圆截面轴对称等离子体环平衡的分析解和极限比压$\langle\beta_\tau\rangle$

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本文根据Mercier 的基本方程用反纵横比ϵ的幂级数展开方法求出了胖 Tokamak 的平衡分析解。为了考查ϵ的影响，本文计算到了ϵ^3的量级，并在没有负压的限制下求得了相应的极限比压$\langle\beta_\tau\rangle$。求得的平衡解亦可作为不稳定分析之用。

一、前 言

矩形截面等离子体的平衡和稳定性问题已经由Maschke[1]，Herrnegger[2]，Oshiyama及Fnkutani[3]等人用分析方法研究过。Mercier及Luc[4]求过圆形截面等离子体环在特殊情况下的平衡解，但是只算到ϵ的量级。因此，他们的结果只适用于小反纵横比ϵ的情况。但是，近来装置的反纵横比不断地在增大，为了考查ϵ对平衡和极限比压的影响，要求更高阶ϵ的解。为此，本文在[4]的基础上求出ϵ^3阶的解并计算了相应的极限比压。

座标系如图1所示，R_0为磁轴半径，a为等离子体环截面半径，V为其体积。若为非圆截面，则a的定义为$V=2\pi^2a^2R_0$。并定义任意量F对体积的平均为$\langle F\rangle=\frac{1}{V}\int_v F\mathrm{d}\tau$。为节省篇幅，本文采用文[4]中使用的符号，其规范化后的理想等离子体平衡方程和边界条件为：

$$\left.\begin{aligned}&\mathscr{L}\mathscr{F}=-\frac{1}{2\epsilon^2\langle\eta\rangle^2}\left[\frac{\mathrm{d}}{\mathrm{d}\mathscr{F}}(\bar{f}^2+\langle\beta_\phi\rangle\bar{\beta})+\left(\frac{R^2}{R_0^2}-1\right)\frac{\mathrm{d}}{\mathrm{d}\mathscr{F}}(\langle\beta_\phi\rangle\bar{\beta})\right],\\&\bar{\beta}_b=0,\quad \bar{f}_b=1.\end{aligned}\right\}\qquad(1)$$

其中$\mathscr{L}=\frac{1-\epsilon\rho\cos\theta}{\rho}\left[\frac{\partial}{\partial\rho}\left(\frac{\rho}{1-\epsilon\rho\cos\theta}\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho}\frac{\partial}{\partial\theta}\left(\frac{1}{1-\epsilon\rho\cos\theta}\frac{\partial}{\partial\theta}\right)\right]$，

$\epsilon=a/R_0$，

定义$\bar{f}^2+\langle\beta_\phi\rangle\bar{\beta}=1+\epsilon^2\langle\eta\rangle^2(2m_1\mathscr{F}+m_2\mathscr{F}^2)$，

$\langle\beta_\phi\rangle\bar{\beta}=\epsilon^2\langle\eta\rangle^2(p_1\mathscr{F}+p_2\mathscr{F})$，或$\beta_\tau=p_1\mathscr{F}+p_2\mathscr{F}^2$。为解方程时减少两个参变量，令$\mathscr{F}=m_1G+p_1H$，将以上诸关系代入(1)得如下关系

$$\left.\begin{aligned}&\left[\mathscr{L}+m_2+p_2\left(\frac{R^2}{R_0^2}-1\right)\right]G=-1\\&\left[\mathscr{L}+m_2+p_2\left(\frac{R^2}{R_0^2}-1\right)\right]H=\left(1-\frac{R^2}{R_0^2}\right)/2,\\&G_b=0,\qquad H_b=0.\end{aligned}\right\}\qquad(2)$$

二、求G的 解

在Mercier和Luc[4]的推导中为避免数学上的困难，假设$p_2=0$，并且只算到ϵ量级。鉴于Tokamak的ϵ日益增加，有必要计算到ϵ的高次量级，同时为探索$p_2\neq 0$的影响，本文对p_2不加限制。引进$\mu^2=m_2$，$\mu^2\alpha=p_2$，$\xi=\mu\rho$； (3)

令 $$G=\sum_{m=0}^{\infty}\epsilon^m G_m(\xi,\theta), \tag{4}$$

将(3)，(4)代入(2)得

$$\epsilon^0:\quad \mathscr{L}_0 G_0=-\frac{1}{\mu}, \tag{5}$$

$$\epsilon^m:\quad \mathscr{L}_0 G_m=\frac{2\alpha\xi\cos\theta}{\mu}G_{m-1}-\frac{\alpha\xi^2}{\mu^2}\cos^2\theta\, G_{m-2}$$

$$-\frac{1}{\xi}\sum_{n=0}^{m}\left(\frac{\xi\cos\theta}{\mu}\right)^{n+1}\frac{\partial G_{m-n-1}}{\partial\xi}+\frac{\sin\theta}{\mu\xi}\sum_{n=0}^{m}\left(\frac{\xi\cos\theta}{\mu}\right)^{n}\frac{\partial G_{m-n-1}}{\partial\theta}, \tag{6}$$

($G_s=0$ 当$s<0$)

$$\mathscr{L}_0=\frac{\partial^2}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial}{\partial\xi}+1+\frac{\partial^2}{\xi^2\partial\theta^2}.$$

$$\left.\begin{array}{l}\text{边界条件：}G_m(\mu,\ \theta)=0,\\ \text{周期性条件：}G_m(\xi,\ \theta)=G_m(\xi,\ \theta+2\pi).\end{array}\right\} \tag{7}$$

