第3期　　固体力学学报　　№3
1983年9月　　ACTA MECHANICA SOLIDA SINICA　　September 1983

承受弯曲的板在裂纹顶端附近的应力和变形

柳春图

（中国科学院力学研究所）

1. 引言 板壳结构是工程中常用的一种受力结构元件，对于有裂纹板的弯曲问题已经发表了一批著作，绝大部分是采用了克希霍夫经典板理论，应用此理论进行断裂分析在理论上是有重大缺陷的．这是由于经典板理论对于裂纹面边界条件只是近似地满足，因此不能正确反映裂纹尖端附近的力学性质．对于Reissner[1]理论的分析工作从六十年代以来就开始了．由于问题的复杂性，长期进展不快．有代表性的是Knowlts和Wang[2]以及Hartranft和Sih[3]的工作，他们用积分变换的方法求解问题，他们得到了问题的首项表达式．这是一个近似的形式．为了更好的了解有裂纹情况下板的力学性能以及更可靠的进行近似分析和计算应力强度因子的目的，类似于平面问题的Williams展开，本文对含有裂纹的Reissner型板的弯曲问题进行了分析，具体给出了前若干项广义位移和广义内力的表达式．

2. 基本方程和求解步骤 考虑一块含有一直线裂纹板的弯曲问题，取原点在裂纹顶端上，x轴在裂纹方向，y轴垂直于裂纹方向，若取极坐标r,θ（图1所示），考虑Reissner型板的三广义位移的弯曲理论，取ψ_r,ψ_θ，w为基本未知量，此时有平衡方程：

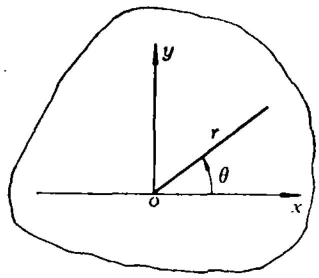

图1

$$\begin{aligned}&\frac{\partial \theta_r}{\partial r}+\frac{1}{r}\left(Q_r+\frac{\partial Q_\theta}{\partial \theta}\right)+P=0\\&\frac{\partial M_r}{\partial r}+\frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta}+\frac{M_r-M_\theta}{r}-Q_r=0 \qquad (1)\\&\frac{1}{r}\frac{\partial M_\theta}{\partial \theta}+\frac{\partial M_{r\theta}}{\partial r}+\frac{2M_{r\theta}}{r}-Q_\theta=0\end{aligned}$$

其中M_r，M_θ，$M_{r\theta}$为内力矩，Q_r，Q_θ为横向剪力，P为横向载荷，广义力与广义位移关系为

$$M_r=-D\left[\frac{\partial \psi_r}{\partial r}+\nu\left(\frac{1}{r}\frac{\partial \psi_\theta}{\partial \theta}+\frac{\psi_r}{r}\right)\right]$$

本文于1981年8月1日收到。

$$M_\theta=-D\left(\frac{1}{r}\frac{\partial\psi_\theta}{\partial\theta}+\frac{\psi_r}{r}+\nu\frac{\partial\psi_r}{\partial r}\right)$$

$$M_{r\theta}=-\frac{D}{2}(1-\nu)\left[\frac{1}{r}\frac{\partial\psi_r}{\partial\theta}+\frac{\partial\psi_\theta}{\partial r}-\frac{\psi_\theta}{r}\right]\tag{2}$$

$$Q_r=C\left(\frac{\partial w}{\partial r}-\psi_r\right)$$

$$Q_\theta=C\left(\frac{1}{r}\frac{\partial w}{\partial\theta}-\psi_\theta\right)$$

