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理想塑性材料扩展裂纹顶端塑性区内应力场

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提要 扩展裂纹顶端弹塑性应力应变场分析是近年来断裂力学发展的重要课题．本文对弹性不可压缩的理想塑性材料中定常扩展裂纹顶端塑性区内的应力应变场进行了系统的数学分析．讨论了速度场和应力场的高阶渐近方程和解的形式，证实高阶渐近场依赖于塑性区和卸载弹性区交界线 Γ_B 的曲线方程．一旦 Γ_B 的曲线方程确定下来，高阶渐近场的解答就能确定下来．

关键词 扩展裂纹；理想塑性；高阶渐近方程；应力场．

一、引　言

扩展裂纹顶端的弹塑性应力应变场分析是建立弹塑性断裂理论的基础．近十年来，该课题的研究引起了很多学者的兴趣[1-4]，这些工作都是寻求裂纹顶端的弹塑性应力应变场的渐近解．1974年，Слепян 采用屈斯卡准则[1]，首先得到了关于弹性不可压缩的理想塑性材料扩展裂纹顶端应力应变场的渐近解．

1980年，Rice 等人[2]和高玉臣[3]采用Mises屈服准则得到了类似的结果．

但是渐近解通常只适用于非常接近裂纹顶端区域，为了确定精确的应力应变场的全场解，需要求得适用于裂纹顶端塑性区内，应力应变场的更详细的解．

本文对弹性不可压缩的理想塑性材料中定常扩展裂纹顶端区域内的应力场和速度场进行了系统分析，建立了应力场和速度场的高阶渐近方程，讨论了解的具体形式，证实了高阶渐近场依赖于塑性区和卸载弹性区交界线 Γ_B 的曲线方程．

二、扩展裂纹顶端塑性区内的应力场

考察弹性不可压缩的理想塑性材料中的纯Ⅰ型的定常扩展裂纹，端部区域的应力场渐近解已由〔1〕给出．如图1所示，区域 A 是均匀应力区，B 是中心扇形区，C 是卸载后弹性区，D 是二次塑性区．渐近解的应力函数 ϕ_0 为：

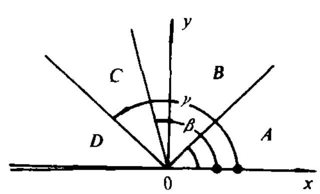

图　1

$$\phi_0 = r^2 F_0(\theta) \tag{2.1}$$

$$\begin{cases} \dfrac{k}{2}\left[b+\cos 2\theta\right], & A\text{区} \\ \dfrac{k}{2}\left[b-2\left(\theta-\dfrac{\pi}{4}\right)\right], & B\text{区} \end{cases} \tag{2.2}$$

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$$F_0(\theta)=\begin{cases}\frac{1}{4}C_2\theta+\frac{C_1}{4}F_s(\theta)+C_3+C_4(1-\cos 2(\theta-\gamma))+C_5\sin 2(\theta-\gamma) & C\text{区}\\ \frac{k}{2}(1-\cos 2\theta) & D\text{区}\end{cases}$$

其中，
$$b=2\left(\beta-\frac{\pi}{4}\right)+1-\frac{1}{2k}\left[C_2(\gamma-\beta)+C_1\ln\frac{\sin\gamma}{\sin\beta}\right]$$

$$F_s(\theta)=(1-\cos 2\theta)\ln\sin\theta-\theta\sin 2\theta-\cos^2\theta \tag{2.3}$$

$$C_1=-4k\sin\beta/\sqrt{2}$$

$$C_2=-4k(1-\cos\beta/\sqrt{2})$$

$$C_3=\frac{k}{2}(1-\cos 2\gamma)-\left(\frac{C_1}{4}F_s(\gamma)+\frac{C_2}{4}\gamma\right)$$

$$C_5=\frac{k}{2}\sin 2\gamma-\left(\frac{C_1}{4}F_s'(\gamma)+\frac{C_2}{4}\right)\frac{1}{2}$$

$$C_4=\frac{k}{2}\cos 2\gamma-\frac{C_1}{16}F_s''(\gamma)$$

$$\beta=112.1^\circ,\quad \gamma=162.1^\circ$$

现在来讨论下述定理．

定理 如果在裂纹顶端的塑性区和卸载弹性区内，应力函数ϕ可展成下列级数：

$$\phi=\sum_{n=0,1}^{\infty}\phi_n(\gamma,\theta)=\sum_{n=0,1}^{\infty}r^{\lambda n+2}\psi_n(t,\theta) \tag{2.4}$$

