_{\theta r}^0\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}-u_\theta\right)=0\\&\tau_{\theta r}+\sigma_r^0\frac{\partial u_\theta}{\partial r}+\tau_{\theta r}^0\frac{1}{r}\left(\frac{\partial u_\theta}{\partial\theta}+u_r\right)=0\\&\tau_{rz}+\sigma_r^0\frac{\partial u_z}{\partial r}+\tau_{\theta r}^0\frac{1}{r}\frac{\partial u_z}{\partial\theta}=0\end{aligned}\right\}\tag{2.6}$$

三、基本方程



设多层圆柱壳的内、外半径分别为 r_0 和 r_s，各层连结处半径分别为 $r_1, r_2\cdots, r_{s-1}$；壳长为 l. 取圆柱坐标如图所示. 设轴向受均匀应变的作用，各层所受均匀轴压分别为 $p_1, p_2\cdots, p_s$. 并假设各层材料的泊桑比相同，则可得第 i 层壳的初始应力解为：

$$\left.\begin{aligned}&\sigma_z^{0i}=-p_i\\&\sigma_r^{0i}=\sigma_\theta^{0i}=\tau_{r\theta}^{0i}=\tau_{\theta z}^{0i}=\tau_{rz}^{0i}=0\end{aligned}\right\}\tag{3.1}$$

其中，$\dfrac{p_i}{E_i}=\dfrac{p_{i+1}}{E_{i+1}}\qquad(i=1,2,\cdots,s-1)$　　　　(3.2)

附加应力为 σ_r^i、σ_θ^i、σ_z^i、$\tau_{r\theta}^i$、τ_{rz}^i、$\tau_{\theta z}^i$；附加位移为 u_r^i、u_θ^i、u_z^i(为简洁起见，以下在单独讨论第 i 层壳时省掉 i 标)，代入(2.5)、(2.6)式可得平衡方程：

$$\left.\begin{aligned}&\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta}+\frac{\partial\tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}-p\frac{\partial^2u_r}{\partial z^2}=0\\&\frac{\partial\tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\tau_{z\theta}}{\partial z}+\frac{2\tau_{r\theta}}{r}-p\frac{\partial^2u_\theta}{\partial z^2}=0\\&\frac{\partial\tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta z}}{\partial\theta}+\frac{\partial\sigma_z}{\partial z}+\frac{\tau_{rz}}{r}-p\frac{\partial^2u_z}{\partial z^2}=0\end{aligned}\right\}\tag{3.3}$$

和边界条件：

$$\sigma_r=\tau_{r\theta}=\tau_{rz}=0\qquad(r=r_0,\ r_s)\tag{3.4}$$

用位移表示的简支边界条件为

$$u_r=u_\theta=\frac{\partial u_z}{\partial z}=0\qquad(z=0,\ l)\tag{3.5}$$

位移-应变关系为

$$\left.\begin{aligned}&\varepsilon_r=\frac{\partial u_r}{\partial r},\quad\varepsilon_\theta=\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta},\quad\varepsilon_z=\frac{\partial u_z}{\partial z}\\&\gamma_{r\theta}=\frac{1}{r}\frac{\partial u_r}{\partial\theta}+\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r},\quad\gamma_{rz}=\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}\\&\gamma_{\theta z}=\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial\theta}\end{aligned}\right\}\tag{3.6}$$

应力-应变关系为

$$\left.\begin{aligned}&\sigma_r=\lambda e+2G\,\varepsilon_r,\quad\sigma_\theta=\lambda e+2G\,\varepsilon_\theta,\quad\sigma_z=\lambda e+2G\,\varepsilon_z\\&\tau_{r\theta}=G\,\gamma_{r\theta},\quad\tau_{rz}=G\,\gamma_{rz},\quad\tau_{\theta z}=G\gamma_{\theta z}\end{aligned}\right\}\tag{3.7}$$

其中 $G=\dfrac{E}{2(1+\nu)}$，$\lambda=\dfrac{\nu E}{(1+\nu)(1-2\nu)}$，$e=\dfrac{\partial u_r}{\partial r}+\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta}+\dfrac{\partial u_z}{\partial z}+\dfrac{u_r}{r}$

在各层壳的连结处，应满足位移连续条件

$$u_r^i=u_r^{i+1},\quad u_\theta^i=u_\theta^{i+1},\quad u_z^i=u_z^{i+1}\qquad(i=1,\ 2,\ \cdots,\ s-1)\tag{3.8}$$