其中D为板的弯曲刚度，C为板的剪切刚度，ν为泊松比，把(2)式代入(1)式得

$$D\left[\frac{\partial^2\psi_r}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_r}{\partial r}-\frac{\psi_r}{r^2}+\frac{1-\nu}{2r^2}\frac{\partial^2\psi_r}{2\theta^2}+\frac{1+\nu}{2r}\frac{\partial^2\psi_\theta}{\partial r\partial\theta}-\frac{3-\nu}{2r^2}\frac{\partial\psi_\theta}{\partial\theta}\right]$$
$$+C\left(\frac{\partial w}{\partial r}-\psi_r\right)=0$$

$$D\left[\frac{1+\nu}{2r}\frac{\partial^2\psi_r}{\partial r\partial\theta}+\frac{3-\nu}{2r^2}\frac{\partial\psi_r}{\partial\theta}+\frac{1-\nu}{2}\frac{\partial^2\psi_\theta}{\partial r^2}+\frac{1-\nu}{2r}\frac{\partial\psi_\theta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi_\theta}{\partial\theta^2}-\frac{1-\nu}{2r^2}\psi_\theta\right]$$
$$+C\left(\frac{1}{r}\frac{\partial w}{\partial\theta}-\psi_\theta\right)=0\tag{3}$$

$$C\left[\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}-\left(\frac{\partial\psi_r}{\partial r}+\frac{\psi_r}{r}+\frac{1}{r}\frac{\partial\psi_\theta}{\partial\theta}\right)\right]+P=0$$

裂纹边界条件，当$\theta=\pm\pi$时

$$M_\theta=M_{r\theta}=Q_\theta=0\tag{4}$$

(3)式是以ψ_r，ψ_θ，w为未知量的极坐标形式的基本方程，(4)式为裂纹边界条件．本文考虑$P=0$的情况即只考虑由于边界力的作用而使板弯曲，对于上述方程，我们把ψ_r，ψ_θ，w展开为推广幂级数的形式代入基本方程和边界条件按逐项分别处理求得．由特征展开可求得全解的展开式．令

$$\begin{aligned}\psi_r^{(\lambda)}&=r^\lambda(a_0^\lambda(\theta)+a_1^\lambda(\theta)r+a_2^\lambda(\theta)r^2+\cdots)\\ \psi_\theta^{(\lambda)}&=r^\lambda(b_0^\lambda(\theta)+b_1^\lambda(\theta)r+b_2^\lambda(\theta)r^2+\cdots)\\ w^{(\lambda)}&=r^\lambda(c_0^\lambda(\theta)+c_1^\lambda(\theta)r+c_2^\lambda(\theta)r^2+\cdots)\end{aligned}\tag{5}$$

以下求解步骤可按上述分别求得$a_i(\theta)$，$b_i(\theta)$，$c_i(\theta)$再按λ求和即得ψ_r，ψ_θ，w的全解．

3．前几项展开式的具体表达式　1)特征值和$a_0(\theta)$，$b_0(\theta)$，$c_0(\theta)$的表示式

由(5)式代入基本方程中$r^{\lambda-2}$次有关项整理可得

$$(\lambda^2-1)a_0+\frac{1-\nu}{2}a_0''+\left(\frac{1+\nu}{2}\lambda-\frac{3-\nu}{2}\right)b_0'=0$$

$$\left(\frac{1+\nu}{2}\lambda+\frac{3-\nu}{2}\right)a_0'+\frac{1-\nu}{2}(\lambda^2-1)b_0+b_0''=0\tag{6}$$

$$c_0''+\lambda^2c_0=0$$

(6)式经一些数学运算可求得

$$\begin{aligned}a_0(\theta)&=A_{01}\cos(\lambda+1)\theta+A_{02}\sin(\lambda+1)\theta+A_{03}\cos(\lambda-1)\theta+A_{04}\sin(\lambda-1)\theta\\ b_0(\theta)&=B_{01}\sin(\lambda+1)\theta+B_{02}\cos(\lambda+1)\theta+B_{03}\sin(\lambda-1)\theta+B_{04}\cos(\lambda-1)\theta\\ c_0(\theta)&=D_{01}\cos\lambda\theta+D_{02}\sin\lambda\theta\end{aligned}\tag{7}$$

其中

$$B_{01}=-A_{01},\qquad B_{02}=A_{02}$$
$$B_{03}=-\frac{(1+\nu)\lambda+3-\nu}{(1+\nu)\lambda-(3-\nu)}A_{03},\qquad B_{04}=\frac{(1+\nu)\lambda+3-\nu}{(1+\nu)\lambda-(3-\nu)}A_{04} \tag{8}$$

