具有粘滞阻尼与非线性结构阻尼的四边固支板的振动响应

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提要　本文以正则化的完全满足四边固支板边条件的广义傅立叶坐标基 $f_i(x, y)$ 为基函数，将各阶振型展成广义傅立叶级数 $\Phi_j(x, y)=\sum_{i=1}^{\infty} A_{ij}f_i(x, y)$，求解了四边固支板在无阻尼情况下的自由振动．以后，以应力应变滞后一位相 ν 的非线性结构阻尼模型，建立了粘滞阻尼和结构阻尼共同影响下的振动微分方程，求得了它的自由振动解，正弦响应和随机响应．已编制程序，对长短比为 $a/b=2$ 的板进行了计算．结果表明：低阶频率和振型收敛较快，较高阶频率则要取足够的项数，方能有较为令人满意的结果．

一、无阻尼的自由振动

不考虑阻尼时，板的自由振动方程为：

$$D\nabla^2\nabla^2\Phi(x, y)-\rho h\omega^2\Phi(x, y)=0 \tag{1}$$

此处：

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

ω 薄板的固有频率

$\Phi(x, y)$ 薄板的固有振型

ρ 薄板的材料密度

h 薄板的厚度

设

$$\Phi(x, y)=\sum_{k=1}^{\infty} A_k f_k(x, y) \tag{2}$$

此处 $f_k(x, y)$ 为正则化的，完全满足四边固支板边条件的广义傅立叶坐标基．显然，选择 $f_k(x, y)$ 为下述基函数可满足上述要求．

$$f_k(x, y)=\left[U\left(\frac{\lambda_i}{a}x\right)-\frac{U(\lambda_i)}{V(\lambda_i)}V\left(\frac{\lambda_i}{a}x\right)\right]\left[V\left(\frac{\lambda_j}{b}y\right)-\frac{U(\lambda_j)}{V(\lambda_j)}V\left(\frac{\lambda_j}{b}y\right)\right]\Big/R_k \tag{3}$$

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其中：$i = k - \frac{1}{2} INT[(1+\sqrt{1+8(k-1)})/2]\{INT[(1+\sqrt{8(k-1)})/2]-1\}$

$j = INT[(1+\sqrt{1+8(k-1)})/2]+1-i$

INT 取整函数

a, b 矩形板的长和宽.

S, T, U, V 为克雷洛夫函数，其表达式为：

$$S(\xi) = \frac{1}{2}(\text{ch}\xi + \cos\xi)$$

$$T(\xi) = \frac{1}{2}(\text{sh}\xi + \sin\xi)$$

$$U(\xi) = \frac{1}{2}(\text{ch}\xi - \cos\xi)$$

$$V(\xi) = \frac{1}{2}(\text{sh}\xi - \sin\xi)$$

$$R_k = \left\{\frac{1}{4} + \frac{1}{8\lambda_i}\left[12U(\lambda_i)V(\lambda_i) + 4U^2(\lambda_i)\frac{S(\lambda_i)}{V(\lambda_i)} - (\text{ch}\lambda_i \sin\lambda_i + \text{sh}\lambda_i \cos\lambda_i) - \frac{U^2(\lambda_i)}{V^2(\lambda_i)}(\text{ch}\lambda_i \sin\lambda_i - \text{sh}\lambda_i \cos\lambda_i)\right]\right\}^{\frac{1}{2}} \cdot$$

$$\left\{\frac{1}{4} + \frac{1}{8\lambda_j}\left[12U(\lambda_j)V(\lambda_j) + 4U^2\frac{S(\lambda_j)}{V(\lambda_j)} - (\text{ch}\lambda_j \sin\lambda_j + \text{sh}\lambda_j \cos\lambda_j) - \frac{U^2(\lambda_j)}{V^2(\lambda_j)}(\text{ch}\lambda_j \sin\lambda_j - \text{sh}\lambda_j \cos\lambda_j)\right]\right\}^{\frac{1}{2}}$$