和应力连续条件

$$\sigma_r^i=\sigma_r^{i+1},\quad \tau_{rz}^i=\tau_{rz}^{i+1},\quad \tau_{r\theta}^i=\tau_{r\theta}^{i+1}\qquad(i=1,\ 2,\ \cdots,\ s-1)\tag{3.9}$$

将(3.6)式代入(3.7)式再代入(3.3)式可得到以位移分量表示的平衡方程：

$$\left.\begin{aligned}&(\lambda+G)\frac{\partial e}{\partial r}+G\nabla^2u_r-G\frac{1}{r^2}\left(2\frac{\partial u_\theta}{\partial\theta}+u_r\right)-p\frac{\partial^2u_r}{\partial z^2}=0\\&(\lambda+G)\frac{1}{r}\frac{\partial e}{\partial\theta}+G\nabla^2u_\theta+G\frac{1}{r^2}\left(2\frac{\partial u_r}{\partial\theta}-u_\theta\right)-p\frac{\partial^2u_\theta}{\partial z^2}=0\\&(\lambda+G)\frac{\partial e}{\partial z}+G\nabla^2u_z-p\frac{\partial^2u_z}{\partial z^2}=0\end{aligned}\right\}\tag{3.10}$$

其中：$\nabla^2=\dfrac{\partial^2}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2}+\dfrac{\partial^2}{\partial z^2}$

采用胡海昌的方法[6]，引进位移函数 φ_0、φ_1、φ_2，使得

$$\left.\begin{aligned}u_r&=\frac{\partial}{\partial r}(\varphi_1+\varphi_2)+\frac{1}{r}\frac{\partial\varphi_0}{\partial\theta}\\u_\theta&=\frac{1}{r}\frac{\partial}{\partial\theta}(\varphi_1+\varphi_2)-\frac{\partial\varphi_0}{\partial r}\\u_z&=\frac{G-p}{G}\frac{\partial\varphi_1}{\partial z}+\frac{\partial\varphi_2}{\partial z}\end{aligned}\right\}\tag{3.11}$$

将(3.11)代入(3.10)可得已分离变量的一组偏微分方程

$$\left.\begin{aligned}&\frac{\partial^2\varphi_0}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi_0}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi_0}{\partial\theta^2}+\frac{G-p}{G}\frac{\partial^2\varphi_0}{\partial z^2}=0\\&\frac{\partial^2\varphi_1}{\partial r^2}+\frac{1}{r}\frac{2\varphi_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi_1}{\partial\theta^2}+\frac{G-p}{G}\frac{\partial^2\varphi_1}{\partial z^2}=0\\&\frac{\partial^2\varphi_2}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi_2}{\partial r}+\frac{1}{r^2}\frac{\partial\varphi_2}{\partial\theta^2}+\frac{\lambda+2G-p}{\lambda+2G}\frac{\partial^2\varphi_2}{\partial z^2}=0\end{aligned}\right\}\tag{3.12}$$

考虑到(3.12)式及简支边界条件(3.5)，设

$$\left.\begin{aligned}\varphi_0&=f_0(r)\sin\frac{m\pi z}{l}\sin n\theta\\\varphi_1&=f_1(r)\sin\frac{m\pi z}{l}\cos n\theta\\\varphi_2&=f_2(r)\sin\frac{m\pi z}{l}\cos n\theta\end{aligned}\right\}\tag{3.13}$$

其中，m 为失稳时轴向形成的半波数；n 为周向形成的半波数.

将(3.13)代入(3.12)式可得

$$\left.\begin{aligned}&f_0''(r)+\frac{1}{r}f_0'(r)-\left(\frac{n^2}{r^2}+\frac{G-p}{G}\frac{m^2\pi^2}{l}\right)f_0(r)=0\\&f_1''(r)+\frac{1}{r}f_1'(r)-\left(\frac{n^2}{r^2}+\frac{G-p}{G}\frac{m^2\pi^2}{l^2}\right)f_1(r)=0\\&f_2''(r)+\frac{1}{r}f_2'(r)-\left(\frac{n^2}{r^2}+\frac{\lambda+2G-p}{\lambda+2G}\frac{m^2\pi^2}{l^2}\right)f_2(r)=0\end{aligned}\right\}\tag{3.14}$$

这是一组虚宗变量的Bessel方程，其解为

$$\left.\begin{aligned}f_0(r)&=AI_n(m_0r)+BK_n(m_0r)\\f_1(r)&=CI_n(m_1r)+DK_n(m_1r)\\f_2(r)&=EI_n(m_2r)+FK_n(m_2r)\end{aligned}\right\}\tag{3.15}$$