由边界条件取与方程的对应项，即

当 $\theta=\pm\pi$ 时，

$$\begin{aligned}&(1+\nu\lambda)a_0+b_0'=0\\&a_0'+(\lambda-1)b_0=0\\&c_0'=0\end{aligned} \tag{9}$$

由(7)式和(9)式经一些运算即可求得以下解答：

当 $\lambda=\frac{1}{2},\ \frac{3}{2},\ \cdots$时，

$$\begin{aligned}a_0(\theta)&=A_{01}\left[\cos(\lambda+1)\theta-\frac{(1+\nu)\lambda-(3-\nu)}{(1+\nu)(\lambda-1)}\cos(\lambda-1)\theta\right]\\&+A_{02}\left[\sin(\lambda+1)\theta-\frac{(1+\nu)\lambda-(3-\nu)}{(1+\nu)(\lambda+1)}\sin(\lambda-1)\theta\right]\\b_0(\theta)&=A_{01}\left[-\sin(\lambda+1)\theta+\frac{(1+\nu)\lambda+3-\nu}{(1+\nu)(\lambda-1)}\sin(\lambda-1)\theta\right]\\&+A_{02}\left[\cos(\lambda+1)\theta\frac{(1+\nu)\lambda+3-\nu}{(1+\nu)(\lambda+1)}\cos(\lambda-1)\theta\right]\\c_0(\theta)&=D_{02}\sin\lambda\theta\end{aligned} \tag{10}$$

当 $\lambda=1,2,\cdots$时，

$$\begin{aligned}a_0(\theta)&=A_{01}\left[\cos(\lambda+1)\theta-\frac{(1+\nu)\lambda-(3-\nu)}{(1+\nu)(\lambda+1)}\cos(\lambda-1)\theta\right]\\&+A_{02}\left[\sin(\lambda+1)\theta-\frac{(1+\nu)\lambda-(3-\nu)}{(1+\nu)(\lambda-1)}\sin(\lambda-1)\theta\right]\\b_0(\theta)&=A_{01}\left[-\sin(\lambda+1)\theta+\frac{(1+\nu)\lambda+3-\nu}{(1+\nu)(\lambda+1)}\sin(\lambda-1)\theta\right]\\&+A_{02}\left[\cos(\lambda+1)\theta-\frac{(1+\nu)\lambda+3-\nu}{(1+\nu)(\lambda-1)}\cos(\lambda-1)\theta\right]\\c_0(\theta)&=D_{01}\cos\lambda\theta\end{aligned} \tag{11}$$

2) $a_1(\theta)$，$b_1(\theta)$，$c_1(\theta)$的表示式

与上节类似整理基本方程中 $r^{\lambda-1}$ 次有关项，有

$$\begin{aligned}&D\left[a_1\lambda(\lambda+2)+\frac{1-\nu}{2}a_1''+\left(\frac{\lambda(1+\nu)}{2}+\nu-1\right)b_1'\right]+Cc_0\lambda=0\\&D\left[\left(\frac{\lambda(1+\nu)}{2}+2\right)a_1'+b_1\lambda(\lambda+2)\ \frac{1-\nu}{2}+b_1''\right]+Cc_0'=0\\&c_1(\lambda+1)^2+c_1''-(\lambda+1)a_0-b_0'=0\end{aligned} \tag{12}$$

方程(12)的齐次方程解为

$$a_1(\theta)=A_{11}\cos(\lambda+2)\theta+A_{12}\sin(\lambda+2)\theta+A_{13}\cos\lambda\theta+A_{14}\sin\lambda\theta$$
$$b_1(\theta)=-A_{11}\sin(\lambda+2)\theta+A_{12}\cos(\lambda+2)\theta-A_{13}\frac{\lambda(1+\nu)+4}{\lambda(1+\nu)+2(\nu-1)}$$

$$+A_{14}\frac{\lambda(1+\nu)+4}{\lambda(1+\nu)+2(\nu-1)}\cos\lambda\theta$$

$$c_1(\theta)=D_{11}\cos(\lambda+1)\theta+D_{12}\sin(\lambda+1)\theta \tag{13}$$

特解可取为

$$a_{1特}=-\frac{\frac{C}{D}D_{01}}{\frac{\lambda(1+\nu)}{2}+2}\cos\lambda\theta-\frac{\frac{C}{D}D_{02}}{\frac{\lambda(1+\nu)}{2}+2}\sin\lambda\theta$$

$$b_{1特}=0 \tag{14}$$

$$c_{1特}=\frac{\nu-1}{\lambda(1+\nu)-(3-\nu)}[A_{03}\cos(\lambda-1)\theta+A_{02}\sin(\lambda-1)\theta]$$