$$\psi_n(t,\theta)=\sum_{m=0,1}^{\infty}t^{-m}F_{nm}(\theta) \tag{2.5}$$

其中

$$\lambda_0=0,\qquad 0<\lambda_1<\lambda_2\cdots$$

$$t=\ln(R/r),\qquad F_{00}(\theta)=F_0(\theta)$$

R 是表征塑性区尺寸的参量．

则有：
$$\psi_0(t,\theta)=F_0(\theta) \tag{2.6}$$

先证明（2.6）式，对ϕ_0有如下屈服条件：

$$S_{r0}^2+\tau_{r\theta 0}^2=k^2 \tag{2.7}$$

其中S_{r0}，$\tau_{\theta 0}$是应力函数ϕ_0所产生的应力偏量．

$$\left.\begin{aligned}S_{r0}&=(\sigma_{r0}-\sigma_{\theta 0})/2=\left(\psi_0''+2\frac{\partial\psi_0}{\partial t}-\frac{\partial^2\psi_0}{\partial t^2}\right)\cdot\frac{1}{2}\\ \tau_{r\theta 0}&=-\left(\psi_0'-\frac{\partial\psi_0'}{\partial t}\right)\end{aligned}\right\} \tag{2.8}$$

记

$$S_{r0}=\sum_{m=0,1}^{\infty}S_{r0m}t^{-m},\qquad \tau_{r\theta 0}=\sum_{m=0,1}^{\infty}\tau_{r\theta 0m}t^{-m} \tag{2.9}$$

则有：

$$\left.\begin{array}{l} S_{r0m}=\{F_{0m}''-2(m-1)F_{0m-1}-(m-2)(m-1)F_{0m-2}\}/2 \\ \tau_{r\theta 0m}=-\{F_{0m}'+(m-1)F_{0m-1}'\},\quad m=0,1, \end{array}\right\} \tag{2.10}$$

这里约定

$$F_{0K}(\theta)=0,\qquad K=-1,-2,$$

将（2.9）代入（2.8），比较不同的t幂次系数，得：

$$S_{r00}^2+\tau_{r\theta 00}^2=k^2 \tag{2.11}$$

$$\sum_{\substack{i=0,1\\ j=m-i}}^{m}(S_{r0i}S_{r0j}+\tau_{r\theta 0i}\tau_{r\theta 0j})=0,\quad m\geqslant 1 \tag{2.12}$$

方程（2.11）已满足．现考虑$m=1$的情况．

$$S_{r01}=F_{01}''/2,\quad \tau_{r\theta 01}=-F_{01}'$$

控制方程为：

$$S_{r00}S_{r01}+\tau_{r\theta 00}\tau_{r\theta 01}=0$$

经分析得，

$$F_{01}=\begin{cases} a_{01} & \text{A 区} \\ b_{01}, & B\text{ 区} \\ 0, & D\text{ 区} \end{cases} \tag{2.13}$$

因此，$S_{r01}(\theta)=\tau_{r\theta 01}(\theta)=0$，在$A$，$B$，$D$区．另外由屈服条件（2.12）推得：

$$F_{02}=b_{02},\qquad \text{在}B\text{ 区}$$

渐近解（2.1）－（2.3）所对应的速度分量v_s在Γ_A上是有间断的．因此，Γ_A是间断线．对理想塑性材料，速度场的间断线也就是特征线．所以，在Γ_A上必有：

$$S_n=-S_r\cos 2\epsilon+\tau_{r\theta}\sin 2\epsilon=0 \tag{2.14}$$

方程（2.14）也可改写为[1)]：

$$S_r(1-\mathrm{tg}^2\epsilon)=2\tau_{r\theta}\,\mathrm{tg}\,\epsilon \tag{2.15}$$

设想Γ_A的曲线方程为，

$$\frac{\pi}{4}-\theta=\sum_{n=0,1}^{\infty}p_n(t)r^{\lambda_n} \tag{2.16}$$

$$p_n(t)=\sum_{m=0,1}^{\infty}p_{nm}t^{-m} \tag{2.17}$$

其中

$$t=\ln(R/r),\quad p_{00}=0,$$

1）ϵ是图2中，径向射线与Γ_A切线之间的夹角．

由几何学分析得知，沿曲线 Γ_A有:

$$\mathrm{tg}\,\vartheta=\mathrm{tg}(\theta-\epsilon)=\frac{dy}{dx}=$$

$$=\frac{(dr/d\theta)\sin\theta+r\cos\theta}{(dr/d\theta)\cos\theta-r\sin\theta}$$

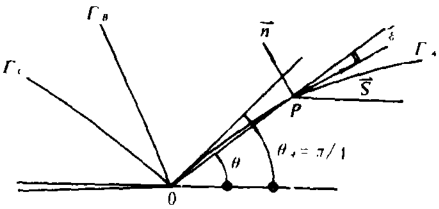

图 2

因此，

$$\mathrm{tg}\,\epsilon=\frac{d\theta}{dt}=-\xi^2\frac{d\theta}{d\xi} \tag{2.18}$$