现在来解方程(5)，显然G_0可以展为Fourier级数

$$G_0(\xi,\ \theta)=\sum_{n=0}^{\infty}\left(G_0^{(n)}(\xi)\cos n\theta+V_0^{(n)}(\xi)\sin n\theta\right), \tag{8}$$

$$-\frac{1}{\mu^2}=\frac{A_0}{2}+\sum_{n=1}^{\infty}(A_n\cos n\theta+B_n\sin n\theta), \tag{9}$$

其中 $$G_0^{(n)}(\xi)=\frac{1}{\pi}\int_0^{2\pi}G_0(\xi,\theta)\cos n\theta\,d\theta,\ V_0^{(n)}=\frac{1}{\pi}\int_0^{2\pi}G_0(\xi,\theta)\sin n\theta\,d\theta,$$

$$A_n=-\frac{1}{\pi}\int_0^{2\pi}\frac{1}{\mu^2}\cos n\theta d\theta,\quad B_n=-\frac{1}{\pi}\int_0^{2\pi}\frac{1}{\mu^2}\cos n\theta d\theta,$$

因此有 $A_0=-\frac{2}{\mu^2}$，$A_n=B_n=0$。($n=1$，2，3…) (10)

把(8)、(9)代入(5)，并比较得

$$\left.\begin{array}{ll}L_0G_0^{(0)}=-\frac{1}{\mu^2}, & G_0^{(0)}(\mu)=0,\\ L_nG_0^{(n)}=0, & G_0^{(n)}(\mu)=0,\\ L_nV_0^{(n)}=0, & V_0^{(n)}(\mu)=0.\end{array}\right\} \tag{11}$$

其中L_n是第n阶贝塞尔算子，因此方程(11)的解为：

$$G_0{}^{(0)}=\frac{1}{\mu^2}\left(\frac{J_0}{J_{0b}}-1\right),\quad G_0{}^{(n)}=V_0{}^{(n)}=0, \tag{12}$$

其中$J_{0b}=J_0(\mu)$，J_m是m阶贝塞尔函数。最后得

$$G_0(\xi,\ \theta)=G_0{}^{(0)}=\frac{1}{\mu}\left(\frac{J_0}{J_{0b}}-1\right). \tag{13}$$

方程(6)的解可以按求解(5)同样方法求解，故下面只给出(6)的解：

$$G_1=G_1{}^{(1)}\cos\theta, \tag{14}$$

其中

$$\mu^3G_1{}^{(1)}=a_0\xi J_0+(a_1\xi^2+a_2)J_1+a_3\xi,$$

$$a_0=-1/2J_{0b},\quad a_1=\alpha/2J_{0b},$$

$$a_2=\mu(1+4\alpha)/2J_{1b}-\alpha\mu^2/2J_{0b},\quad a_3=-2\alpha.$$

$$G_2=G_2{}^{(0)}+G_2{}^{(2)}\cos2\theta, \tag{15}$$

其中：$\mu^4G_2{}^{(0)}=(c_0+c_2\xi^2+c_4\xi^4)J_0+(c_1\xi+c_3\xi^3)J_1+c_5\xi^2+8\alpha^2$;

$$c_0=-[8\alpha^2+c_5\mu^2+(c_2\mu^2+c_4\mu^4)J_{0b}+(c_1\mu+c_3\mu^3)J_{1b}]/J_{0b},$$

$$c_1=-2c_2-a_0/2-a_2/4,\ c_2=c_3+\frac{a_1}{4}+\frac{a_0}{8}-\frac{\alpha a_2}{4},\quad c_3=-(5\alpha-4\alpha^2)/24J_{0b},$$

$$c_4=-\alpha^2/16J_{0b},\quad c_5=\frac{\alpha}{2}-2\alpha^2;\ \mu^4G_2{}^{(2)}=(b_0\xi^2+b_1\xi^4)J_0+(b_2\xi+b_3\xi^3)J_1+b_4J_2+b_5\xi^2;$$

$$b_0=(a_0-2\alpha a_2)/8,\quad b_1=-\alpha a_1/8,\quad b_2=-a_2/4,\quad b_3=-2b_1-5a_1/6,$$

$$b_4=-[(b_0\mu^2+b_1\mu^4)J_{0b}+(b_2\mu+b_3\mu^3)J_{1b}+b_5\mu^2]/J_{2b},\quad b_5=\alpha/2-2\alpha^2.$$

$$G_3=G_3{}^{(1)}\cos\theta+G_3{}^{(3)}\cos3\theta, \tag{16}$$

其中

$$\mu^5G_3{}^{(1)}=(e_1\xi+e_3\xi^3+e_5\xi^5)J_0+(e_0+e_2\xi^2+e_4\xi^4+e_6\xi^6)J_1+e_8\xi+e_9\xi^3,$$

$$e_0=-[(e_1\mu+e_3\mu^3+e_5\mu^5)J_{0b}+(e_2\mu^2+e_4\mu^4+e_6\mu^6)J_{1b}+e_8\mu+e_9\mu^3]/J_{1b},$$

$$e_1=-\frac{c_0}{2}+\frac{b_4}{4}+\frac{3}{2}b_2-\alpha b_4,\quad e_2=-2e_3+\frac{\alpha a_2}{8}-\frac{15}{16}a_0+\frac{7}{32}a_2+\frac{\alpha c_0}{2}-\frac{\alpha b_4}{2};$$

$$e_3=\frac{4}{3}e_4-\frac{c_2}{6}+\frac{c_3}{3}-\frac{b_0}{12}+\frac{b_3}{3}+\frac{a_1}{6}-\frac{\alpha c_1}{3}-\frac{5}{2}\alpha b_2,$$

$$e_4=-4\alpha b_1+\frac{35}{8}b_1-\frac{c_3}{8}-\frac{b_3}{16}+\frac{1}{4}\alpha c_2+\frac{\alpha b_0}{8},\quad e_5=\frac{4}{3}\alpha b_1-\frac{7}{4}b_1,$$