把解代入下列边界条件

当 $\theta=\pm\pi$ 时

$$a_1[1+\nu(\lambda+1)]+b_1'=0$$

$$a_1'+\lambda b_1=0 \tag{15}$$

$$c_1'-b_0=0$$

再适当选取多余参数，可得 $a_1(\theta)$，$b_1(\theta)$，$c_1(\theta)$ 表达式如下：

当 $\lambda=\frac{1}{2}$，$\frac{3}{2}$，…时

$$a_1(\theta)=D_{02}\,k^2\frac{\lambda}{\lambda+1}[\sin(\lambda+2)\theta-\sin\lambda\theta]$$

$$b_1(\theta)=-D_{02}\,k^2\frac{\lambda+2}{\lambda+1}\left\{\cos\lambda\theta-\frac{\lambda}{\lambda+2}\cos(\lambda+2)\theta\right\}$$

$$c_1(\theta)=\frac{A_{02}(1-\nu)}{(\lambda+1)(1+\nu)}\left\{\sin(\lambda-1)\theta+\frac{4-(\lambda+1)(1-\nu)}{(\lambda+1)(1-\nu)}\sin(\lambda+1)\theta\right\} \tag{16}$$

$$+\frac{A_{01}(1-\nu)}{(\lambda-1)(1+\nu)}\left\{\cos(\lambda-1)\theta-\frac{4+(\lambda-1)(1-\nu)}{(\lambda+1)(1-\nu)}\cos(\lambda+1)\theta\right\}$$

当 $\lambda=1$，2，3…时

$$a_1(\theta)=D_{01}\,k^2\frac{\lambda}{\lambda+1}[\cos(\lambda+2)\theta-\cos\lambda\theta]$$

$$b_1(\theta)=D_{01}\,k^2\frac{\lambda+2}{\lambda+1}\left\{\sin\lambda\theta-\frac{\lambda}{\lambda+2}\sin(\lambda+2)\theta\right\}$$

$$c_1(\theta)=\frac{A_{01}(1-\nu)}{(\lambda+1)(1+\nu)}\left\{\cos(\lambda-1)\theta+\frac{4-(\lambda+1)(1-\nu)}{(\lambda+1)(1-\nu)}\cos(\lambda+1)\theta\right\} \tag{17}$$

$$+\frac{A_{02}(1-\nu)}{(\lambda-1)(1+\nu)}\left\{\sin(\lambda-1)\theta-\frac{4+(\lambda-1)(1-\nu)}{(\lambda+1)(1-\nu)}\sin(\lambda+1)\theta\right\}$$

式中
$$k^2=\frac{C}{2D(1-\nu)}$$

类似于以上做法，逐项可求得 a_i，b_i，c_i 各个函数，在求解各项函数的微分方程时，齐次方程的解有一定规律，可一次求得解答．系数有一定关系，表示如下：

$$a_i(\theta)=A_{i1}\cos(\lambda+i+1)\theta+A_{i2}\sin(\lambda+i+1)\theta+A_{i3}\cos(\lambda+i-1)\theta$$
$$+A_{i4}\sin(\lambda+i-1)\theta$$

$$b_i(\theta)=B_{i1}\sin(\lambda+i+1)\theta+B_{i2}\cos(\lambda+i+1)\theta+B_{i3}\sin(\lambda+i-1)\theta$$

$$+B_{i4}\cos(\lambda+i-1)\theta \tag{18}$$

$$c_i(\theta)=D_{i1}\cos(\lambda+i)\theta+D_{i2}\sin(\lambda+i)\theta$$

其中：

$$B_{i1}=-A_{i1},\quad B_{i2}=A_{i2}$$

$$B_{i3}=-\frac{(\lambda+i)(\nu-1)+3-\nu}{(\lambda+i)(\nu+1)-(3-\nu)},\quad B_{i4}=\frac{(\lambda+i)(\nu+1)+3-\nu}{(\lambda+i)(\nu+1)-(3-\nu)} \tag{19}$$

