第17卷 第1期　　力 学 学 报　　Vol. 17, No. 1
1985 年 1 月　　ACTA MECHANICA SINICA　　Jan., 1985

有基本流动的渠道中的孤立波

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迄今为止，大多数关于渠道中的孤立波的研究都作了静水的假定，从而避免了波与流相互作用的难题.但在自然界的江河中大都有基本流动，这就要求我们研究有基本流动情况下渠道中的孤立波. Benjamin[1]. Peters[2], Turpin[3] 等人先后在这方面做了一些工作.

我们研究无粘、不可压缩流体的运动，基本流动沿着渠道方向，渠道断面及坐标如图1所示. 我们假定：(1)渠道的深度和宽度为同量级的；(2)水深远小于波长，即 $1 \gg \epsilon = \left(\frac{h_0}{\lambda}\right)^2 = O\left(\frac{a}{h_0}\right)$，其中 a 为波幅，λ 为波长 h_0 为特征水深；(3)渠道是缓变的，即

$$y' = l_\pm(z', \epsilon x').$$

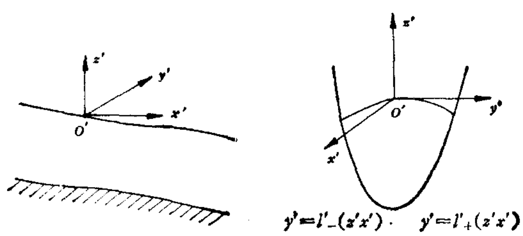

图 1

引进无量纲量

$$x = x'/\lambda \qquad y = y'/h_0 \qquad z = z'/h_0$$

$$t = t'\sqrt{gh_0}/\lambda, \quad \eta = \eta'/h_0, \qquad l_\pm = l'_\pm/h_0$$

$$u = u'/\sqrt{gh_0}, \quad v = v'\frac{\lambda}{h_0}\frac{1}{\sqrt{gh_0}}, \quad w = w'\frac{\lambda}{h_0}\frac{1}{\sqrt{gh_0}}$$

$$p = \frac{p'}{\rho g h_0} + z$$

我们求行波解，引进坐标变换

$$\xi = f(x) - t \qquad X = \epsilon x \tag{1}$$

流体力学基本方程组及边界条件变为

$$\left.\begin{aligned} &-u_\xi + u(f'u_\xi + \epsilon u_X) + vu_y + wu_z + f'p_\xi + \epsilon p_X = 0 \\ &\epsilon[-v_\xi + u(f'v_\xi + \epsilon v_X) + vv_y + wv_z] + p_y = 0 \end{aligned}\right|$$

本文于 1983 年 6 月 14 日收到.

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$$\left.\begin{array}{l}\varepsilon[-w_\xi + u(f'w_\xi + \varepsilon w_X) + vw_y + ww_z] + p_z = 0\\ f'u_\xi + \varepsilon u_X + v_y + w_z = 0\end{array}\right\} \tag{2}$$

$$\left.\begin{array}{l}\left.\begin{array}{l}p = \eta\\ w = -\eta_\xi + u(f'\eta_\xi + \varepsilon\eta_X) + v\eta_y\end{array}\right\} \text{在 } z = \eta \text{ 上}\\ v = wl_{\pm z} + \varepsilon ul_{\pm X} \qquad \text{在 } y = l_\pm(z, \varepsilon x) \text{ 上}\end{array}\right\} \tag{3}$$

把各个量形式地展开如下：

$$\left.\begin{array}{l}u = u_0(X, y, z) + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots\\ p = p_0(X) + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots\\ \eta = \eta_0(X) + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \cdots\\ v = \varepsilon v_1 + \varepsilon^2 v_2 + \cdots\\ w = \varepsilon w_1 + \varepsilon^2 w_2 + \cdots\end{array}\right\} \tag{4}$$

下标 0 表示基本流动，v, w 的基本流动为一阶小量，并入 v_1, w_1 中. 把 (4) 式代入方程 (2) 及边界条件 (3)，从 ε^0 项及 ε^1 项的方程可得：

$$p_0(X) = \eta_0(X), \quad p_1(X, \xi) = \eta_1(X, \xi) \tag{5}$$

$$f'^2 \iint_{(\sigma)} \frac{dydz}{(f'u_0 - 1)^2} = l_+(\eta_0, X) - l_-(\eta_0, X) = l_0 \tag{6}$$

$$\eta_0'\left[f'\iint_{(\sigma)} \frac{dydz}{(f'u_0 - 1)^2} - \int_{l_-(\eta_0,X)}^{l_+(\eta_0,X)} \frac{u_0}{f'u_0 - 1}\Big|_{z=\eta_0} dy\right] + \iint_{(\sigma)} \frac{u_{0,X}}{(f'u_0 - 1)^2} dydz$$
$$= \int_{\eta_0}^{-h} \frac{u_0 l_{-X}}{f'u_0 - 1}\Big|_{y=l_-} dz + \int_{-h}^{\eta_0} \frac{u_0 l_{+X}}{f'u_0 - 1}\Big|_{y=l_+} dz \tag{7}$$

(7) 式为基本流动应满足的关系，(6) 式可定出 $f(x)$.