$$e_6=\frac{1}{4}\alpha b_1,\ e_8=-16\alpha^2+64\alpha^3,\quad e_9=3\alpha^2-6\alpha^3,$$

$$\mu^5G_3{}^{(3)}=(d_3\xi^3+d_5\xi^5)J_0+(d_2\xi^2+d_4\xi^4+d_6\xi^6)J_1+d_7\xi J_2+d_9\xi^3+d_{10}J_3,$$

$$d_2=\frac{1}{8}b_2-\frac{1}{4}\alpha b_4,\quad d_3=\frac{1}{12}\alpha a_2+\frac{a_0}{24}-\frac{b_0}{12},\ d_4=\frac{\alpha}{8}b_0-2d_5-\frac{1}{8}b_1-\frac{1}{16}b_3,$$

$$d_5=\frac{2}{15}\alpha b_1-\frac{b_1}{4}-\frac{1}{10}\alpha b_3,\quad d_6=\frac{1}{12}\alpha b_1,\quad d_7=-b_4/4,\quad d_9=\alpha^2-2\alpha^3,$$

$$d_{10}=-[d_3\mu^3J_{0b}+d_5\mu^5J_{0b}+d_2\mu^2J_{1b}+d_4\mu^4J_{1b}+d_6\mu^6J_{1b}+d_7\mu J_{2b}+d_9\mu^3]/J_{3b}.$$

三、求H的解

解H和解G原则上完全一样，令

$$H=\sum_{m=0}^{\infty}\epsilon^m H_m(\xi,\ \theta). \tag{17}$$

代(17)到(2)得

ϵ^0:
$$\mathscr{L}_0 H_0=0, \tag{18}$$

ϵ^m:
$$\mathscr{L}_0 H_m=\frac{z\alpha\xi\cos\theta}{\mu}H_{m-1}-\frac{\alpha\xi^2}{\mu^2}\cos^2\theta\, H_{m-2}-\frac{1}{\mu^2}\left(-\frac{\xi}{s\mu}\cos\theta\right)^s$$
$$-\frac{1}{\xi}\sum_{n=0}^{m}\left(\frac{\xi\cos\theta}{\mu}\right)^{n+1}\frac{\partial H_{m-n-1}}{\partial\xi}+\frac{\sin\theta}{\mu\xi}\sum_{n=0}^{m}\left(\frac{\xi\cos\theta}{\mu}\right)^{n}\frac{\partial H_{m-n-1}}{\partial\theta}, \tag{19}$$

(当$m\leqslant 2$时，$s=m$；当$m>2$时，$s=0$)

$$\left.\begin{array}{ll}\text{边界条件} & H_m(\mu,\ \theta)=0,\\ \text{周期性条件} & H_m(\xi,\ \theta)=H_m(\xi,\ \theta+2\pi),\end{array}\right\} \tag{20}$$

对于方程(18)、(19)经过和求解G的类似计算，得

$$H_0=0. \tag{21}$$

$$H_1=H_1^{(1)}\cos\theta, \tag{22}$$

其中 $\mu^3 H_1^{(1)}=\xi+a_0'J_1$，$a_0'=-\mu/J_{1b}$.

$$H_2=H_2^{(0)}+H_2^{(2)}\cos 2\theta, \tag{23}$$

其中

$\mu^4 H_2^{(0)}=c_4'\xi^2+c_0'J_0+c_1'\xi J_1+c_2'\xi^2 J_0+c_3'$,

$c_0'=-[c_3'+c_4'\mu^2+c_1'\mu J_{1b}+c_2'\mu^2 J_{0b}]/J_{0b}$，$c_3=-4\alpha$,

$c_1=-\dfrac{a_0'}{4}+\dfrac{1}{2}\alpha a_0'$，$c_2'=-\alpha a_0'/4$，$c_4=-\dfrac{1}{4}+\alpha$,

$\mu^4 H_2^{(2)}=b_0'\xi^2 J_0+b_2'\xi J_1+b_4'J_2+b_5'\xi^2$，$b_0'=-\alpha a_0'/4$，$b_2'=-a_0'/4$,

$b_5'=-1/4+\alpha$，$b_4'=-[b_0'\mu^2 J_{0b}+b_2'\mu J_{1b}+b_5'\mu^2]/J_{2b}$.

$$H_3=H_3^{(1)}\cos\theta+H_3^{(3)}\cos 3\theta, \tag{24}$$

其中

$\mu^5 H_3^{(1)}=(e_1'\xi+e_3'\xi^3)J_0+(e_0'+e_2'\xi^2+e_4'\xi^4)J_1+e_8'\xi+e_6'\xi^3$,

$e_0'=-[(e_1'\mu+e_3'\mu^3)J_{0b}+(e_2'\mu^2+e_4'\mu^4)J_{1b}+e_8'\mu+e_6'\mu^3]/J_{1b}$,

$e_1'=-\alpha b_4'-c_0'/2+b_4'/4+b_2'/2-a_0'/4$，$e_2'=-2e_3'-c_1'/4-b_0'-b_2'/8$
$-\alpha b_4'/4-3a_0'/16+\alpha c_0'/2$，$e_3'=\alpha b_0'/2-11b_0'/12-\alpha c_1'/3$,

$e_4'=3\alpha b_0'/8$，$e_6'=-3\alpha/2+3\alpha^2$，$e_8'=8\alpha-32\alpha^2$,

$\mu^5 H_3^{(3)}=d_3'\xi^3 J_0+d_2'\xi^2 J_1+d_4'\xi^4 J_1+d_7'\xi J_2+d_{10}'J_3+d_9'\xi^3$,

$d_2'=-b_2'/8-a_0'/16-\alpha b_4'/4$，$d_3'=-b_0'/12+\alpha a_0'/24-\alpha b_2'/6$,

$d_4'=\alpha b_0'/8$，$d_7'=-b_4'/4$，$d_9'=-\alpha/2+\alpha^2$,

$d_{10}'=-[d_3'\mu^3 J_{0b}+(d_2'\mu^2+d_4'\mu^4)J_{1b}+d_7'\mu J_{2b}+d_9'\mu^3]/J_{3b}$.