4．广义位移和广义内力展开式　若取 ψ_r，ψ_θ，w 至 r^3 项，再改变前面符号把 A_{01} 写为 A_1，A_{02} 写为 A_2，D_{01} 写为 D_1，D_{02} 写为 D_2，则广义位移和广义内力展开式为：

$$\begin{aligned}\psi_r=&r^{\frac12}\left[A_1^{(\frac12)}\left(\cos\frac32\theta+\frac{3\nu-5}{1+\nu}\cos\frac\theta2\right)+A_2^{(\frac12)}\left(\sin\frac32\theta+\frac{3\nu-5}{3(1+\nu)}\sin\frac\theta2\right)\right]\\&+r^{\frac32}\frac{k^2}{3}D_2^{(\frac12)}\left(\sin\frac52\theta-\sin\frac\theta2\right)+A_1^{(1)}r\left(\cos2\theta+\frac{1-\nu}{1+\nu}\right)\\&+D_1^{(1)}r^2\frac{k^2}{2}(\cos3\theta-\cos\theta)+r^{\frac32}\left[A_1^{(\frac32)}\left(\cos\frac52\theta-\frac{5\nu-3}{1+\nu}\cos\frac\theta2\right)\right.\\&\left.+A_2^{(\frac32)}\left(\sin\frac52\theta-\frac{5\nu-3}{5(1+\nu)}\sin\frac\theta2\right)\right]+D_2^{(\frac32)}r^{\frac52}\frac{3k^2}{5}\left(\sin\frac72\theta-\sin\frac32\theta\right)\\&+r^2\left[A_1^{(2)}\left(\cos3\theta-\frac{3\nu-1}{3(1+\nu)}\cos\theta\right)+A_2^{(2)}\left(\sin3\theta-\frac{3\nu-1}{1+\nu}\sin\theta\right)\right]\\&+D_1^{(2)}r^3\frac{2k^2}{3}(\cos4\theta-\cos2\theta)+O\left(r^{\frac72}\right)\end{aligned} \tag{20}$$

$$\begin{aligned}\psi_\theta=&r^{\frac12}\left[A_1^{(\frac12)}\left(-\sin\frac32\theta+\frac{7-\nu}{1+\nu}\sin\frac\theta2\right)+A_2^{(\frac12)}\left(\cos\frac32\theta-\frac{7-\nu}{3(1+\nu)}\cos\frac\theta2\right)\right]\\&-D_2^{(\frac12)}r^{\frac32}\frac{5k^2}{3}\left(\cos\frac\theta2-\frac15\cos\frac52\theta\right)-A_1^{(1)}r\sin2\theta\\&+D_1^{(1)}r^2\frac{3k^2}{2}\left(\sin\theta-\frac13\sin3\theta\right)+r^{\frac32}\left[A_1^{(\frac32)}\left(-\sin\frac52\theta+\frac{9+\nu}{1+\nu}\sin\frac\theta2\right)\right.\\&\left.+A_2^{(\frac32)}\left(\cos\frac52\theta-\frac{9+\nu}{5(1+\nu)}\cos\frac\theta2\right)\right]-D_2^{(\frac32)}r^{\frac52}\frac{7k^2}{5}\left(\cos\frac32\theta-\frac37\cos\frac72\theta\right)\\&+r^2\left[A_1^{(2)}\left(-\sin3\theta+\frac{5+\nu}{3(1+\nu)}\sin\theta\right)+A_2^{(2)}\left(\cos3\theta-\frac{5+\nu}{1+\nu}\cos\theta\right)\right]\\&+D_1^{(2)}r^3\frac{4k^2}{3}\left(\sin2\theta-\frac12\sin4\theta\right)+O\left(r^{\frac72}\right)\end{aligned} \tag{21}$$