其中，A、B、C、D、E、F 为积分常数：

$$m_0=\frac{m\pi}{l}\left(1-\frac{p}{G}\right)^{\frac{1}{2}}=m_1,\quad m_2=\frac{m\pi}{l}\left(1-\frac{p}{\lambda+2G}\right)^{\frac{1}{2}}\tag{3.16}$$

将(3.15)代入(3.13)再代入(3.11)可得位移

$$\left.\begin{aligned}u_r&=\left\{\frac{\partial}{\partial r}[CI_n(m_1r)+DK_n(m_1r)+EI_n(m_2r)+FK_n(m_2r)]\right.\\&\quad\left.+\frac{n}{r}[AI_n(m_0r)+BK_n(m_0r)]\right\}\sin\frac{m\pi z}{l}\cos n\theta\\u_\theta&=-\left\{\frac{n}{r}[CI_n(m_1r)+DK_n(m_1r)+EI_n(m_2r)+FK_n(m_2r)]\right.\\&\quad\left.+\frac{\partial}{\partial r}[AI_n(m_0r)+BK_n(m_0r)]\right\}\sin\frac{m\pi z}{l}\sin n\theta\\u_z&=\left\{\frac{G-p}{G}[CI_n(m_1r)+DK_n(m_1r)]+[EI_n(m_2r)\right.\\&\quad\left.+FK_n(m_2r)]\right\}\frac{m\pi}{l}\cos\frac{m\pi z}{l}\cos n\theta\end{aligned}\right\}\tag{3.17}$$

可以看到，简支边界条件已得到满足．

将(3.11)式代入(3.6)式再代入(3.7)式可得到用位移函数表示的应力分量

$$\left.\begin{aligned}\sigma_r&=\frac{\lambda p}{\lambda+2G}\frac{\partial^2\varphi_2}{\partial z^2}+2G\left[-\frac{1}{r}\frac{\partial}{\partial r}(\varphi_1+\varphi_2)-\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(\varphi_1+\varphi_2)\right.\\&\quad\left.-\frac{\partial^2}{\partial z^2}\left(\frac{G-p}{G}\varphi_1+\frac{\lambda+2G-p}{\lambda+2G}\varphi_2\right)+\frac{1}{r}\frac{\partial^2\varphi_0}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial\varphi_0}{\partial\theta}\right]\\\tau_{rz}&=2G\frac{\partial^2}{\partial r\partial z}\left(\frac{2G-p}{2G}\varphi_1+\varphi_2\right)+\frac{G}{r}\frac{\partial^2\varphi_0}{\partial\theta\partial z}\\\tau_{r\theta}&=2G\left[\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}(\varphi_1+\varphi_2)-\frac{1}{r^2}\frac{\partial}{\partial\theta}(\varphi_1+\varphi_2)+\frac{1}{r^2}\frac{\partial^2\varphi_0}{\partial\theta^2}\right.\\&\quad\left.+\frac{1}{r}\frac{\partial\varphi_0}{\partial r}+\frac{G-p}{2G}\frac{\partial^2\varphi_0}{\partial z^2}\right]\end{aligned}\right\}\tag{3.18}$$

由(3.15)、(3.13)、(3.18)和边界条件(3.4)，并注意到应力连续条件(3.9)和位移连续条件 (3.8)式，可得积分常数的齐次线性方程组：

$$\left.\begin{aligned}&[Q(r_0)]_1=0\\&[R(r_0)]_1=0\\&[G(r_0)]_1=0\\&[Q(r_s)]_s=0\\&[R(r_s)]_s=0\\&[G(r_s)]_s=0\end{aligned}\right\}\tag{3.19a}$$

$$\left.\begin{aligned}&[J(r_i)]_i-[J(r_i)]_{i+1}=0\\&[T(r_i)]_i-[T(r_i)]_{i+1}=0\\&[W(r_i)]_i-[W(r_i)]_{i+1}=0\\&[Q(r_i)]_i-[Q(r_i)]_{i+1}=0\\&[R(r_i)]_i-[R(r_i)]_{i+1}=0\\&[G(r_i)]_i-[G(r_i)]_{i+1}=0\end{aligned}\quad(i=1,2,,\cdots,s-1)\right\}\tag{3.19b}$$