将(2)和(3)代入(1)，整理后可得：

$$A_k\left\{\left[\left(\frac{\lambda_i}{a}\right)^2 + \left(\frac{\lambda_j}{b}\right)^2\right]^2 - \lambda\right\}\left\{\frac{1}{4} + \frac{1}{8\lambda_i}\left[12U(\lambda_i)V(\lambda_i) + 4U^2(\lambda_i)\frac{S(\lambda_i)}{V(\lambda_i)} - (\text{ch}\lambda_i \sin\lambda_i + \text{sh}\lambda_i \cos\lambda_i) - \frac{U^2(\lambda_i)}{V^2(\lambda_i)}(\text{ch}\lambda_i \sin\lambda_i - \text{sh}\lambda_i \cos\lambda_i)\right]\right\} \cdot \left\{\frac{1}{4} + \frac{1}{8\lambda_j} \cdot \left[12U(\lambda_j)V(\lambda_j) + 4U^2(\lambda_j)\frac{S(\lambda_j)}{V(\lambda_j)} - (\text{ch}\lambda_j \sin\lambda_j + \text{sh}\lambda_j \cos\lambda_j) - \frac{U^2(\lambda_j)}{V^2(\lambda_j)}(\text{ch}\lambda_j \sin\lambda_j - \text{sh}\lambda_j \cos\lambda_j)\right]\right\}$$

$$+ \sum_{l=1}^{\infty} A_l \left(\frac{\lambda_m}{a}\right)^2\left(\frac{\lambda_n}{b}\right)^2 \cdot \frac{1}{2}\left\{\frac{\lambda_i}{\lambda_i^2+\lambda_m^2}\text{sh}\lambda_i \cos\lambda_m + \frac{\lambda_m}{\lambda_i^2+\lambda_m^2} \cdot \text{ch}\lambda_i \sin\lambda_m - \frac{V(\lambda_m)}{V(\lambda_m)}\left[\frac{\lambda_i}{\lambda_i^2+\lambda_m^2}\text{sh}\lambda_i \sin\lambda_m - \frac{\lambda_m}{\lambda_i^2+\lambda_m^2} \cdot (\text{ch}\lambda_i \cos\lambda_m - 1)\right] - \frac{1}{2}\left[\frac{\sin(\lambda_i - \lambda_m)}{\lambda_i - \lambda_m} + \frac{\sin(\lambda_i + \lambda_m)}{\lambda_i + \lambda_m}\right] + \frac{1}{2}\frac{U(\lambda_m)}{V(\lambda_m)}\left[\frac{\cos(\lambda_i - \lambda_m)}{\lambda_i - \lambda_m} - \frac{\cos(\lambda_i + \lambda_m)}{\lambda_i + \lambda_m}\right]$$

$$-\frac{U(\lambda_i)}{V(\lambda_i)}\left[\frac{\lambda_i}{\lambda_i^2+\lambda_m^2}(\mathrm{ch}\lambda_i\cdot\cos\lambda_m-1)+\frac{\lambda_m}{\lambda_i^2+\lambda_m^2}\mathrm{sh}\lambda_i\sin\lambda_m\right]$$

$$+\frac{U(\lambda_i)}{V(\lambda_i)}\frac{U(\lambda_m)}{V(\lambda_m)}\left[\frac{\lambda_i}{\lambda_i^2+\lambda_m^2}\cdot\mathrm{ch}\lambda_i\sin\lambda_m\right.$$

$$\left.-\frac{\lambda_m}{\lambda_i^2+\lambda_m^2}\mathrm{sh}\lambda_i\cos\lambda_m\right]-\frac{1}{2}\frac{U(\lambda_i)}{V(\lambda_i)}$$

$$\cdot\left[\frac{\cos(\lambda_i-\lambda_m)}{\lambda_i-\lambda_m}+\frac{\cos(\lambda_i+\lambda_m)}{\lambda_i+\lambda_m}\right]-\frac{1}{2}\frac{U(\lambda_i)}{V(\lambda_i)}\frac{U(\lambda_m)}{V(\lambda_m)}$$

$$\cdot\left[\frac{\sin(\lambda_i-\lambda_m)}{\lambda_i-\lambda_m}-\frac{\sin(\lambda_i+\lambda_m)}{\lambda_i+\lambda_m}\right]\Bigg\}\Bigg\{\frac{\lambda_j}{\lambda_j^2+\lambda_n^2}\mathrm{sh}\lambda_j\cos\lambda_n$$

$$+\frac{\lambda_n}{\lambda_j^2+\lambda_n^2}\mathrm{ch}\lambda_j\sin\lambda_n-\frac{U(\lambda_n)}{V(\lambda_n)}\left[\frac{\lambda_j}{\lambda_j^2+\lambda_n^2}\mathrm{sh}\lambda_j\sin\lambda_n\right.$$