$$\begin{aligned}w=&r^{\frac12}D_2^{(\frac12)}\sin\frac\theta2+rD_1^{(1)}\cos\theta+r^{\frac32}D_2^{(\frac32)}\sin\frac32\theta\\&+r^{\frac32}A_2^{(\frac12)}\cdot\frac{2(1-\nu)}{3(1+\nu)}\left[-\sin\frac\theta2+\frac{5+3\nu}{3(1-\nu)}\sin\frac32\theta\right]\\&-r^{\frac32}A_1^{(\frac12)}\frac{2(1-\nu)}{1+\nu}\left[\cos\frac\theta2-\frac{7+\nu}{3(1-\nu)}\cos\frac32\theta\right]+r^2D_1^{(2)}\cos2\theta\\&+r^2A_1^{(1)}\frac{1-\nu}{2(1+\nu)}\left(1+\frac{1+\nu}{1-\nu}\cos2\theta\right)\\&+r^{\frac52}D_2^{(\frac52)}\sin\frac52\theta+r^{\frac52}A_2^{(\frac32)}\frac{2(1-\nu)}{5(1+\nu)}\left[\sin\frac\theta2+\frac{3+5\nu}{5(1-\nu)}\sin\frac52\theta\right]\end{aligned}$$

$$+r^{\frac{5}{2}}A_1^{(\frac{3}{2})}\frac{2(1-\nu)}{1+\nu}\left[\cos\frac{\theta}{2}-\frac{9-\nu}{5(1-\nu)}\cos\frac{5}{2}\theta\right]$$

$$+r^3D_1^{(3)}\cos 3\theta+r^3A_1^{(2)}\frac{1-\nu}{3(1+\nu)}\left[\cos\theta+\frac{1+3\nu}{3(1-\nu)}\cos 3\theta\right]$$

$$+r^3A_2^{(2)}\frac{1-\nu}{1+\nu}\left[\sin\theta-\frac{5-\nu}{3(1-\nu)}\sin 3\theta\right]+O(r^{\frac{7}{2}}) \tag{22}$$

$$-\frac{M_r}{D}=r^{-\frac{1}{2}}\frac{1}{2}(1-\nu)\left[A_1^{(\frac{1}{2})}(\cos\frac{3}{2}\theta-5\cos\frac{\theta}{2})+A_2^{(\frac{1}{2})}(\sin\frac{3}{2}\theta-\frac{5}{3}\sin\frac{\theta}{2})\right]$$

$$+\frac{C}{4D}D_2^{(\frac{1}{2})}r^{\frac{1}{2}}(\sin\frac{5}{2}\theta-\sin\frac{\theta}{2})+A_1^{(1)}(1-\nu)(\cos 2\theta+1)$$

$$+D_1^{(1)}\frac{C}{2D}r(\cos 3\theta-\cos\theta)+r^{\frac{1}{2}}\frac{3}{2}(1-\nu)\left[A_1^{(\frac{3}{2})}(\cos\frac{5}{2}\theta+3\cos\frac{\theta}{2})\right.$$

$$\left.+A_2^{(\frac{3}{2})}(\sin\frac{5}{2}\theta+\frac{3}{5}\sin\frac{\theta}{2})\right]+2r(1-\nu)\left[A_1^{(2)}(\cos 3\theta+\frac{1}{3}\cos\theta)\right.$$

$$\left.+A_2^{(2)}(\sin 3\theta+\sin\theta)\right]+O(r^{\frac{3}{2}}) \tag{23}$$

$$-\frac{M_\theta}{D}=r^{-\frac{1}{2}}\frac{1}{2}(\nu-1)\left[A_1^{(\frac{1}{2})}(\cos\frac{3}{2}\theta+3\cos\frac{\theta}{2})+A_2^{(\frac{1}{2})}(\sin\frac{3}{2}\theta+\sin\frac{\theta}{2})\right]$$

$$-D_2^{(\frac{1}{2})}\frac{C}{4D}r^{\frac{1}{2}}(\sin\frac{5}{2}\theta-\sin\frac{\theta}{2})+A_1^{(1)}(1-\nu)(1-\cos 2\theta)$$

$$-D_1^{(1)}\frac{C}{2D}r(\cos 3\theta-\cos\theta)+r^{\frac{1}{2}}\frac{3}{2}(\nu-1)\left[A_1^{(\frac{3}{2})}(\cos\frac{5}{2}\theta-5\cos\frac{\theta}{2})\right.$$