其中

$$
\left.
\begin{aligned}
&[Q(r_j)]_j=\Big\{\Big[-\frac{n}{r_j^2}I_n(m_{0i}r_j)+\frac{n}{r_j}m_{0i}I'_n(m_{0i}r_j)\Big]A_i\\
&\quad+\Big[-\frac{n}{r_j^2}K_n(m_{0i}r_j)-\frac{n}{r_j}m_{0i}K'_n(m_{0i}r_j)\Big]B_i\\
&\quad+\Big[\Big(\frac{n^2}{r_j^2}+\frac{G_i-p_i}{G_i}\Big(\frac{m\pi}{l}\Big)^2\Big)I_n(m_{1i}r_j)-\frac{m_{1i}}{r_j}I'_n(m_{1i}r_j)\Big]C_i\\
&\quad+\Big[\Big(\frac{n^2}{r_j^2}+\frac{G_i-p_i}{G_i}\Big(\frac{m\pi}{l}\Big)^2\Big)K_n(m_{1i}r_j)+\frac{m_{1i}}{r_j}K'_n(m_{1i}r_j)\Big]D_i\\
&\quad+\Big[\Big(\frac{n^2}{r_j^2}+\frac{2G_i-p_i}{2G_i}\Big(\frac{m\pi}{l}\Big)^2\Big)I_n(m_{2i}r_j)-\frac{m_{2i}}{r_j}I'_n(m_{2i}r_j)\Big]E_i\\
&\quad+\Big[\Big(\frac{n^2}{r_j^2}+\frac{2G_i-p_i}{2G_i}\Big(\frac{m\pi}{l}\Big)^2\Big)K_n(m_{2i}r_j)+\frac{m_{2i}}{r_j}K'_n(m_{2i}r_j)\Big]F_i\Big\}\\
&\quad\cdot\sin\frac{m\pi z}{l}\cos n\theta\\
&[R(r_j)]_i=\Big\{\Big[\frac{n}{2r_j}I_n(m_{0i}r_j)\Big]A_i+\Big[\frac{n}{2r_j}K_n(m_{0i}r_j)\Big]B_i\\
&\quad+\Big[\frac{2G_i-p_i}{2G_i}m_{1i}I'_n(m_{1i}r_j)\Big]C_i+\Big[-\frac{2G_i-p_i}{2G_i}m_{1i}K'_n(m_{1i}r_j)\Big]D_i\\
&\quad+[m_{2i}I'_n(m_{2i}r_j)]E_i+[-m_{2i}K'_n(m_{2i}r_j)]F_i\Big\}\cos\frac{m\pi z}{l}\cos n\theta\\
&[G(r_j)]_i=\Big\{\Big[\frac{m_{0i}}{r_j}I'_n(m_{0i}r_j)-\Big(\frac{n^2}{r_j^2}-\frac{G_i-p_i}{2G_i}\Big(\frac{m\pi}{l}\Big)^2\Big)I_n(m_{0i}r_j)\Big]A_i\\
&\quad+\Big[-\frac{m_{0i}}{r_j}K'_n(m_{0i}r_j)-\Big(\frac{n^2}{r_j^2}+\frac{G_i-p_i}{2G_i}\Big(\frac{m\pi}{l}\Big)^2\Big)K_n(m_{0i}r_j)\Big]B_i\\
&\quad+\Big[\frac{n}{r_j^2}I_n(m_{1i}r_j)-\frac{n}{r_j}m_{1i}I'_n(m_{1i}r_j)\Big]C_i\\
&\quad+\Big[\frac{n}{r_j^2}K_n(m_{1i}r_j)+\frac{n}{r_j}m_{1i}K'_n(m_{1i}r_j)\Big]D_i\\
&\quad+\Big[\frac{n}{r_j^2}I_n(m_{2i}r_j)-\frac{n}{r_j}m_{2i}I'_n(m_{2i}r_j)\Big]E_i\\
&\quad+\Big[\frac{n}{r_j^2}K_n(m_{2i}r_j)+\frac{n}{r_j}m_{2i}K'_n(m_{2i}r_j)\Big]F_i\Big\}\sin\frac{m\pi z}{l}\cos n\theta\\
&[J(r_j)]_i=\Big\{\Big[\frac{n}{r_j}I_n(m_{0i}r_j)\Big]A_i+\Big[\frac{n}{r_j}K_n(m_{0i}r_j)\Big]B_i\\
&\quad+[m_{1i}I'_n(m_{1i}r_j)]C_i+[-m_{1i}K'_n(m_{1i}r_j)]D_i\\
&\quad+[m_{2i}I'_n(m_{2i}r_j)]E_i+[-m_{2i}K'_n(m_{2i}r_j)]E_i\Big\}\sin\frac{m\pi z}{l}\sin n\theta\\
&[T(r_j)]_i=\{[m_{0i}I'_n(m_{0i}r_j)]A_i+[-m_{0i}K'_n(m_{0i}r_j)]B_i\\
&\quad+\Big[-\frac{n}{r_j}I_n(m_{1i}r_j)\Big]C_i+\Big[-\frac{n}{r_j}K_n(m_{1i}r_j)\Big]D_i\\
&\quad+\Big[-\frac{n}{r_j}I_n(m_{2i}r_j)\Big]E_i+\Big[-\frac{n}{r_j}K_n(m_{2i}r_j)\Big]F_i\Big\}\sin\frac{m\pi z}{l}\sin n\theta\\
&[W(r_j)]_i=\Big\{\Big[\frac{G_i-p_i}{G_i}I_n(m_{1i}r_j)\Big]C_i+\Big[\frac{G_i-p_i}{G_i}K_n(m_{1i}r_j)\Big]D_i\\
&\quad+[I_n(m_{2i}r_j)]E_i+[K_n(m_{2i}r_j)]F_i\Big\}\cos\frac{m\pi z}{l}\cos\theta n
\end{aligned}
\right\}\qquad(3.20)
$$