$$\left.-\frac{\lambda_n}{\lambda_j^2+\lambda_n^2}(\mathrm{ch}\lambda_j\cos\lambda_n-1)\right]-\frac{1}{2}\left[\frac{\sin(\lambda_j-\lambda_n)}{\lambda_j-\lambda_n}\right.$$

$$\left.+\frac{\sin(\lambda_j+\lambda_n)}{\lambda_j+\lambda_n}\right]+\frac{1}{2}\frac{U(\lambda_n)}{V(\lambda_n)}\left[\frac{\cos(\lambda_j-\lambda_n)}{\lambda_j-\lambda_n}\right.$$

$$\left.-\frac{\cos(\lambda_j+\lambda_n)}{\lambda_j+\lambda_n}\right]-\frac{U(\lambda_j)}{V(\lambda_j)}\left[\frac{\lambda_j}{\lambda_j^2+\lambda_n^2}(\mathrm{ch}\lambda_j\cos\lambda_n-1)\right.$$

$$\left.+\frac{\lambda_n}{\lambda_j^2+\lambda_n^2}\mathrm{sh}\lambda_j\sin\lambda_n\right]+\frac{U(\lambda_j)}{V(\lambda_j)}\frac{U(\lambda_n)}{V(\lambda_n)}\left[\frac{\lambda_j}{\lambda_j^2+\lambda_n^2}\right.$$

$$\left.\cdot\mathrm{ch}\lambda_j\sin\lambda_n-\frac{\lambda_n}{\lambda_j^2+\lambda_n^2}\mathrm{sh}\lambda_j\cos\lambda_n\right]-\frac{1}{2}\frac{U(\lambda_j)}{V(\lambda_j)}$$

$$\cdot\left[\frac{\cos(\lambda_j-\lambda_n)}{\lambda_j-\lambda_n}+\frac{\cos(\lambda_j+\lambda_n)}{\lambda_j+\lambda_n}\right]-\frac{1}{2}\frac{U(\lambda_j)}{V(\lambda_j)}\frac{U(\lambda_n)}{V(\lambda_n)}$$

$$\cdot\left[\frac{\sin(\lambda_j-\lambda_n)}{\lambda_j-\lambda_n}-\frac{\sin(\lambda_j+\lambda_n)}{\lambda_j+\lambda_n}\right]\Bigg\}+\sum_{l_1=1}^{\infty}A_{l_1}\left(\frac{\lambda_{m_1}}{a}\right)^2$$

$$\cdot\left(\frac{\lambda_j}{b}\right)^2\Bigg\{\frac{\lambda_i}{\lambda_i^2+\lambda_{m_1}^2}\mathrm{sh}\lambda_i\cos\lambda_{m_1}+\frac{\lambda_{m_1}}{\lambda_i^2+\lambda_{m_1}^2}\mathrm{ch}\lambda_i\sin\lambda_{m_1}$$

$$-\frac{U(\lambda_{m_1})}{V(\lambda_{m_1})}\left[\frac{\lambda_i}{\lambda_i^2+\lambda_{m_1}^2}\mathrm{sh}\lambda_1\sin\lambda_{m_1}-\frac{\lambda_{m_1}}{\lambda_i^2+\lambda_{m_1}^2}(\mathrm{ch}\lambda_i\cos\lambda_{m_1}-1)\right]$$

$$-\frac{1}{2}\left[\frac{\sin(\lambda_i-\lambda_{m_1})}{\lambda_i-\lambda_{m_1}}+\frac{\sin(\lambda_i+\lambda_{m_1})}{\lambda_i+\lambda_{m_1}}\right]+\frac{1}{2}\frac{U(\lambda_{m_1})}{V(\lambda_{m_1})}$$

$$\cdot\left[\frac{\cos(\lambda_i-\lambda_{m_1})}{\lambda_i-\lambda_{m_1}}-\frac{\cos(\lambda_i+\lambda_{m_1})}{\lambda_i+\lambda_{m_1}}\right]-\frac{U(\lambda_i)}{V(\lambda_i)}\left[\frac{\lambda_i}{\lambda_i^2+\lambda_{m_1}^2}\right.$$

$$\left.\cdot(\mathrm{ch}\lambda_i\cos\lambda_{m_1}-1)+\frac{\lambda_{m_1}}{\lambda_i^2+\lambda_{m_1}^2}\mathrm{sh}\lambda_i\sin\lambda_{m_1}\right]+\frac{U(\lambda_i)}{V(\lambda_i)}\frac{U(\lambda_{m_1})}{V(\lambda_{m_1})}$$