这里 $\xi=1/t$，将 (2.18) 代入 (2.15)，并利用 (2.16)、(2.17) 式，得:

$$\left\{1-\left[\sum_{m=1}^{\infty}(m-1)p_{0m-1}\xi^m+r^{\lambda_1}\sum_{m=0,1}^{\infty}p_{1m}^{*}\xi^m+O(r^{\lambda_2})\right]^2\right\}$$

$$\times\left\{\sum_{m=0,1}^{\infty}S_{r0m}\xi^m+r^{\lambda_1}\sum_{m=0,1}^{\infty}S_{r1m}\xi^m+O(r^{\lambda_2})\right\}$$

$$=2\left\{\sum_{m=0,1}^{\infty}\tau_{r\theta 0m}\xi^m+r^{\lambda_1}\sum_{m=0,1}^{\infty}\tau_{r\theta 1m}\xi^m+O(r^{\lambda_2})\right\}$$

$$\times\left\{\sum_{m=1}^{\infty}(m-1)p_{0m-1}\xi^m+r^{\lambda_1}\sum_{m=0,1}^{\infty}p_{1m}^{*}\xi^m+O(r^{\lambda_2})\right\} \tag{2.19}$$

由此推得:

$$\left\{1-\left(\sum_{m=1}^{\infty}(m-1)p_{0m-1}\xi^m\right)^2\right\}\cdot\left(\sum_{m=0,1}^{\infty}S_{r0m}\xi^m\right)$$

$$=2\left(\sum_{m=0,1}^{\infty}\tau_{r\theta 0m}\xi^m\right)\left(\sum_{m=1,2}^{\infty}(m-1)p_{0m-1}\xi^m\right)+O(r^{\lambda_1}) \tag{2.20}$$

比较两边 ξ 相同阶幂次的系数，立即推得:

$$\left.\begin{array}{l}S_{r00}(\theta_A)=0\\ S_{r01}(\theta_A)-p_{01}S'_{r00}(\theta_A)=0\end{array}\right\} \tag{2.21}$$

上述公式必须在 Γ_A的两侧均成立、由(2.21)式及应力场的连续性，立即推得:

$$p_{01}=0, \tag{2.22}$$

进一步利用公式 (2.20)，将S_{r0m}，$\tau_{r\theta 0m}$展开成$(\theta-\theta_A)$的台劳级数，并利用公式 (2.16)，得:

$$S_{r02}(\theta_A)-p_{02}S'_{r00}(\theta_A)=0 \tag{2.23}$$

由此推出，

$$S_{r02}(\theta_A^{+})=0 \tag{2.24}$$

利用公式 (2.10)，不难证实，

$$\left.\begin{array}{l}S_{r02}(\theta_A^{+})=-F_{01}(\theta_A^{+})=0\\ a_{01}=b_{01}=0\end{array}\right\} \tag{2.25}$$

$$F_{01}(\theta)=0,\quad \text{在区域}A\text{和}B\text{中},$$

$$p_{02}=S_{r02}(\theta_A^{-})=0 \tag{2.26}$$

利用数学归纳法，可以得知，

$$p_{0m}=F_{0m}=0,\quad m\geq 1,\text{在区域}A\text{、}B\text{中} \tag{2.27}$$

利用上面已经导出的结果，由（2.19）得：

$$S_{r00}'(\theta_i)(\theta-\theta_i)/r^{\lambda_1}+\sum_{m=0,1}^{\infty}S_{r1m}(\theta_i)\xi^m$$

$$=2\tau_{r\theta00}(\theta_i)\left(\sum_{m=0,1}^{\infty}p_{1m}^{*}\xi^m\right)+\cdots \tag{2.28}$$

这里

$$p_{10}^{*}=\lambda_1 p_{10}$$

$$p_{1m}^{*}=\lambda_1 p_{1m}+(m-1)p_{1m-1}\quad m\geq 1$$

由式(2.28)中右端的余项是$o(r^{\lambda_1})$或$o(r^{\lambda_2-\lambda_1})$量级，由（2.28）式，得：

$$2k\,p_{10}^{*}+S_{r00}'(\theta_i)p_{10}=S_{r10}(\theta_i) \tag{2.29}$$

鉴于（2.27）的结果，由屈服条件不难推得：

$$\left.\begin{array}{l}S_{r0}S_{r1}+\tau_{r\theta0}\tau_{r\theta1}=0\\ S_{y0}S_{y1}+\tau_{xy0}\tau_{xy1}=0\end{array}\right\} \tag{2.30}$$