(3.19)式为 $6\times s$ 阶超越方程组，关于待定系数 A_1，…，A_s，…，F_1 …，F_s 是线性的．该方程组具有非零解的充要条件为其系数行列式等于零．由此，通过试算的方法可求得两端简支多层圆柱壳在轴压下，以不同形态失稳时的临界载荷和屈曲半波数．

为了尽快求得所需的结果，变化参数 p(即临界载荷，等于 $\sum_{i=1}^{s} p_i(r_i-r_{i-1})$；其中，$p_i(i=1,2,\cdots,s)$ 之间的关系式见(3.2)式)和 r_i 可以按壳理论算得的值为基础，分别代入不同的数值进行试算．使得行列式的值最接近于零的那一组试算值即为所求的解答．(3.19)式的系数矩阵为聚集在主对角线附近的稀疏矩阵，经过适当的处理用高斯消去法将行列式化为上三角形矩阵，可求得行列式的值．另外，还可采用文献[2]中的方法，将超越函数展开成幂级数，最终得到求解临界载荷的方程(隐式)．不过这种方法当壳的层数较多时很难奏效，因为这时级数的项次甚多，数量级的大小不易区分，不如直接用数值计算方法来得方便．

四、三层圆柱壳的轴对称失稳

轴对称失稳是圆柱壳失稳的一种特殊形式．这时

$$u_\theta=\gamma_{r\theta}=\gamma_{\theta z}=n=\varphi_0=0 \tag{4.1}$$

考虑到(4.1)式，令(3.19)式中 $s=3$ 可得求解三层圆柱壳轴对称失稳时的临界载荷和屈曲半波数的方程：

$$\begin{vmatrix}
T_{11} & T_{12} & T_{13} & T_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
T_{21} & T_{22} & T_{23} & T_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
T_{31} & T_{32} & T_{33} & T_{34} & T_{35} & T_{36} & T_{37} & T_{38} & 0 & 0 & 0 & 0 \\
T_{41} & T_{42} & T_{43} & T_{44} & T_{45} & T_{46} & T_{47} & T_{48} & 0 & 0 & 0 & 0 \\
T_{51} & T_{52} & T_{53} & T_{54} & T_{55} & T_{56} & T_{57} & T_{58} & 0 & 0 & 0 & 0 \\
T_{61} & T_{62} & T_{63} & T_{64} & T_{65} & T_{66} & T_{67} & T_{68} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & T_{75} & T_{76} & T_{77} & T_{78} & T_{79} & T_{7,10} & T_{7,11} & T_{7,12} \\
0 & 0 & 0 & 0 & T_{85} & T_{86} & T_{87} & T_{88} & T_{89} & T_{8,10} & T_{8,11} & T_{8,12} \\
0 & 0 & 0 & 0 & T_{95} & T_{96} & T_{97} & T_{98} & T_{99} & T_{9,10} & T_{9,11} & T_{9,12} \\
0 & 0 & 0 & 0 & T_{10,5} & T_{10,6} & T_{10,7} & T_{10,8} & T_{10,9} & T_{10,10} & T_{10,11} & T_{10,12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{11,9} & T_{11,10} & T_{11,11} & T_{11,12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T_{12,9} & T_{12,10} & T_{10,11} & T_{12,12}
\end{vmatrix}=0 \tag{4.2}$$