四、m_1和p_1的计算

根据文献[4]，基本方程(1)可以化为

$$\tilde{\eta}=-\frac{1}{2\epsilon^2\langle\eta\rangle^2}\left[\left(\frac{R_0}{R}\right)^2\frac{d\tilde{f}^2}{d\mathscr{F}}+\langle\beta_\phi\rangle\frac{d\tilde{\beta}}{d\mathscr{F}}\right]=\frac{R_0{}^2}{2R^2}\mathscr{L}\mathscr{F}. \tag{25}$$

取(25)的体积分得

$$a_1m_1+a_2p_1+1=0, \tag{26}$$

式中 $a_1=\frac{1}{2}\left[\langle\frac{R_0{}^2}{R^2}\rangle+m_2\langle\frac{R_0{}^2G}{R^2}\rangle+p_2\langle\left(1-\frac{R_0{}^2}{R^2}\right)G\rangle\right]$,

$$a_2=\frac{1}{4}\langle 1-\frac{R_0{}^2}{R^2}\rangle+\frac{m^2}{2}\langle\frac{R_0{}^2H}{R^2}\rangle+\frac{p_2}{2}\langle(1-\frac{R_0{}^2}{R^2})H\rangle].$$

对β_τ取体积分得

$$a_3m_1p_1+a_4m_1{}^2+a_5p_1{}^2-\langle\beta_\tau\rangle=0, \tag{27}$$

式中 $a_3=\langle G\rangle+2p_2\langle GH\rangle$，$a_4=p_2\langle G^2\rangle$，$a_5=\langle H\rangle+p_2\langle H^2\rangle$.

利用第二节G的解和第三节H的解计算a_i到ϵ^3的量级，得

$$a_1=a_1{}^{(0)}+\epsilon^2a_1{}^{(2)}+0(\epsilon^4), \tag{28}$$

式中 $a_1{}^{(0)}=J_{1b}/\mu J_{0b}$，$a_1{}^{(2)}=(1/4+\alpha+4\alpha^2)\mu^{-2}+(\alpha/12+\alpha^2/12)-8\alpha^2J_{1b}/\mu^3J_{0b}$
$+(3/8-\alpha/3+2\alpha^2/3)J_{1b}/\mu J_{0b}-(1/8+\alpha/2)J_{0b}/\mu J_{1b}-(3/8+\alpha/6-\alpha^2/3)J_{1b}{}^2/\mu^2J_{0b}{}^2$
$+(\alpha/12+\alpha^2/12)J_{1b}{}^2/J_{0b}{}^2$.

$$a_2=\epsilon^2a_2{}^{(2)}+0(\epsilon^4), \tag{29}$$

式中 $a_2{}^{(2)}=-2\alpha\mu^{-2}-J_{2b}/4\mu J_{1b}+4\alpha J_{1b}/\mu^3J_{0b}-\alpha J_{1b}/2\mu J_{0b}$,

$$a_3=a_3{}^{(0)}+\epsilon^2a_3{}^{(2)}+0(\epsilon^4), \tag{30}$$

式中 $a_3{}^{(0)}=J_{2b}/\mu^2J_{0b}$，$a_3{}^{(2)}=-3/2\mu^4-16\alpha/\mu^4+72\alpha^2/\mu^4+7\alpha/6\mu^2-35\alpha^2/6\mu^2$
$+32\alpha J_{1b}/\mu^5J_{0b}-96\alpha^2J_{1b}/\mu^5J_{0b}+(3/4-14\alpha/3+57\alpha^2/6)J_{1b}/\mu^3J_{0b}$
$+(3/4+5\alpha-12\alpha^2)J_{0b}/\mu^3J_{1b}+\alpha^2J_{0b}/2\mu J_{1b}-(3/4+\alpha/3-26\alpha^2/3)J_{1b}{}^2/\mu^4J_{0b}{}^2$
$+(\alpha/6-5\alpha^2/6)J_{1b}{}^2/\mu^2J_{0b}{}^2-(\alpha/2+2\alpha^2)J_{0b}{}^2/\mu^2J_{1b}{}^2$.

$$a_4=a_4{}^{(0)}+\epsilon^2a_4{}^{(2)}+0(\epsilon^4), \tag{31}$$

式中 $a_4{}^{(0)}=2\alpha(\mu+\mu J_{1b}{}^2/2J_{0b}{}^2-2J_{1b}/J_{0b})/\mu^3$,

$a_4{}^{(2)}=\frac{2\alpha}{\mu^6}[(3+1301\alpha/72-1345\alpha^2/24)\mu^2+(1/3-35\alpha/72+55\alpha^2/18)\mu^4$
$-(4+32\alpha-80\alpha^2)\mu J_{1b}/J_{0b}-(35/24-25\alpha/6+19\alpha^2/3)\mu^3J_{1b}/J_{0b}$
$+(\alpha/12+\alpha^2/2)\mu^5J_{1b}/J_{0b}-(7/8+2\alpha-6\alpha^2)\mu^3J_{0b}/J_{1b}+(11/24$
$+23\alpha/18-115\alpha^2/12)\mu^2J_{1b}{}^2/J_{0b}{}^2+(13/48-\alpha/4+\alpha^2/2)\mu^4J_{1b}{}^2/J_{0b}{}^2$
$+(1/16+\alpha/2+\alpha^2)\mu^4J_{0b}{}^2/J_{1b}{}^2-(3/8+\alpha/6-\alpha^2/3)\mu^3J_{1b}{}^3/J_{2b}{}^3$
$+(\alpha/12+\alpha^2/12)\mu^5J_{1b}{}^3/J_{0b}{}^3$.