这里S_{r1}，$\tau_{r\theta1}$是应力函数ϕ_1所产生的应力偏量.

$$\left.\begin{array}{l}S_{r1}=r^{\lambda_1}\left\{[\psi_1''-\lambda_1(\lambda_1+2)\psi_1]+2(\lambda_1+1)\dfrac{\partial\psi_1}{\partial t}-\dfrac{\partial^2\psi_1}{\partial t^2}\right\}\cdot\dfrac{1}{2}\\ \tau_{r\theta1}=-r^{\lambda_1}\left\{(1+\lambda_1)\psi_1'-\dfrac{\partial^2\psi_1}{\partial t\partial\theta}\right\}\end{array}\right\} \tag{2.31}$$

记

$$\left.\begin{array}{l}S_{r1}=r^{\lambda_1}\sum\limits_{m=0,1}^{\infty}S_{r1m}t^{-m}\\ \tau_{r\theta1}=r^{\lambda_1}\sum\limits_{m=0,1}^{\infty}\tau_{r\theta1m}t^{-m}\end{array}\right\} \tag{2.32}$$

则有：

$$\left.\begin{array}{l}S_{r1m}=\{[F_{1m}''-\lambda_1(\lambda_1+2)F_{1m}]-2(m-1)(\lambda_1+1)F_{1m-1}\\ \qquad -(m-2)(m-1)F_{1m-2}\}\cdot\dfrac{1}{2},\\ \tau_{r\theta1m}=-\{(1+\lambda_1)F_{1m}'+(m-1)F_{1m-1}'\},\quad m=0,1,\cdots\end{array}\right\} \tag{2.33}$$

这里约定，

$$F_{1K}(\theta)=0,\qquad K=-1,-2$$

代入（2.30），在A、B两区得：

$$S_{r0}S_{r1m}+\tau_{r\theta0}\tau_{r\theta1m}=0,\quad m\geq 0, \tag{2.34}$$

由此推得（当 $m=0,1$ 时）：

$$F_{1m}=\begin{cases} a_{1m}\{(\cos\theta+\sin\theta)^{2+\lambda_1}+(\cos\theta-\sin\theta)^{2+\lambda_1}\}, & A\text{ 区} \\ b_{1m}, & B\text{ 区} \\ 0 & D\text{ 区} \end{cases} \tag{2.35}$$

进一步分析可得：

$$b_{10}=(\sqrt{2})^{2+\lambda_1}a_{10},\quad p_{10}=-\frac{(\lambda_1+2)}{4k}b_{10} \tag{2.36}$$

$$b_{11}=(\sqrt{2})^{2+\lambda_1}a_{11},\quad p_{11}=-\frac{(\lambda_1+2)}{4k}b_{11} \tag{2.37}$$

现在来证明，在 C 区也有

$$F_{0m}=0, \qquad m\geqslant 1$$

为此讨论速度场，我们有

$$\left.\begin{aligned} &\dot{\phi}=\dot{a}\sum_{n=0,1}^{\infty} r^{1+\lambda_n}\overline{\Psi}_n(t,\theta), \\ &\overline{\Psi}_n=\psi_n'\sin\theta-(2+\lambda_n)\psi_n\cos\theta+\frac{\partial\psi_n}{\partial t}\cos\theta \end{aligned}\right\} \tag{2.38}$$

$$\left.\begin{aligned} &\overline{\Psi}_0(t,\theta)=\sum_{m=0,1}^{\infty} t^{-m} f_{0m}(\theta) \\ &f_{0m}=F_{0m}'\sin\theta-2F_{0m}\cos\theta-(m-1)F_{0m-1}\cos\theta \end{aligned}\right\} \tag{2.39}$$

$$\left.\begin{aligned} &\dot{S}_{r0}=\frac{\dot{a}}{r}\sum_{m=0,1}^{\infty} t^{-m}\overline{h}_{0m}(\theta) \\ &\dot{\tau}_{r\theta 0}=-\frac{\dot{a}}{r}\sum_{m=0,1}^{\infty} t^{-m}(m-1)f_{0m-1}' \end{aligned}\right\} \tag{2.40}$$

$$\overline{h}_{0m}=(f_{0m}''+f_{0m}-(m-2)(m-1)f_{0m-2})^{1/2} \tag{2.41}$$

这里 $\dot{S}_{r0}$，$\dot{\tau}_{r\theta 0}$ 是应力率在极坐标中的分量．由此得到下列速度场：

$$\left.\begin{aligned} &v_r=\frac{\dot{a}}{2\mu}\sum_{n=0,1}^{\infty} r^{\lambda_n}v_{rn}, \\ &v_\theta=\frac{\dot{a}}{2\mu}\sum_{n=0,1}^{\infty} r^{\lambda_n}v_{\theta n} \end{aligned}\right\} \tag{2.42}$$