其中

$$\left.\begin{aligned}
&T_{11}=\left(1-\frac{p_1}{G_1}\right)\left(\frac{m\pi}{l}\right)^2 I_0(m_{11}r_0)-\frac{m_{11}}{r_0}I_1(m_{11}r_0)\\
&T_{12}=\left(1-\frac{p_1}{G_1}\right)\left(\frac{m\pi}{l}\right)^2 K_0(m_{11}r_0)+\frac{m_{11}}{r_0}K_1(m_{11}r_0)\\
&T_{13}=\left(1-\frac{p_1}{2G_1}\right)\left(\frac{m\pi}{l}\right)^2 I_0(m_{21}r_0)-\frac{m_{21}}{r_0}I_1(m_{21}r_0)\\
&T_{14}=\left(1-\frac{p_1}{2G_1}\right)\left(\frac{m\pi}{l}\right)^2 K_0(m_{21}r_0)+\frac{m_{21}}{r_0}K_1(m_{21}r_0)\\
&T_{21}=\left(1-\frac{p_1}{2G_1}\right)m_{11}I_1(m_{11}r_0),\quad T_{22}=-\left(1-\frac{p_1}{2G_1}\right)m_{11}K_1(m_{11}r_0)\\
&T_{23}=m_{21}I_1(m_{21}r_0),\quad T_{24}=-m_{21}K_1(m_{21}r_0)\\
&T_{31}=-\frac{m_{11}}{r_1}I_1(m_{11}r_1)+\left(1-\frac{p_1}{G_1}\right)\left(\frac{m\pi}{l}\right)^2 I_0(m_{11}r_1)\\
&T_{32}=\frac{m_{11}}{r_1}K_1(m_{11}r_1)+\left(1-\frac{p_1}{G_1}\right)\left(\frac{m\pi}{l}\right)^2 K_0(m_{11}r_1)\\
&T_{33}=-\frac{m_{21}}{r_1}I_1(m_{21}r_1)+\left(1-\frac{p_1}{2G_1}\right)\left(\frac{m\pi}{l}\right)^2 I_0(m_{21}r_1)\\
&T_{34}=\frac{m_{21}}{r_1}K_1(m_{21}r_1)+\left(1-\frac{p_1}{2G_1}\right)\left(\frac{m\pi}{l}\right)^2 K_0(m_{21}r_1)\\
&T_{35}=\frac{m_{12}}{r_1}I_1(m_{12}r_1)-\left(1-\frac{p_2}{G_2}\right)\left(\frac{m\pi}{l}\right)^2 I_0(m_{12}r_1)\\
&T_{36}=-\frac{m_{12}}{r_1}K_1(m_{12}r_1)-\left(1-\frac{p_2}{G_2}\right)\left(\frac{m\pi}{l}\right)^2 K_0(m_{12}r_1)\\
&T_{37}=\frac{m_{22}}{r_1}I_1(m_{22}r_1)-\left(1-\frac{p_2}{2G_2}\right)\left(\frac{m\pi}{l}\right)^2 I_0(m_{22}r_1)\\
&T_{38}=-\frac{m_{22}}{r_1}K_1(m_{22}r_1)-\left(1-\frac{p_2}{2G_2}\right)\left(\frac{m\pi}{l}\right)^2 K_0(m_{22}r_1)\\
&T_{41}=\left(1-\frac{p_1}{2G_1}\right)m_{11}I_1(m_{11}r_1),\quad T_{42}=-\left(1-\frac{p_1}{2G_1}\right)m_{11}K_1(m_{11}r_1)\\
&T_{43}=m_{21}I_1(m_{21}r_1),\quad T_{44}=-m_{21}K_1(m_{21}r_1)\\
&T_{45}=-\left(1-\frac{p_2}{2G_2}\right)m_{12}I_1(m_{12}r_1),\quad T_{46}=\left(1-\frac{p_2}{2G_2}\right)m_{12}K_1(m_{12}r_1)\\
&T_{47}=-m_{22}I_1(m_{22}r_1),\quad T_{48}=m_{22}K_1(m_{22}r_1)\\
&T_{51}=m_{11}I_1(m_{11}r_1),\quad T_{52}=-m_{11}K_1(m_{11}r_1)\\
&T_{53}=m_{21}I_1(m_{21}r_1),\qquad T_{54}=-m_{21}K_1(m_{21}r_1)\\
&T_{55}=-m_{12}I_1(m_{12}r_1),\qquad T_{56}=m_{12}K_1(m_{12}r_1)\\
&T_{57}=-m_{22}I_1(m_{22}r_1),\qquad T_{58}=m_{22}K_1(m_{22}r_1)\\
&T_{61}=\left(1-\frac{p_1}{G_1}\right)I_0(m_{11}r_1),\qquad T_{62}=\left(1-\frac{p_1}{G_1}\right)K_0(m_{11}r_1)\\
&T_{63}=I_0(m_{21}r_1),\quad T_{64}=K_0(m_{21}r_1),\quad T_{65}=-\left(1-\frac{p_2}{G_2}\right)I_0(m_{12}r_1)\\
&T_{66}=-\left(1-\frac{p_2}{G_2}\right)K_0(m_{12}r_1),\quad T_{67}=-I_0(m_{22}r_1),\quad T_{68}=-K_0(m_{22}r_1)\\
&T_{75}=-\frac{m_{12}}{r_2}I_1(m_{12}r_2)+\left(1-\frac{p_2}{G_2}\right)\left(\frac{m\pi}{l}\right)^2 I_0(m_{12}r_2)
\end{aligned}\right\}\qquad(4.3)$$