$$a_5=\epsilon^2a_5{}^{(2)}+0(\epsilon^4), \tag{32}$$

上式 $a_5{}^{(2)}=-3/8\mu^2+3J_{2b}/2\mu^3J_{1b}-4\alpha/\mu^4+8\alpha J_{1b}/\mu^5J_{0b}+5\alpha/4\mu^2$
$-4\alpha J_{2b}/\mu^3J_{1b}-\alpha J_{0b}/\mu^3J_{1b}-\alpha J_{1b}/\mu^3J_{0b}+\alpha J_{0b}{}^2/2\mu^2J_{1b}{}^2$.

根据定义

$$\beta_\tau = p_1\mathscr{F} + p_2\mathscr{F}^2 = m_1p_1G + p_1{}^2H + p_2m_1{}^2G^2 + 2p_2m_1p_1GH + p_2p_1{}^2H^2, \tag{33}$$

$$\frac{d\beta_\tau}{d\rho} = m_1p_1\frac{dG}{d\rho} + p_1{}^2\frac{dH}{d\rho} + 2m_1{}^2p_2G\frac{dG}{d\rho} + 2p_2m_1p_1\left(G\frac{dH}{d\rho} + H\frac{dG}{d\rho}\right) + 2p_2p_1{}^2H\frac{dH}{d\rho}, \tag{34}$$

由于在环的内侧压力不能为负值，并在 $\rho=1$（磁流面）上压力为零。因此，必须有不等式

$$\left.\frac{d\beta_\tau}{d\rho}\right|_{\rho=1,\theta=0} \leqslant 0, \tag{35}$$

记 $$\left.\frac{dG}{d\rho}\right|_{\rho=1,\theta=0} = g_1 = g_1{}^{(0)} + g_1{}^{(1)}\epsilon + g_1{}^{(2)}\epsilon^2 + \cdots\cdots, \tag{36}$$

$$\left.\frac{dH}{d\rho}\right|_{\rho=1,\theta=0} = g_2 = g_2{}^{(0)} + g_2{}^{(1)}\epsilon + g_2{}^{(2)}\epsilon^2 + \cdots\cdots. \tag{37}$$

经计算得到

$g_1{}^{(0)} = -J_{1b}/\mu J_{0b}$, $g_1{}^{(1)} = -(1+4\alpha)J_{2b}/2\mu J_{0b} + (1+2\alpha)J_{1b}/2\mu J_{0b}$,

$$g_1{}^{(2)} = \frac{1}{\mu^3}[-(1/4-2\alpha+12\alpha^2)\mu + (3/32-\alpha/24+5\alpha^2/3)\mu^3 + (\alpha/48-\alpha^2/16)\mu^5 + 8\alpha^2J_{1b}/J_{0b} - (1/16+3\alpha/4+\alpha^2/6)\mu^2J_{1b}/J_{0b} - (1/4+3\alpha/2+2\alpha^2)\mu^2J_{0b}/J_{1b} + (3/32+\alpha/8+3\alpha^2/2)\mu^3J_{0b}/J_{2b} + (\alpha/24-\alpha^2/8)\mu^4J_{1b}/J_{0b} + (\alpha/48-\alpha^2/16)\mu^5J_{0b}/J_{2b} + (3/8+\alpha/6-\alpha^2/3)\mu J_{1b}{}^2/J_{0b}{}^2 - (\alpha/12+\alpha^2/12)\mu^3J_{1b}{}^2/J_{0b}{}^2,$$

$$g_1{}^{(3)} = \frac{1}{\mu^4}[(2e_1+e_0)J_{0b} + (4e_3+e_2+6d_3+d_2)\mu^2J_{0b} + (6e_5+e_4+d_4 + 8d_5)\mu^4J_{0b} + (e_6+d_6)\mu^6J_{0b} + (2e_2-e_1+4d_2+d_7)\mu J_{1b} + (4e_4-e_3-d_3+6d_4)\mu^3J_{1b} + (6e_6-e_5-d_5+8d_6)\mu^5J_{1b} + (2d_7+d_{10})J_{2b} + (4e_9+6d_9)\mu^2 + 2e_8],$$ $g_2{}^{(0)}=0$,

$g_2{}^{(1)} = J_{2b}/\mu J_{1b}$,

$$g_2{}^{(2)} = \frac{1}{\mu^3}[(1+2\alpha)\mu^2J_{0b}/2J_{1b} - 3\alpha\mu^2J_{1b}/2J_{2b} - (8-\mu^2)\alpha J_{1b}/2J_{0b} - (1-6\alpha)\mu],$$

$$g_2{}^{(3)} = \frac{1}{\mu^4}[12\alpha - 64\alpha^2 - (1/16+7\alpha-101\alpha^2/6)\mu^2 + (13\alpha/24+13\alpha^2/48)\mu^4 + (3/4-6\alpha+32\alpha^2)\mu J_{0b}/J_{1b} + (1/32-\alpha/2-5\alpha^2/6)\mu^3J_{0b}/J_{1b} + (3\alpha/8 + 3\alpha^2/4)\mu^4J_{0b}/J_{2b} - \alpha^2\mu^5J_{0b}/32J_{1b} + (2\alpha+4\alpha^2)\mu J_{1b}/J_{0b} - (\alpha/4+\alpha^2/2)\mu^3J_{1b}/J_{0b} - 3\mu^2J_{0b}{}^2/8J_{1b}{}^2 + \alpha\mu^4J_{0b}J_{2b}/24J_{1b}J_{3b} + (\alpha/8-\alpha^2/24)\mu^4J_{0b}{}^2/J_{1b}{}^2 - 3\alpha^2\mu^4J_{1b}/8J_{3b} + \alpha^2\mu^5J_{2b}/32J_{3b} - (1/32-\alpha/4+\alpha^2)\mu^3J_{2b}/J_{3b}.$$