$$\left.\begin{aligned} &v_{r0}=\{-t h_{00}-\ln t\cdot h_{01}+\sum h_{0m+1}t^{-m}/m\}-f'(\theta), \\ &v_{\theta 0}=\{t\cdot H_{00}-H_{00}+\ln t\, H_{01}-\sum_{m=1,2}^{\infty} t^{-m}(H_{0m}+\frac{1}{m}H_{0m+1})\}+f(\theta), \end{aligned}\right\} \tag{2.43}$$

$$H_{0m}=\int h_{0m}\,d\theta, \qquad h_{0m}=\overline{h}_{0m}, \tag{2.44}$$

h_{0m} 满足下述协调方程：

$$H_{00}+h'_{00}=-2k_{0\bar{1}},\tag{2.45}$$

$$H_{01}+h'_{01}=0,\tag{2.46}$$

$$f''+f=-2k_{00},\tag{2.47}$$

$$h'_{0m+1}+H_{0m+1}=m(m-1)H_{0m-1}+2mk_{0m},\quad m\geqslant 1,\tag{2.48}$$

这里：

$$k_{0\bar{1}}=0,$$

$$k_{0m}=-(m-1)f'_{0m-1},\tag{2.49}$$

由此求得：

$$\left.\begin{aligned}&h_{00}=C_{01}\cos\theta+C_{02}\sin\theta\\&F_{00}=\frac{C_{01}}{4}F_{\iota}(\theta)+\frac{C_{02}}{4}(\theta+\frac{1}{2}\sin2\theta)+C_{03}\\&\quad+C_{04}[1-\cos2(\theta-\gamma)]+C_{05}\sin2(\theta-\gamma)\end{aligned}\right\}\tag{2.50}$$

$$\left.\begin{aligned}&h_{01}=C_{11}\cos\theta+C_{12}\sin\theta\\&F_{01}=\frac{C_{11}}{4}F_{\iota}(\theta)+\frac{C_{12}}{4}(\theta+\frac{1}{2}\sin2\theta)-\frac{C_{13}}{2}\\&\quad-\frac{C_{14}}{2}\sin2\theta+C_{15}\sin2\theta\ ,\end{aligned}\right\}\tag{2.51}$$

由应力分量及速度分量的连续条件得：

$$\left.\begin{aligned}&[F_{01}(\beta)]_{\Gamma_B}=[F'_{01}(\beta)]_{\Gamma_B}=[h_{01}(\beta)]_{\Gamma_B}=0\\&[F_{01}(\gamma)]_{\Gamma_C}=[F'_{01}(\gamma)]_{\Gamma_C}=0\end{aligned}\right\}\tag{2.52}$$

为了使（2.59）得到满足必有：

$$C_{1i}=0\qquad i=1,2\cdots5$$

$$F_{01}(\theta)=0\qquad \text{在}C\text{区}$$

类似的可证明：

$$F_{0m}(\theta)=0\qquad \text{在}C\text{区}$$

（2.6）式证毕．

三、关于A区的应力场的一般表达式

业已证明 $F_{0m}(\theta)=0, m\geqslant 1$，现在转向直角坐标，我们有：

$$\left.\begin{aligned}&S_y=\sum_{n=0,1}^{\infty}S_{yn}r^{\lambda_n}\\&\tau_{xy}=\sum_{n=0,1}^{\infty}\tau_{xyn}r^{\lambda_n}\end{aligned}\right\}\tag{3.1}$$