$$\cdot\left[\frac{\lambda_i}{\lambda_i^2+\lambda_{m_1}^2}\mathrm{ch}\lambda_i\sin\lambda_{m_1}-\frac{\lambda_{m_1}}{\lambda_i^2+\lambda_{m_1}^2}\mathrm{sh}\lambda_i\cos\lambda_{m_1}\right]-\frac{1}{2}\frac{U(\lambda_i)}{V(\lambda_i)}$$

$$\cdot\left[\frac{\cos(\lambda_i-\lambda_{m_1})}{\lambda_i-\lambda_{m_1}}+\frac{\cos(\lambda_i+\lambda_{m_1})}{\lambda_i+\lambda_{m_1}}\right]-\frac{1}{2}\frac{U(\lambda_i)}{V(\lambda_i)}\frac{U(\lambda_{m_1})}{V(\lambda_{m_1})}$$

$$\cdot\left[\frac{\sin(\lambda_i-\lambda_{m_1})}{\lambda_i-\lambda_{m_1}}-\frac{\sin(\lambda_i+\lambda_{m_1})}{\lambda_i+\lambda_{m_1}}\right]\Big\}\Big\{\frac{1}{4}+\frac{1}{8\lambda_j}\Big[12U(\lambda_j)$$

$$\cdot V(\lambda_j)+4U^2(\lambda_j)\frac{S(\lambda_j)}{V(\lambda_j)}-(\mathrm{ch}\lambda_j\sin\lambda_j+\mathrm{sh}\lambda_j\cos\lambda_j)$$

$$-\frac{U^2(\lambda_j)}{V(\lambda_j)}(\mathrm{ch}\lambda_j\sin\lambda_j-\mathrm{sh}\lambda_j\cos\lambda_j)\Big]\Big\}+\sum_{l_2=1}^{\infty}A_{l_2}$$

$$\cdot\left(\frac{\lambda_i}{a}\right)^2\left(\frac{\lambda_{n_1}}{b}\right)^2\Big\{\frac{1}{4}+\frac{1}{8\lambda_i}\Big[12U(\lambda_i)V(\lambda_i)+4U^2(\lambda_i)\frac{S(\lambda_i)}{V(\lambda_i)}$$

$$-(\mathrm{ch}\lambda_i\sin\lambda_i+\mathrm{sh}\lambda_i\cos\lambda_i)-\frac{U^2(\lambda_i)}{V^2(\lambda_i)}(\mathrm{ch}\lambda_i\sin\lambda_i$$

$$-\mathrm{sh}\lambda_i\cos\lambda_i)\Big]\Big\}\Big\{\frac{\lambda_j}{\lambda_j^2+\lambda_{n_1}^2}\mathrm{sh}\lambda_j\cos\lambda_{n_1}+\frac{\lambda_{n_1}}{\lambda_j^2+\lambda_{n_1}^2}$$

$$\cdot\mathrm{ch}\lambda_j\sin\lambda_{n_1}-\frac{U(\lambda_{n_1})}{V(\lambda_{n_1})}\left[\frac{\lambda_j}{\lambda_j^2+\lambda_{n_1}^2}\mathrm{sh}\lambda_j\sin\lambda_{n_1}-\frac{\lambda_{n_1}}{\lambda_j^2+\lambda_{n_1}^2}\right.$$

$$\left.\cdot(\mathrm{ch}\lambda_j\cos\lambda_{n_1}-1)\right]-\frac{1}{2}\left[\frac{\sin(\lambda_j-\lambda_{n_1})}{\lambda_j-\lambda_{n_1}}+\frac{\sin(\lambda_j+\lambda_{n_1})}{\lambda_j+\lambda_{n_1}}\right]$$

$$+\frac{1}{2}\frac{U(\lambda_{n_1})}{V(\lambda_{n_1})}\left[\frac{\cos(\lambda_j-\lambda_{n_1})}{\lambda_j-\lambda_{n_1}}-\frac{\cos(\lambda_j+\lambda_{n_1})}{\lambda_j+\lambda_{n_1}}\right]-\frac{U(\lambda_j)}{V(\lambda_j)}$$

$$\cdot\left[\frac{\lambda_j}{\lambda_j^2+\lambda_{n_1}^2}(\mathrm{ch}\lambda_j\cos\lambda_{n_1}-1)-\frac{\lambda_{n_1}}{\lambda_j^2+\lambda_{n_1}^2}\mathrm{sh}\lambda_j\sin\lambda_{n_1}\right]$$