$$T_{76}=\frac{m_{12}}{r_2}K_1(m_{12}r_2)+\left(1-\frac{p_2}{G_2}\right)\left(\frac{m\pi}{l}\right)^2K_0(m_{12}r_2)$$

$$T_{77}=-\frac{m_{22}}{r_2}I_1(m_{22}r_2)+\left(1-\frac{p_2}{2G_2}\right)\left(\frac{m\pi}{l}\right)^2I_0(m_{22}r_2)$$

$$T_{78}=\frac{m_{22}}{r_2}K_1(m_{22}r_2)+\left(1-\frac{p_2}{2G_2}\right)\left(\frac{m\pi}{l}\right)^2K_0(m_{22}r_2)$$

$$T_{79}=\frac{m_{13}}{r_2}I_1(m_{13}r_2)-\left(1-\frac{p_3}{G_3}\right)\left(\frac{m\pi}{l}\right)^2I_0(m_{13}r_2)$$

$$T_{7,10}=-\frac{m_{13}}{r_2}K_1(m_{13}r_2)-\left(1-\frac{p_3}{G_3}\right)\left(\frac{m\pi}{l}\right)^2K_0(m_{13}r_2)$$

$$T_{7,11}=\frac{m_{23}}{r_2}I_1(m_{23}r_2)-\left(1-\frac{p_3}{2G_3}\right)\left(\frac{m\pi}{l}\right)^2I_0(m_{23}r_2)$$

$$T_{7,12}=-\frac{m_{23}}{r_2}K_1(m_{23}r_2)-\left(1-\frac{p_3}{2G_3}\right)\left(\frac{m\pi}{l}\right)^2K_0(m_{23}r_2)$$

$$T_{85}=\left(1-\frac{p_2}{2G_2}\right)m_{12}I_1(m_{12}r_2),\quad T_{86}=-\left(1-\frac{p_2}{2G_2}\right)m_{12}K_1(m_{22}r_2)$$

$$T_{87}=m_{22}I_1(m_{22}r_2),\quad T_{88}=-m_{22}K_1(m_{22}r_2)$$

$$T_{89}=-\left(1-\frac{p_3}{2G_3}\right)m_{13}I_1(m_{13}r_2),\quad T_{8,10}=\left(1-\frac{p_3}{2G_3}\right)m_{13}K_1(m_{13}r_2)$$

$$T_{8,11}=-m_{23}I_1(m_{23}r_2),\quad T_{8,12}=m_{23}K_1(m_{23}r_2)$$

$$T_{95}=m_{12}I_1(m_{12}r_2),\quad T_{96}=-m_{12}K_1(m_{12}r_2),\quad T_{97}=m_{22}I_1(m_{22}r_2)$$

$$T_{98}=-m_{22}K_1(m_{22}r_2),\quad T_{99}=-m_{13}I_1(m_{13}r_2),\quad T_{9,10}=m_{13}K_1(m_{13}r_2)$$

$$T_{9,11}=-m_{23}I_1(m_{23}r_2),\quad T_{9,12}=m_{23}K_1(m_{23}r_2)$$

$$T_{10,5}=\left(1-\frac{p_2}{G_2}\right)I_0(m_{12}r_2),\quad T_{10,6}=\left(1-\frac{p_2}{G_2}\right)K_0(m_{12}r_2)$$

$$T_{10,7}=I_0(m_{22}r_2),\quad T_{10,8}=K_0(m_{22}r_2),\quad T_{10,9}=-\left(1-\frac{p_3}{G_3}\right)I_0(m_{13}r_2)$$

$$T_{10,10}=-\left(1-\frac{p_3}{G_3}\right)K_0(m_{13}r_2),\quad T_{10,11}=-I_0(m_{23}r_2),$$