将上式代入屈服条件，得：

$$S_{y0}^2+\tau_{xy0}^2=k^2\tag{3.2}$$

$$S_{y0}S_{y1}+\tau_{xy0}\tau_{xy_1}=0\tag{3.3}$$

在A区有　$\tau_{xy0}=0,\quad S_{y0}=k$

由此导出
$$S_{\nu_1}=0$$
另一方面，
$$r^{\lambda_1}S_{\nu_1}=\frac{\partial^2\phi_1}{\partial x^2}-\frac{\partial^2\phi_1}{\partial y^2}=0 \tag{3.4}$$
注意到解的对称性，我们有：
$$\phi_1=(x+y)^{2+\lambda_1}\sum_{m=0,1}^{\infty}\frac{\bar{a}_{1m}}{[\ln(R/(x+y))]^m}+(x-y)^{2+\lambda_1}\sum_{m=0,1}^{\infty}\frac{\bar{a}_{1m}}{[\ln(R/(x-y))]^m}. \tag{3.5}$$
$$\phi_1=r^{2+\lambda_1}\left\{(\cos\theta+\sin\theta)^{2+\lambda_1}\sum_{m=0,1}^{\infty}\frac{\bar{a}_{1m}\xi^m}{[1-\xi\ln(\cos\theta+\sin\theta)]^m}+(\cos\theta-\sin\theta)^{2+\lambda_1}\sum_{m=0,1}^{\infty}\frac{\bar{a}_{1m}\xi^m}{[1-\xi\ln(\cos\theta-\sin\theta)]^m}\right\} \tag{3.6}$$
由（3.6）式不难推得：
$$F_{10}=a_{10}\{(\cos\theta+\sin\theta)^{2+\lambda_1}+(\cos\theta-\sin\theta)^{2+\lambda_1}\}, \tag{3.7}$$
$$F_{11}=a_{11}\{(\cos\theta+\sin\theta)^{2+\lambda_1}+(\cos\theta-\sin\theta)^{2+\lambda_1}\}, \tag{3.8}$$
$$\begin{aligned}F_{12}=&a_{12}\{(\cos\theta+\sin\theta)^{2+\lambda_1}+(\cos\theta-\sin\theta)^{2+\lambda_1}\}\\&+a_{11}\{(\cos\theta+\sin\theta)^{2+\lambda_1}\ln(\cos\theta+\sin\theta)\\&+(\cos\theta-\sin\theta)^{2+\lambda_1}\ln(\cos\theta-\sin\theta)\}\end{aligned} \tag{3.9}$$
……

另一方面在B区，由（2.33）和（2.34）式得：
$$F_{1m}=b_{1m} \tag{3.10}$$

四、高阶渐近解

讨论速度场，已有公式(2.42)－(2.51).从公式(2.40)－(2.42)，推得[2]：
$$\left.\begin{aligned}&h_{00}=-2k\sin\theta,\ H_{00}=2k\left(\cos\theta-\frac{1}{\sqrt{2}}\right)\\&v_{r0}=(2k\sin\theta)t-f_B'(\theta),\qquad 在B区\\&v_{\theta0}=2k\left(\cos\theta-\frac{1}{\sqrt{2}}\right)(t-1)+f_B(\theta)\\&A_{01}=\frac{1}{\sqrt{2}},\end{aligned}\right\} \tag{4.1}$$
在C区，
$$\begin{aligned}&h_{00}=-2k\cdot\left[\frac{\sin\beta}{\sqrt{2}}\cos\theta+\left(1-\frac{\cos\beta}{\sqrt{2}}\right)\sin\theta\right]\\&H_{00}=2k\left[\left(1-\frac{1}{\sqrt{2}}\cos\beta\right)\cos\theta-\frac{1}{\sqrt{2}}\sin\beta\sin\theta\right]\end{aligned}$$

$$\left.\begin{aligned} v_{r0} &= 2k\left[\frac{\sin\beta}{\sqrt{2}}\cos\theta+\left(1-\frac{\cos\beta}{\sqrt{2}}\right)\sin\theta\right]t - f_C'(\theta) \\ v_{\theta 0} &= 2k\left[\left(1-\frac{1}{\sqrt{2}}\cos\beta\right)\cos\theta-\frac{\sin\beta}{\sqrt{2}}\sin\theta\right](t-1)+f_C(\theta) \end{aligned}\right\} \tag{4.2}$$

进一步讨论应力函数 ϕ_1 所产生的应力率场.

$$\left.\begin{aligned} \dot{\phi}_1 &= \dot{a}r^{1+\lambda_1}\Psi_1(t,\theta)=\dot{a}r^{1+\lambda_1}\sum_{m=0,1}^{\infty} f_{1m}(\theta)t^{-m} \\ f_{1m} &= F_{1m}'\sin\theta-(2+\lambda_1)F_{1m}\cos\theta-(m-1)F_{1m-1}\cos\theta \quad m\geqslant 0 \end{aligned}\right\} \tag{4.3}$$

这里约定　$F_{1K}(\theta)=f_{1K}(\theta)=0$,　$K=-1,-2$

$$\left.\begin{aligned} \dot{S}_{r1} &= \frac{\dot{a}}{r}r^{\lambda_1}\sum_{m=0,1}^{\infty}\overline{h}_{1m}t^{-m} \\ \dot{\tau}_{r\theta 1} &= \frac{\dot{a}}{r}r^{\lambda_1}\sum_{m=0,1}^{\infty}\overline{k}_{1m}t^{-m} \end{aligned}\right\} \tag{4.4}$$

$$\left.\begin{aligned} \overline{h}_{1m} &= \frac{1}{2}\{f_{1m}''+(1-\lambda_1^2)f_{1m}-2\lambda_1(m-1)f_{1m-1} \\ &\quad -(m-2)(m-1)f_{1m-2}\} \\ \overline{k}_{1m} &= -\{\lambda_1 f_{1m}'+(m-1)f_{1m-1}'\},\qquad m\geqslant 0 \end{aligned}\right\} \tag{4.5}$$