如果令 $\left.\frac{d\beta_\tau}{d\rho}\right|_{\rho=1,\theta=0} = 0$，我们就得到

$$m_1g_1 + p_1g_2 = 0. \tag{38}$$

考虑到方程(26)，就可以解出m_1，p_1。

$$m_1 = -g_2/(\alpha_1g_2-\alpha_2g_1), \qquad p_1 = g_1/(\alpha_1g_2-\alpha_2g_1). \tag{39}$$

记 $$h = \alpha_1g_2 - \alpha_2g_1 = h^{(0)} + h^{(1)}\epsilon + h^{(2)}\epsilon^2 + h^{(3)}\epsilon^3 + \cdots\cdots, \tag{40}$$

式中 $h^{(0)}=0$，$h^{(1)} = J_{2b}/\mu^2J_{0b}$，$h^{(2)} = \alpha_1{}^{(0)}g_2{}^{(2)} - \alpha_2{}^{(2)}g_1{}^{(0)}$，

$h^{(3)} = \alpha_1{}^{(0)}g_2{}^{(3)} + \alpha_1{}^{(2)}g_2{}^{(1)} - \alpha_2{}^{(2)}g_1{}^{(1)}$,

同样记　$m_1=m_1^{(0)}+m_1^{(1)}\epsilon+m_1^{(2)}\epsilon^2+\cdots,$　(41)

$$p_1=p_1^{(-1)}\epsilon^{-1}+p_1^{(0)}+p_1^{(1)}\epsilon+\cdots\cdots, \tag{42}$$

其中　$m_1^{(0)}=\mu J_{0b}/J_{1b}$，$m_1^{(1)}=g_2^{(1)}h^{(2)}/(h^{(1)})^2-g_2^{(2)}/h^{(1)}$,

$$m_1^{(2)}=-g_2^{(1)}[(h^{(2)})^2/(h^{(1)})^3-h^{(3)}/(h^{(1)})^2]+g_2^{(2)}h^{(2)}/(h^{(1)})^2-g_2^{(3)}/h^{(1)},$$

$$p_1^{(-1)}=-\mu J_{1b}/J_{2b},\quad p_1^{(0)}=-g_1^{(0)}h^{(2)}/(h^{(1)})^2+g_1^{(1)}/h^{(1)},$$

$$p_1^{(1)}=g_1^{(0)}[(h^{(2)})^2/(h^{(1)})^3-h^{(3)}/(h^{(1)})^2]-g_1^{(1)}h^{(2)}/(h^{(1)})^2-g_1^{(2)}/h^{(1)}.$$

五、关于极限比压$\langle\beta\tau\rangle$，$\tilde{\beta}$和$\tilde{\eta}$的计算

现在有了m_1，p_1就可以计算$\langle\beta\tau\rangle$，$\tilde{\beta}$和$\tilde{\eta}$。由(31)式

$$\langle\beta\tau\rangle=\alpha_3m_1p_1+\alpha_4m_1^2+\alpha_5p_1^2=\langle\beta\tau\rangle^{(-1)}\epsilon^{-1}+\langle\beta\tau\rangle^{(0)}+\langle\beta\tau\rangle^{(1)}\epsilon+\cdots. \tag{43}$$

其中　$\langle\beta\tau\rangle^{-1}=1$，$\langle\beta\tau\rangle^{(0)}=\alpha_3^{(0)}(p_1^{(-1)}m_1^{(1)}+p_1^{(0)}m_1^{(0)})+\alpha_4^{(0)}(m_1^{(0)})^2+\alpha_5^{(2)}(p_1^{(-1)})^2$,

$$\langle\beta\tau\rangle^{(1)}=\alpha_3^{(0)}(p_1^{(-1)}m_1^{(2)}+p_1^{(0)}m_1^{(1)}+p_1^{(1)}m_1^{(0)})+\alpha_3^{(2)}p_1^{(-1)}m_1^{(0)}$$
$$+2\alpha_4^{(0)}m_1^{(0)}m_1^{(1)}+2\alpha_5^{(2)}p_1^{(-1)}p_1^{(0)}.$$

从上面可以看到，$\langle\beta\tau\rangle$是μ，α，ϵ的函数，对于固定的ϵ，对所有可能的α，μ要想使$\langle\beta\tau\rangle$达到极限值，α，μ必须满足关系式

$$\left.\begin{aligned}\frac{\partial\langle\beta_\tau\rangle}{\partial\mu}&=\frac{\partial\langle\beta_\tau\rangle^{(0)}}{\partial\mu}+\frac{\partial\langle\beta_\tau\rangle^{(1)}}{\partial\mu}\epsilon=0,\\ \frac{\partial\langle\beta_\tau\rangle}{\partial\alpha}&=\frac{\partial\langle\beta_\tau\rangle^{(0)}}{\partial\alpha}+\frac{\partial\langle\beta_\tau\rangle^{(1)}}{\partial\alpha}\epsilon=0.\end{aligned}\right\} \tag{44}$$