$$T_{10,12}=-K_0(m_{23}r_2)$$

$$T_{11,9}=\left(1-\frac{p_3}{G_3}\right)\left(\frac{m\pi}{l}\right)^2I_0(m_{13}r_3)-\frac{m_{13}}{r_3}I_1(m_{13}r_3)$$

$$T_{11,10}=\left(1-\frac{p_3}{G_3}\right)\left(\frac{m\pi}{l}\right)^2K_0(m_{13}r_3)+\frac{m_{13}}{r_3}K_1(m_{13}r_3)$$

$$T_{11,11}=\left(1-\frac{p_3}{2G_3}\right)\left(\frac{m\pi}{l}\right)^2I_0(m_{23}r_3)-\frac{m_{23}}{r_3}I_1(m_{23}r_3)$$

$$T_{11,12}=\left(1-\frac{p_3}{2G_3}\right)\left(\frac{m\pi}{l}\right)^2K_0(m_{23}r_3)+\frac{m_{23}}{r_3}K_1(m_{23}r_3)$$

$$T_{12,9}=\left(1-\frac{p_3}{2G_3}\right)m_{13}I_1(m_{23}r_3),\quad T_{12,10}=-\left(1-\frac{p_3}{2G_3}\right)m_{13}K_1(m_{13}r_3)$$

$$T_{12,11}=m_{23}I_1(m_{23}r_3),\quad T_{12,12}=-m_{23}K_1(m_{23}r_2)$$

其余的 $T_{ij}(i, j=1, 2, \cdots, 12)$ 均为零.

五、算例和讨论

三层圆柱壳的各种物理参数和几何参数列于表 1，按 (4.2)式算得的数学弹性力学解和按文献[4]夹层壳理论((3.1)式)算得的结果列于表 2。

表 1

No.	E_1	E_2	E_3	ν	r_0	r_1	r_2	r_3	l
	kg/cm²				cm				
1	7.0×10^5	1.3×10^2	7.0×10^5	0.3	28.9	29.0	30.0	30.1	100
2	1.0×10^6	5.2×10^2	7.0×10^5	0.3	19.9	20.0	21.0	21.2	60
3	7.0×10^5	5.2×10^2	7.0×10^5	0.3	8.4	8.5	11.5	11.6	40
4	1.0×10^6	5.2×10^2	7.0×10^5	0.3	14.8	15.0	15.1	15.4	60

表 2

No.	夹层壳理论解		数学弹性力学解		相对误差 %
	临界载荷(kg/cm)	屈曲半波数	临界载荷(kg/cm)	屈曲半波数	
1	3.480×10^2	34	3.459×10^2	34	0.6
2	1.4344×10^3	19	1.4295×10^3	19	0.3
3	1.4969×10^3	23	1.2935×10^3	23	15.7
4	4.4587×10^3	18	3.6027×10^3	18	23.7

注：这里，临界载荷 $p=\sum_{i=1}^{3}p_i(r_i-r_{i-1})$；其中，$\frac{p_1}{E_1}=\frac{p_2}{E_2}=\frac{p_3}{E_3}$。计算时可以夹层壳理论解为基础选择 p 的试算值，逐次迭代修正，直至(4.2)式得到满足或近似满足。

由表 2 可以看到，当三层壳的两表板较薄并且其几何尺寸符合薄壳情况时，用夹层壳理论算得的结果与数学弹性力学解的差别不大(见No.1,2)；当三层壳的几何尺寸为厚壳时，差别较大(见No.3)；当三层壳的表板(相对于夹芯)较厚时，其结果相差也较大(见No.4)。

通过算例可以看出，本文求解多层圆柱壳临界载荷的方法不仅理论上严格，而且实际应用起来也不困难。多层壳层数的增加相应于求解临界载荷的行列式阶数的增加，这对于计算程序的形成没有带来很大困难。本文将多层壳视为弹性组合体，对于壳体的几何参数和物理参数没有限制，只要属于弹性失稳的问题，本文的方法总是适用的。

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A SOLUTION OF MATHEMATICAL THEORY OF ELASTICITY FOR STABILITY OF CYLINDRICAL MULTI-PLY SHELLS UNDER AXIAL LOAD

Dai Fuling and Wang Zhenming

(Institute of Mechanics, Academia Sinica)

Abstract

In this paper, Novozhilov's equilibrium equations and boundary conditions of stability problem in rectangular Cartesian coordinates were translated into tensor equations in general coordinates, simply supported cylindrical multi-ply shells under axial load were analyzed by mathematical theory of elasticity, the transcendental equations for determining critical loads were derived and the numerical results were compared with those of sandwich theory.