ε^2 项的方程及边界条件如下：

$$\left.\begin{array}{l}-u_{2,\xi} + f'(u_0 u_{2,\xi} + u_1 u_{1,\xi}) + (u_0 u_1)_X + v_2 u_{0,y} + v_1 u_{1,y}\\ \qquad + w_2 u_{0,z} + w_1 u_{1,z} + f'p_{2,\xi} + p_{1,X} = 0\\ -v_{1,\xi} + f'u_0 v_{1,\xi} + p_{2,y} = 0\\ -w_{1,\xi} + f'u_0 w_{1,\xi} + p_{2,z} = 0\\ f'u_{2,\xi} + u_{1,X} + v_{2,y} + w_{2,z} = 0\end{array}\right\} \tag{8}$$

$$\left.\begin{array}{l}p_2 = \eta_2, \qquad \text{在 } z = \eta_0 \text{ 上}\\ w_{1,z}\eta_1 + W_2 = -\eta_{2,\xi} + f'(u_0\eta_{2,\xi} + u_1\eta_{1,\xi} + u_{0,z}\eta_1\eta_{1,\xi})\\ \qquad + u_0\eta_{1,X} + u_{0,z}\eta_1\eta_{0,X} + u_1\eta_{0,X}, \text{ 在 } z = \eta_0 \text{ 上}\\ v_2 = w_2 l_{\pm z} + u_1 l_{\pm X}, \text{ 在 } y = l_\pm(z, X) \text{上}\end{array}\right\} \tag{9}$$

下面先求解 p_2. 把 p_2 写为

$$p_2 = -\phi(y, z)\eta_{1\xi\xi}(\xi, X) + F(\xi, X) \tag{10}$$

从方程 (8) 及边界条件 (9) 可得 ϕ 的方程

$$\left.\begin{array}{l}\left[\dfrac{\phi_y}{(f'u_0 - 1)^2}\right]_y + \left[\dfrac{\phi_z}{(f'u_0 - 1)^2}\right]_z = \dfrac{f'^2}{(f'u_0 - 1)^2}\\ \phi_z = (f'u_0 - 1)^2, \qquad \text{在 } z = \eta_0 \text{ 上}\\ \phi_n = 0, \qquad \text{在 } y = l_\pm(z, X) \text{ 上}\end{array}\right\} \tag{11}$$

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这是典型的泊松方程的牛曼边值问题，可以用解析方法或数值方法解出．由方程(8)还可解得

$$\left.\begin{aligned} v_1 &= P\eta_{1\xi} + R_1 \\ w_1 &= Q\eta_{1\xi} + R_2 \\ u_1 &= -f'^{-1}(P_y + Q_z)\eta_1 \end{aligned}\right\} \tag{12}$$

这里

$$\left.\begin{aligned} P &= \psi_y/(f'u_0 - 1) \\ Q &= \psi_z/(f'u_0 - 1) \end{aligned}\right\} \tag{13}$$

R_1、R_2 与 ξ 无关，它们是由 u_0 引起的基本流动．由(8)式消去 $u_{2,\xi}$，然后把所得之结果在过水断面上积分，并化简，可得 η_1 的方程

$$m_0\eta_{1,\xi\xi\xi} + m_1\eta_1\eta_{1,\xi} + m_2\eta_{1,X} + m_3\eta_1 = 0 \tag{14}$$

其中

$$m_0 = \iint\limits_{(\sigma)} \frac{\psi_y^2 + \psi_z^2}{(f'u_0 - 1)^2}\, dydz \tag{15}$$

$$m_1 = \int_{l_-(\eta_0,X)}^{l_+(\eta_0,X)} - \left[\frac{\psi_y}{(f'u_0 - 1)^2}\right]_y \Bigg|_{z=\eta_0} dy + 3f'^4 \iint\limits_{(\sigma)} \frac{dydz}{(f'u_0 - 1)^4} \tag{16}$$