$$\left.+A_2^{(\frac{3}{2})}(\sin\frac{5}{2}\theta-\sin\frac{\theta}{2})\right]+2r(\nu-1)[A_1^{(2)}(\cos 3\theta-\cos\theta)$$

$$+A_2^{(2)}(\sin 3\theta-3\sin\theta)]+O(r^{\frac{3}{2}}) \tag{24}$$

$$-\frac{2M_{r\theta}}{D(1-\nu)}=r^{-\frac{1}{2}}\left[-A_1^{(\frac{1}{2})}(\sin\frac{\theta}{2}+\sin\frac{3}{2}\theta)+A_2^{(\frac{1}{2})}(\cos\frac{3}{2}\theta+\frac{1}{3}\cos\frac{\theta}{2})\right]$$

$$+D_2^{(\frac{1}{2})}r^{\frac{1}{2}}k^2(\cos\frac{5}{2}\theta-\cos\frac{\theta}{2})-2A_1^{(1)}\sin 2\theta$$

$$+D_1^{(1)}r\,2k^2(\sin\theta-\sin 3\theta)+3r^{\frac{1}{2}}\left[A_1^{(\frac{3}{2})}(\sin\frac{\theta}{2}-\sin\frac{5}{2}\theta)\right.$$

$$\left.+A_2^{(\frac{3}{2})}(\cos\frac{5}{2}\theta-\frac{1}{5}\cos\frac{\theta}{2})\right]+4r\left[A_1^{(2)}(-\sin 3\theta+\frac{1}{3}\sin\theta)\right.$$

$$\left.+A_2^{(2)}(\cos 3\theta-\cos\theta)\right]+O(r^{\frac{3}{2}}) \tag{25}$$

$$\frac{Q_r}{C}=r^{\frac{1}{2}}\left[A_1^{(\frac{1}{2})}\frac{2}{1+\nu}(\cos\frac{\theta}{2}+3\cos\frac{3}{2}\theta)+\frac{2}{3(1+\nu)}A_2^{(\frac{1}{2})}(\sin\frac{\theta}{2}+\sin\frac{3}{2}\theta)\right]$$

$$+D_2^{(\frac{1}{2})}\left[\frac{1}{2}r^{-\frac{1}{2}}\sin\frac{\theta}{2}+\frac{k^2}{3}r^{\frac{3}{2}}(\sin\frac{\theta}{2}-\sin\frac{5\theta}{2})\right]$$

$$+D_1^{(1)}\left[\cos\theta+\frac{k^2}{2}r^2(\cos\theta-\cos 3\theta)\right]$$

$$+r^{\frac{3}{2}}\left[A_1^{(\frac{3}{2})}\frac{2}{1+\nu}(\cos\frac{\theta}{2}-5\cos\frac{5}{2}\theta)+A_2^{(\frac{3}{2})}\frac{2}{5(1+\nu)}(\sin\frac{\theta}{2}-\sin\frac{5\theta}{2})\right]$$

$$+D_2^{(\frac{3}{2})}\left[r^{\frac{1}{2}}\frac{3}{2}\sin\frac{3}{2}\theta-r^{\frac{5}{2}}\frac{3k^2}{5}(\sin\frac{7}{2}\theta-\sin\frac{3}{2}\theta)\right]$$

$$
\begin{aligned}
&+r^2\left[A_1^{(2)}\frac{2}{3(1+\nu)}(\cos\theta-\cos 3\theta)+A_2^{(2)}\frac{2}{1+\nu}(\sin\theta-3\sin 3\theta)\right]\\
&+D_1^{(2)}\left[2r\cos 2\theta-r^3\frac{2k^2}{3}(\cos 4\theta-\cos 2\theta)\right]+D_2^{(\frac{5}{2})}r^{\frac{3}{2}}\frac{5}{2}\sin\frac{5}{2}\theta+O(r^{\frac{7}{2}}) \qquad (26)
\end{aligned}
$$