设α_0，μ_0是满足(44)的解，将α_0，μ_0和给定的ϵ代入(43)即得到极限$\langle\beta\tau\rangle$。

把$\beta\tau$按ϵ的级数展开为

$$\beta\tau=\beta\tau^{(-1)}\epsilon^{-1}+\beta\tau^{(0)}+\beta\tau^{(1)}\epsilon+\cdots\cdots, \tag{45}$$

由(33)式　$\beta\tau^{(-1)}=m_1^{(0)}p_1^{(-1)}G_0+(p_1^{(-1)})^2H_1$,

$$\beta\tau^{(0)}=m_1^{(0)}p_1^{(-1)}G_1+m_1^{(1)}p_1^{(-1)}G_0+m_1^{(0)}p_1^{(0)}G_0+(p_1^{(-1)})^2H_2+2p_1^{(-1)}p_1^{(0)}H_1$$
$$+p_2(m_1^{(0)})^2G_0^2+2p_2m_1^{(0)}p_1^{(-1)}G_0H_1+p_2(p_1^{(-1)})^2H_1^2,$$

$$\beta\tau^{(1)}=m_1^{(0)}p_1^{(1)}G_0+m_1^{(2)}p_1^{(-1)}G_0+m_1^{(1)}p_1^{(0)}G_1+m_1^{(0)}p_1^{(0)}G_1+m_1^{(0)}p_1^{(-1)}G_2$$
$$+m_1^{(1)}p_1^{(-1)}G_1$$
$$+[(p_1^{(0)})^2+2p_1^{(-1)}p_1^{(1)}]H_1+2p_1^{(-1)}p_1^{(0)}H_2+(p_1^{(-1)})^2H_3+2p_2[m_1^{(0)}m_1^{(1)}G_0^2$$
$$+(m_1^{(0)})^2G_0G_1]+2p_2m_1^{(0)}p_1^{(-1)}G_1H_1+2p_2m_1^{(0)}p_1^{(-1)}G_0H_2+2p_2m_1^{(0)}p_1^{(0)}G_0H_1$$
$$+2p_2m_1^{(1)}p_1^{(-1)}G_0H_1+2p_2p_1^{(0)}p_1^{(-1)}H_1^2+2p_2(p_1^{(-1)})^2H_1H_2.$$

由定义　$\tilde{\beta}=\beta_\tau/\langle\beta_\tau\rangle=\tilde{\beta}^{(0)}+\tilde{\beta}^{(1)}\epsilon+\tilde{\beta}^{(2)}\epsilon^2+\cdots\cdots,$　(46)

其中　$\tilde{\beta}^{(0)}=\beta_\tau^{(-1)}/\langle\beta_\tau\rangle^{(-1)}$，$\tilde{\beta}^{(1)}=\beta\tau^{(0)}\langle\beta_\tau\rangle^{(-1)}-\beta^{(-1)}\langle\beta_\tau\rangle^{(0)}$,

$$\tilde{\beta}^{(2)}=2[\beta_\tau^{(1)}-\beta_\tau^{(-1)}\langle\beta_\tau\rangle^{(1)}-\beta_\tau^{(0)}\langle\beta_\tau\rangle^{(0)}+\beta_\tau^{(-1)}(\langle\beta_\tau\rangle^{(0)})^2].$$

把$\tilde{\eta}$按ϵ的级数展开为

$$\tilde{\eta}=\tilde{\eta}^{(0)}+\tilde{\eta}^{(1)}\epsilon+\tilde{\eta}^{(2)}\epsilon^2+\cdots\cdots \tag{47}$$

由(25)式得　$\tilde{\eta}^{(0)}=\frac{\mu}{2}(J_0/J_{1b}-J_1\cos\theta/J_{2b})$,

$$\tilde{\eta}^{(1)}=-\frac{1}{2}\{(1+\mu^2G_0)m_1^{(1)}+[2\xi\cos\theta/\mu+2(1-\alpha)\mu\xi G_0\cos\theta+\mu^2G_1]m_1^{(0)}$$

$$+(\mu^2-\xi\cos\theta/\mu)p_1^{(0)}-[3\xi\cos\theta/2\mu-\mu^2H_2-2\mu(1-\alpha)\xi H_1\cos\theta]p_1^{(-1)}\},$$

$$\tilde{\eta}^{(2)}=-\frac{1}{2}\{(1+\mu^2G_0)m_1^{(2)}+[2\xi\cos\theta/\mu+\mu^2G_1+2\mu(1-\alpha)\xi G_0\cos\theta]m_1^{(1)}$$

$$+[3\xi^2\cos^2\theta/\mu^2+\mu^2G_2+3(1-\alpha)\xi^2G_0\cos^2\theta+2(1-\alpha)\mu\xi G_1\cos\theta]m_1^{(0)}$$

$$+(\mu^2H_1-\xi\cos\theta/\mu)p_1^{(1)}+[\mu^2H_2-3\xi^2\cos^2\theta/2\mu^2+2(1-\alpha)\mu\xi H_1\cos\theta]p_1^{(0)}$$

$$+[\mu^2H_3-2\xi^3\cos^3\theta/\mu^3+2(1-\alpha)\mu\xi H_2\cos\theta+3(1-\alpha)\xi^2\cos^2\theta H_1]p_1^{(-)}\}.$$

六、结　　论

图2给出了极限$\langle\beta_\tau\rangle$随ϵ变化的关系。从图可以看出当ϵ增加时$\langle\beta_\tau\rangle$减少，当ϵ较小时$1/\epsilon$阶的解已足够精确了。但是，当ϵ较大时(比如说$\epsilon=0.3$)，高阶解给出了约20%的修正。因此，对大的ϵ必须将流函数展到ϵ的较高的量级。