$$m_2 = -2f' \iint\limits_{(\sigma)} \frac{dydz}{(f'u_0 - 1)^3} \tag{17}$$

$$\begin{aligned} m_3 = &\iint\limits_{(\sigma)} \frac{-1}{(f'u_0 - 1)^2} \frac{\partial}{\partial X}\left[\frac{1}{f'}(P_y + Q_z)\right] dydz \\ &+ 2\iint\limits_{(\sigma)} \frac{u_{0,X} + f'\eta_0'}{(f'u_0 - 1)^3}(P_y + Q_z)\, dydz \\ &- \int_{l_-(\eta_0,X)}^{l_+(\eta_0,X)} dy \left\{\frac{\eta_0'}{f'(f'u_0 - 1)^2}(P_y + Q_z) + \frac{u_{0z}\eta_0'}{(f'u_0 - 1)} \right. \\ &\left. - \frac{R_{2,z}}{f'u_0 - 1}\right\}\Bigg|_{z=\eta_0} - \int_{\eta_0}^{-h} \frac{l_{-X}(P_y + Q_z)}{f'(f'u_0 - 1)^2}\Bigg|_{y=l_-(z,X)} dz \\ &- \int_{-h}^{\eta_0} \frac{l_{+X}(P_y + Q_z)}{f'(f'u_0 - 1)^2}\Bigg|_{y=l_+(z,X)} dz \end{aligned} \tag{18}$$

作变换

$$\eta_1 = m_4\zeta = e^{-\int (m_3/m_2) dX}\zeta \tag{19}$$

方程(14)化为

$$\zeta_X + \alpha(X)\zeta\zeta_\xi + \beta(X)\zeta_{\xi\xi\xi} = 0 \tag{20}$$

其中

$$\left.\begin{aligned} \alpha &= m_1 m_4/m_2 \\ \beta &= m_0/m_2 \end{aligned}\right\} \tag{21}$$

方程(20)就是我们得到的基本方程——缓变系数 KdV 方程．

渠道均匀时，$u_{0,X} = 0$，$h' = 0$，$\eta_0 = 0$，(15)—(18)式和 Peters 的结果一致．渠道中没有流动时，$u_0 = \eta_0 = 0$，本文的结果(14)—(18)式和周显初[4]的结果一致．

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当 α,β 为 X 的缓变函数，即 $\alpha(\mu X)$，$\beta(\mu X)$ （$\mu \ll 1$） 时，我们可以把方程(20)的解按 μ 展开，求出首项近似解.

$$\zeta = \zeta_0(\theta, T) + \mu\zeta_1(\theta, T) + \cdots \tag{22}$$

其中

$$\left.\begin{aligned} d\theta &= k d\xi - \omega(T) dX \\ T &= \mu X \end{aligned}\right\} \tag{23}$$

ζ_0 应满足的方程为

$$-\omega\zeta_{0,\theta} + \alpha(T)k\zeta_0\zeta_{0,\theta} + \beta(T)k^3\zeta_{0,\theta\theta\theta} = 0 \tag{24}$$

其解为

$$\zeta = a\operatorname{sech}^2 \sqrt{\frac{a\alpha}{12\beta}}\left(\xi - \int\frac{\varepsilon a m_1 m_4}{3m_2}dx\right) \tag{25}$$

所以孤立波的相速度为

$$c = [f' - (\varepsilon a m_1 m_4/3m_2)]^{-1} \tag{26}$$

ζ_1 应满足的方程为

$$-\omega\zeta_{1,\theta} + \alpha(T)k(\zeta_0\zeta_1)_\theta + \beta(T)\zeta_{1,\theta\theta\theta} = -\zeta_{0,T} \tag{27}$$

它的相容性条件为

$$\frac{9\omega^2}{\alpha^2}\sqrt{\frac{4\beta}{\omega}} = \text{const} \tag{28}$$

由此得出孤立波振幅增长的规律

$$\zeta \sim (\alpha/\beta)^{1/3} \tag{29}$$

作为一个例子，我们考虑宽度固定的、底部缓变的矩形渠道中的孤立波. 基本流动的速度为

$$u_0(x) = Q_0/(\eta_0 - h) \tag{30}$$

其中 Q_0 为流量，$z = h(X)$ 为渠底的方程. 由(6)式可得

$$f'^{-1} = u_0 + \sqrt{\eta_0 - h} \tag{31}$$

这正是线化理论的结论：有流动时的局部临界速度为基本流速与静水中的局部临界速度之和.

(7)式即为基本流动的 Bernulli 方程

$$\eta_0 + Q_0^2/[2(\eta_0 - h)^2] = \text{const} \tag{32}$$

由(15)式、(16)式可知

$$\frac{m_1}{m_0} = \frac{9(u_0 + \sqrt{\eta_0 - h})^2}{(\eta_0 - h)^3} \tag{33}$$

由此可知：与有同样水深的静水中的波形相比，孤立波与流动同向时的波形更窄，孤立波与流动反向时的波形变宽.

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SOLITARY WAVES ON A STREAM IN A CHANNEL

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Abstract

In this paper, the propagation of solitary waves on a basic stream in a channel with slowly varying cross-section is studied. The KdV equation with variable coefficients is derived, and its first-order approximate solution and the relation among wave velocity, surface elevation, channel dimensions and parameters of the basic flow are obtained. The results herein are in agreement with Peter's results in the case of uniform cross-section and the author's previous results without considering the basic stream.

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