又由应力率和应变率关系得：

$$D_r=\frac{\dot{a}}{2\mu}\frac{1}{r}\sum_{n=0,1}^{\infty}r^{\lambda_n}\left(\sum_{m=-1,0}^{\infty}h_{nm}t^{-m}\right) \tag{4.6}$$

另有：

$$h_{1\bar{1}}=S_{r00}\Lambda_{1\bar{1}}+S_{r10}\Lambda_{0\bar{1}} \tag{4.7}$$

$$h_{1m}=\overline{h}_{1m}+\sum_{\substack{i=-1,0\\ j=m-i}}^{m}S_{r1i}\Lambda_{0j}+S_{r00}\Lambda_{1m},\qquad m\geqslant 0 \tag{4.8}$$

$$\left.\begin{aligned} v_{r1} &= \sum_{m=-1,0}^{\infty}v_{r1m}t^{-m} \\ v_{\theta 1} &= \sum_{m=-1,0}^{\infty}v_{\theta 1m}t^{-m} \end{aligned}\right\} \tag{4.9}$$

$$\left.\begin{aligned} &\lambda_1 v_{r1m}+(m-1)v_{r1m-1}=h_{1m} \\ &v_{\theta 1m}=-\{H_{1m}+\int v_{r1m}d\theta\},\quad m\geqslant -1 \\ &H_{1m}=\int h_{1m}d\theta \end{aligned}\right\} \tag{4.10}$$

协调方程为

$$v_{r1m}'+(\lambda_1-1)v_{\theta 1m}+(m-1)v_{\theta 1m-1}=2k_{1m},\qquad m\geqslant -1 \tag{4.11}$$

这里约定 $v_{r1K}=v_{\theta 1K}=0$, $K=-2$.

$$\left.\begin{aligned}
k_{1\bar{1}} &= \tau_{r\theta 00}\,\Lambda_{1\bar{1}}+\tau_{r\theta 10}\,\Lambda_{0\bar{1}} \\
k_{1m} &= -\{\lambda_1 f'_{1m}+(m-1)f'_{1m-1}\}+\tau_{r\theta 00}\,\Lambda_{1m} \\
&\quad+\sum_{\substack{i=-1,0\\ j=m-i}}^{m}\tau_{r\theta 1j}\,\Lambda_{0i},\quad m\geqslant 0
\end{aligned}\right\}\tag{4.12}$$

在C区，$\Lambda\equiv 0$, 由（4.3）—(4.10) 可以推得:

$$\left.\begin{aligned}
F_{10} &= \mathring{C}_1\cos\lambda_1\theta+\mathring{C}_2\sin\lambda_1\theta+\mathring{C}_3\cos(\lambda_1+2)\theta+\mathring{C}_4\sin(\lambda_1+2)\theta \\
&\quad+\mathring{C}_5(\sin\theta)^{\lambda_1+2} \\
f_{10} &= -(1+\lambda_1)\mathring{C}_1\cos(\lambda_1-1)\theta-(1+\lambda_1)\mathring{C}_2\sin(\lambda_1-1)\theta \\
&\quad-[\mathring{C}_2+(2+\lambda_1)\mathring{C}_3]\cos(\lambda_1+1)\theta-[(2+\lambda_1)\mathring{C}_4+\mathring{C}_2]\sin(\lambda_1+1)\theta \\
h_{10} &= 2\lambda_1\{-(1-\lambda_1^2)[\mathring{C}_1\cos(\lambda_1-1)\theta+\mathring{C}_2\sin(\lambda_1-1)\theta]+(1+\lambda_1)[(\mathring{C}_1+ \\
&\quad+(2+\lambda_1)\mathring{C}_3)\cos(\lambda_1+1)\theta+((2+\lambda_1)\mathring{C}_4+\mathring{C}_2)\sin(\lambda_1+1)\theta]\}
\end{aligned}\right\}\tag{4.13}$$

以上公式适用于 $\lambda_1\neq 1$ 的情况，对于 $\lambda_1=1$ 的情况，限于篇幅，不再列出具体公式.