图3给出了$\langle\beta_\phi\rangle/\langle\eta\rangle^2$对$\epsilon$的关系。同图2一样，当$\epsilon$较大时必须考虑较高阶的解。图4、5，分别是$\tilde{\beta}$与$\tilde{\eta}$的分布。图6是流面图。$\tilde{\beta}$最大的地方在磁轴中心处。从图5的环向电流分布可以看出在环的内侧存在负电流。由(25)

$$\tilde{\eta}+\frac{1}{2\epsilon^2\langle\eta\rangle^2}\left[\frac{R_0^2}{R^2}\frac{d\tilde{f}^2}{d\mathscr{F}}+\langle\beta_\phi\rangle\frac{d\tilde{\beta}}{d\mathscr{F}}\right]=0\ .$$

$\frac{d\tilde{\beta}}{d\mathscr{F}}<0$说明压力总是指向流面的外法线的正方向。压力是由洛伦兹力来平衡，$\tilde{\eta}$是环向洛伦兹力，$\tilde{\eta}>0$表明环向洛伦兹力是压缩和限制等离子体向外扩张。$\tilde{\eta}<0$是加强等离子体向外扩张，$\tilde{\eta}<0$的出现，说明逆磁电流的存在。关于如何在计算上防止负的$\tilde{\eta}$出现，可以用条件$\tilde{\eta}_{\rho=1,\theta=0}\geqslant0$代替条件(35)来实现。

本文的平衡解可作为不稳定性分析之用。

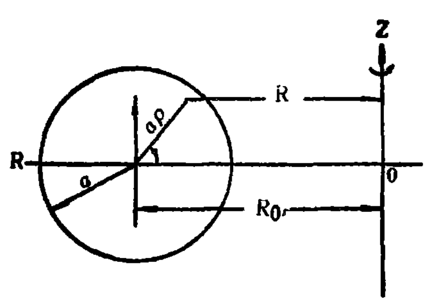

图1　坐标系

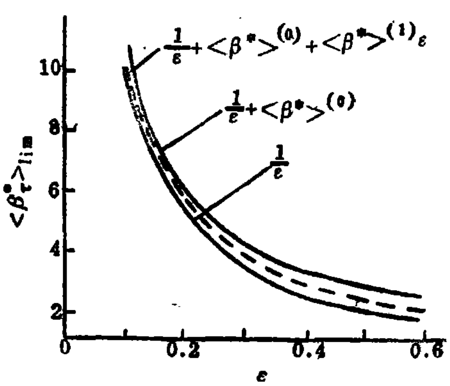

图2　$\langle\beta_\tau^*\rangle_{lim}$与$\epsilon$的关系

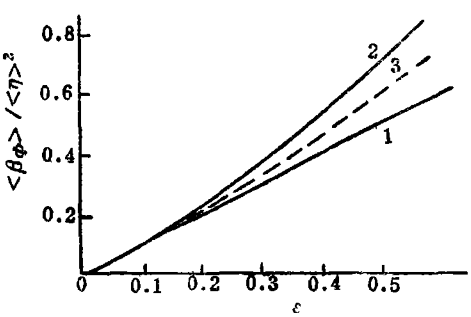

图3　$\langle\beta_\phi\rangle/\langle\eta\rangle^2$高阶解的影响

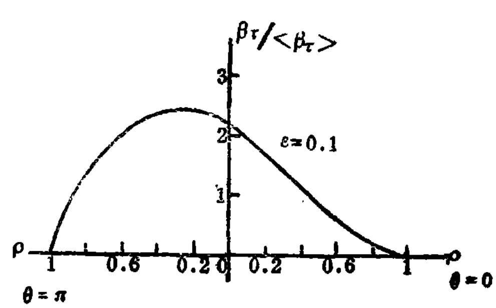

图4　$\beta_\tau/\langle\beta_\tau\rangle$的分布

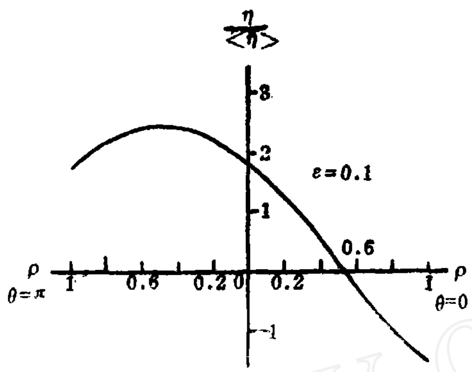

图5 $\eta/\langle\eta\rangle$的分布

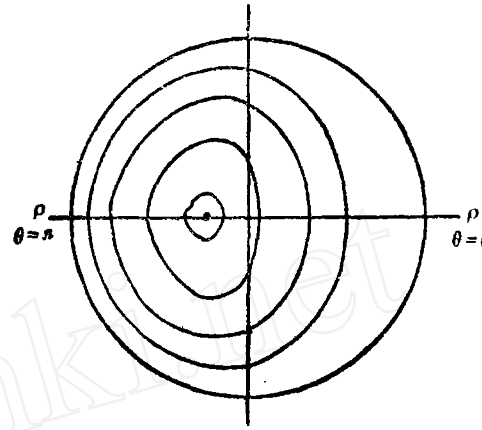

图6 流函数

张雅琴同志参加了本文数值计算的部分工作，特此致谢。

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THE ANALYTICAL SOLUTION AND LIMITING$\langle\beta\tau\rangle$ FOR THE EQUILIBRIUM OF A FAT TOKAMAK WITH CIRCULAR CROSS-SECTION

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ABSTRACT

The equilibrium of a fat tokamak with circular cross-section is treated analytically to order ϵ^3, by the method of expansion in powers of the inverse aspect ratio ϵ, based on Mercier's fundamental equations. The limiting $\langle\beta_\tau\rangle$ is obtained with restriction that no negative pressure appears. The analytical solution may be used to study the properties of instabilities.