$$+\frac{U(\lambda_j)}{V(\lambda_j)}\frac{U(\lambda_{n_1})}{V(\lambda_{n_1})}\left[\frac{\lambda_j}{\lambda_j^2+\lambda_{n_1}^2}\mathrm{ch}\lambda_j\sin\lambda_{n_1}-\frac{\lambda_{n_1}}{\lambda_j^2+\lambda_{n_1}^2}\right.$$

$$\left.\cdot\mathrm{sh}\lambda_j\cos\lambda_{n_1}\right]-\frac{1}{2}\frac{U(\lambda_j)}{V(\lambda_j)}\left[\frac{\cos(\lambda_j-\lambda_{n_1})}{\lambda_j-\lambda_{n_1}}+\frac{\cos(\lambda_j+\lambda_{n_1})}{\lambda_j+\lambda_{n_1}}\right]$$

$$-\frac{1}{2}\frac{U(\lambda_j)}{V(\lambda_j)}\frac{U(\lambda_{n_1})}{V(\lambda_{n_1})}\left[\frac{\sin(\lambda_j-\lambda_{n_1})}{\lambda_j-\lambda_{n_1}}-\frac{\sin(\lambda_j+\lambda_{n_1})}{\lambda_j+\lambda_{n_1}}\right]\Big\}=0$$

$$(k=1,\ 2,\ \cdots) \qquad (4)$$

其中:

$$m=l-\frac{1}{2}\cdot INT\{[1+\sqrt{1+8(l-1)}]/2\}$$

$$\cdot\{INT[(1+\sqrt{1+8(l-1)})/2]-1\}$$

$$n=INT\{[1+\sqrt{1+8(l-1)}]/2\}+1-m$$

$$m_1=INT\{[1+\sqrt{1+8(l_1-1)}]/2\}+1-i$$

$$n_1=INT\{[1+\sqrt{1+8(l_2-1)}]/2\}+1-j$$

$$\lambda=\rho h\omega^2/D$$

求解无穷联立方程(4),可求得固有频率 ω_α,特征向量

$$A^{\alpha^T}=\{A_1^\alpha,\ A_2^\alpha,\cdots\},\ (\alpha=1,\ 2,\ \cdots).$$

由此，可求得各阶振型．例如第 α 阶振型为：

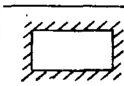
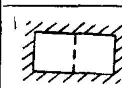
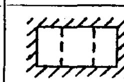
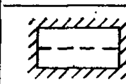
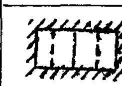
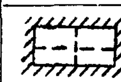
$$\Phi_\alpha(x,\ y)=\sum_{k=1}^{\infty}f_k(x,\ y)A_k^\alpha\quad(\alpha=1,2,\cdots)\tag{5}$$

通常，在计算方程（4）时，k，l，l_1，l_2，只需取有限项数 β．

已编制程序对长短比 $a/b=2$ 的板进行了计算，结果表明：低阶频率和振型随 β 值的增大，收敛较快，而高阶频率，则要有足够的项数，才能有较为令人满意的结果．

表 1 列出不同 β 下的前六阶频率，以便和文献[1]的计算结果作比较．

表 1

β	$\sqrt{\frac{\rho h}{D}}\frac{\omega_1}{b^2}$	$\sqrt{\frac{\rho h}{D}}\frac{\omega_2}{b^2}$	$\sqrt{\frac{\rho h}{D}}\frac{\omega_3}{b^2}$	$\sqrt{\frac{\rho h}{D}}\frac{\omega_4}{b^2}$	$\sqrt{\frac{\rho h}{D}}\frac{\omega_5}{b^2}$	$\sqrt{\frac{\rho h}{D}}\frac{\omega_6}{b^2}$
10	24.49	31.81	44.56	63.20	64.00	71.32
15	24.49	31.79	44.54	62.71	63.83	71.21
21	24.49	31.79	44.50	62.69	63.82	68.60
28	24.49	31.79	44.50	62.60	63.82	
36	24.49	31.79	44.50	62.60	63.82	68.37
45	24.49	31.79	44.50	62.60	63.82	65.95
文献[1]的结果	24.56			65.41		72.66
板的振型节线						