$$
\begin{aligned}
\frac{Q_\theta}{C}=&\,r^{\frac{1}{2}}\left[-A_1^{(\frac{1}{2})}\frac{6}{1+\nu}\left(\sin\frac{\theta}{2}+\sin\frac{3}{2}\theta\right)+A_2^{(\frac{1}{2})}\frac{2}{1+\nu}\left(\cos\frac{\theta}{2}+\frac{1}{3}\cos\frac{3}{2}\theta\right)\right]\\
&+D_2^{(\frac{1}{2})}\left[r^{-\frac{1}{2}}\frac{1}{2}\cos\frac{\theta}{2}+r^{\frac{3}{2}}\frac{5k^2}{3}\left(\cos\frac{\theta}{2}-\frac{1}{5}\cos\frac{5}{2}\theta\right)\right]\\
&+D_1^{(1)}\left[-\sin\theta-\frac{3}{2}k^2r^2\left(\sin\theta-\frac{1}{3}\sin 3\theta\right)\right]\\
&+r^{\frac{3}{2}}\left[A_1^{(\frac{3}{2})}\frac{10}{1+\nu}\left(\sin\frac{5}{2}\theta-\sin\frac{\theta}{2}\right)+A_2^{(\frac{3}{2})}\frac{2}{1+\nu}\left(\cos\frac{\theta}{2}-\frac{1}{5}\cos\frac{5}{2}\theta\right)\right]\\
&+D_2^{(\frac{3}{2})}\left[r^{\frac{1}{2}}\frac{3}{2}\cos\frac{3}{2}\theta+r^{\frac{5}{2}}\frac{7k^2}{5}\left(\cos\frac{3}{2}\theta-\frac{3}{7}\cos\frac{7}{2}\theta\right)\right]\\
&+r^2\left[A_1^{(2)}\frac{2}{1+\nu}\left(-\sin\theta+\frac{1}{3}\sin 3\theta\right)+A_2^{(2)}\frac{6}{1+\nu}(\cos\theta-\cos 3\theta)\right]\\
&-D_1^{(2)}\left[2r\sin 2\theta+r^3\frac{4k^2}{3}\left(\sin 2\theta-\frac{1}{2}\sin 4\theta\right)\right]+D_2^{(\frac{5}{2})}r^{\frac{3}{2}}\frac{5}{2}\cos\frac{5}{2}\theta+O(r^{\frac{7}{2}})
\end{aligned}
$$

(27)

以上式中　$k^2=\dfrac{C}{2D(1-\nu)}=\dfrac{5}{2h^2}$

在上式中，系数的上标表示特征值数．由 $u_r=-z\psi_r$ 和 $u_\theta=-z\psi_\theta$ 可求得正内位移展开式．

5．讨论

1）类似于平面问题中的 Williams 展开，本文求得了 Reissner 型中厚板弯曲问题的应力应变场的展开式．

2）本文中求得的 a_0，b_0，c_0 的第一项与文献[2][3]结果一致．

3）在求解微分方程中，未知函数 ψ_r，ψ_θ，是可以与未知函数 w 分离开的，在解齐次方程中，各项未知函数可以直接写出解答，且有一定规律如(18)，(19)两式所示．

4）由 a_i，b_i，c_i 式中可知当距离 r 趋近于零时或当 $\dfrac{C}{D}$ 趋近于零时，板的应力状态接近于平面应力状态．

5）本文结果可以作为进行近似分析和数值计算应力强度因子的基础．原则上可把平面问题中应用 Williams 展开的一套计算方法推广到板的弯曲问题中去，如边界配置法，奇异元分析，应用能量法计算等．

参 考 文 献

〔1〕 Reissner E.,On Bending of Elastic Plates, Quarterly of Applied Mathematics, 5(1947),55—68.

〔2〕 Knowles J. K. and Wang N. M., On the Bending of an Elastic Plate Containing a Crack, Journal of Mathematics and Physics, 39(1960), 223—236.

〔3〕 Hartranft R.J.and Sih G.C., Effect of Plate Thickness on the Bending Stress Distribution around Through Cracks, Journal of Mathematics and Physics, 47(1968), 276—291.

〔4〕 Williams M.L., The Bending Stress Distribution at the Base of Stationary Crack, Journal of Applied Mechanics, 28(1961), 78—82.

STRESSES AND DEFORMATIONS NEAR THE CRACK TIP FOR BENDING PLATE

Liu Chun-tu

(Institute of Mechanics, Academia Sinica)