另外，在C区应有:

$$v_{r1\bar{1}}(\theta)=v_{\theta 1\bar{1}}(\theta)=0$$

$$\left.\begin{aligned}
v_{r10}(\theta) &= \frac{1}{\lambda_1}h_{10} \\
v_{\theta 10}(\theta) &= -\left(1+\frac{1}{\lambda_1}\right)\int h_{10}\,d\theta
\end{aligned}\right\}\tag{4.14}$$

$$v_{\theta 10}(\theta)=\frac{1}{\lambda_1(1-\lambda_1)}\{f'''_{10}+(1+\lambda_1^2)f'_{10}\},\quad \lambda_1\neq 1$$

$$\left.\begin{aligned}
v_{r11}(\theta) &= \frac{1}{\lambda_1}h_{11} \\
v_{\theta 11}(\theta) &= -\left(1+\frac{1}{\lambda_1}\right)\int h_{11}\,d\theta
\end{aligned}\right\}\tag{4.15}$$

$$v_{\theta 11}(\theta)=\frac{1}{\lambda_1(1-\lambda_1)}\{f'''_{11}+(1+\lambda_1^2)f'_{11}\},\quad \lambda_1\neq 1$$

在B区则有:

$$\left.\begin{aligned}
h_{1\bar{1}} &= -\frac{\lambda_1(\lambda_1+2)}{2}b_{10}\,\Lambda_{0\bar{1}}=-\frac{\lambda_1(\lambda_1+2)}{2\sqrt{2}}b_{10} \\
v_{r1\bar{1}} &= -\frac{(\lambda_1+2)}{2\sqrt{2}}b_{10} \\
v_{\theta 1\bar{1}} &= \frac{(\lambda_1+1)(\lambda_1+2)}{2\sqrt{2}}(\theta-\theta_A)b_{10}+b^*_{11},\quad \Lambda_{1\bar{1}}=\frac{(\lambda_1-1)}{2k}v_{\theta 1\bar{1}}
\end{aligned}\right\}\tag{4.16}$$

$$\begin{aligned}
f_{10} &= -(2+\lambda_1)b_{10}\cos\theta \\
h_{10} &= \frac{\lambda_1^2}{2}(2+\lambda_1)b_{10}\cos\theta+S_{r10}\,\Lambda_{00}+S_{r11}\,\Lambda_{0\bar{1}}
\end{aligned}$$

$$
\left.\begin{aligned}
v_{r10} &= \frac{1}{\lambda_1}\{h_{10} + v_{r1\bar{1}}\} \\
v_{\theta 10} &= -\left\{\int h_{10}\, d\theta + \int v_{r10}\, d\theta\right\}
\end{aligned}\right\} \tag{4.17}
$$

设想 Γ_B 的曲线方程为：

$$
\left.\begin{aligned}
\theta - \beta &= \sum_{n=1,2}^{\infty} q_n r^{\lambda_n}, \\
q_n &= \sum_{m=0,1}^{\infty} q_{nm} t^{-m}, \qquad n \geq 1
\end{aligned}\right\} \tag{4.18}
$$

则由应力分量的连续条件推得：

$$[F_{10}(\beta)]_{\Gamma_B} = [F'_{10}(\beta)]_{\Gamma_B} = 0, \tag{4.19}$$

$$[F''_{10}(\beta)]_{\Gamma_B} + [F'''_{00}(\beta)]_{\Gamma_B} q_{10} = 0 \tag{4.20}$$

$$[F_{10}(\gamma)]_{\Gamma_C} = [F'_{10}(\gamma)]_{\Gamma_C} = 0, \tag{4.21}$$

另外由速度分量的连续条件得：

$$q_{10}[v'_{r0\bar{1}}(\beta)]_{\Gamma_B} + [v_{r1\bar{1}}(\beta)]_{\Gamma_B} = 0 \tag{4.22}$$

方程(4.19)－(4.22)共有6 个求解方程而包含7 个未知数　$\ddot{C}_i, i = 1,2,\cdots 5$；$b_{10}$、$q_{10}$．因此可选 q_{10} 作为自由参数．其他系数就完全确定下来．从而 $F_{10}(\theta)$ 也就确定下来．

类似的 F_{1m} 依赖于 q_{1m}．

参　考　文　献

[1] Слепян, Л. И.・ Изв. А. Н. СССР・МТТ・, 9 (1974).

[2] Rice, J. R., Drugan, W. J. and Sham. T. L., ASTM STP 700 (1980), 189－219.

[3] 高玉臣，力学学报，1 (1980), 48.

[4] Drugan, W. J. and Rice, J. R., Mechanics of Material Behavior (1983).

ELASTIC－PLASTIC STRESS FIELD OF GROWING CRACK IN ELASTIC－PERFECTLY PLASTIC SOLIDS

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ABSTRACT　This paper presents a systematic analysis of elastic－plastic stress and strain fields around the tip of a growing crack in the elastic-perfectly plastic solid.

The high order asymptotic equations of stress and velocity fields are developed. The solutions of high order asymptotic equation are discussed. It is proved the high order asymptotic solutions are dependent on the curvilinear equation of boundary between plastic region and unloading elastic region.

KEY WORDS　growing crack, perfectly plasticity, high order asymptotic solution, stress field.