不同 β 值的 A_k'（$k=1,2,\cdots\beta$）值见表 2．

表 2

β	3	6	10	15
A_1^1	0.9929	0.9926	0.9924	0.9924
A_2^1	−0.1178	−0.1195	−0.1202	−0.1203
A_3^1	−0.0131	−0.0130	−0.0128	−0.0127
A_4^1		0.0140	0.0148	0.0149
A_5^1		−0.0020	−0.0022	−0.0022
A_6^1		0.0007	0.0007	0.0007
A_7^1			−0.0149	−0.0145
A_8^1			0.0013	0.0015
A_9^1			0.0002	0.0002
A_{10}^1			−0.0011	−0.0011
A_{11}^1				0.0016
A_{12}^1				−0.0024
A_{13}^1				−0.0001
A_{14}^1				−0.0002
A_{15}^1				0.0000

二、在外粘滞阻尼和结构阻尼共同影响下的自由振动和强迫振动

1. 自由振动.

设结构阻尼形式如图 1 所示，振动一周，结构内摩擦所做之功为图中阴影部分面积，它可看作应变对应力滞后一位相 ν，即应力应变由下式近似描述：

$$\sigma = E\varepsilon_\nu \tag{6}$$

其中：ε_ν 表示对 ε 滞后位相 $-\nu$，而 ν 的大小由材料性质决定.

外粘滞阻尼如常所述，它与速度成正比．今设阻尼系数为 μ．此时板的振动方程为：

$$\rho h\ddot{W} + \mu\dot{W} + \nabla^2\nabla^2 W_\nu = 0 \tag{7}$$

此处：W_ν 表示对 W 滞后位相 $-\nu$．W 为板的横向位移.

由展开定理，固支板的横向位移可表示为：

$$W(x, y, t) = \sum_{k=1}^{\infty} \Phi_k(x,y)\eta_k(t) \tag{8}$$

代入 (7)，整理可得：

$$\ddot{\eta}_k + 2n_k\omega_k\dot{\eta}_k + \omega_k^2\eta_{k\nu} = 0 \tag{9}$$

其中：$n_k = \dfrac{\mu}{2\rho h\omega_k}$，$\eta_{k\nu}$ 表示对 η_k 滞后位相 $-\nu_k$.

设自由振动方程 (9) 的解为下述形式

$$\eta_k(t) = H_k e^{-\omega_k t\left(a_k + \sin\frac{\nu_k}{2}\right)} \sin\left[C_k \cdot \cos\frac{\nu_k}{2} \cdot \omega_k t + \theta_k\right] \tag{10}$$

其中：H_k、θ_k 的值由初始条件决定.

为满足方程 (9)，显然，a_k 必须是下列四次方程中大于 $-\sin\dfrac{\nu_k}{2}$ 的实数解

$$\left[\left(a_k + \sin\frac{\nu_k}{2}\right) - n_k\right]^4 + (\cos\nu_k - n_k^2)\left[\left(a_k + \sin\frac{\nu_k}{2}\right) - n_k\right]^2 - \cos^4\frac{\nu_k}{2} = 0$$

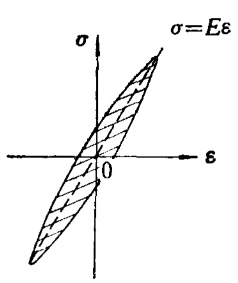


图 1

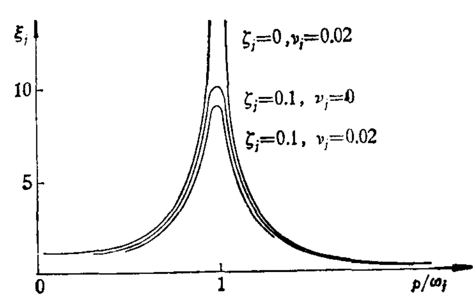


图 2　阻尼对结构强迫振动的影响

$$\xi_i = \frac{1}{\sqrt{\left[\cos\nu_i - \left(\frac{p}{\omega_i}\right)^2\right]^2 + \left(2n_i\frac{p}{\omega_i} + \sin\nu_i\right)^2}}$$

$$\zeta_i = n_i/\omega_i$$

C_k 应由下述表达式决定

$$C_k=\frac{\cos\frac{\nu_k}{2}}{\left(a_k+\sin\frac{\nu_k}{2}\right)-n_k}$$

显然，此时自由振动的频率为

$$\omega'_k=C_k\cdot\cos\frac{\nu_k}{2}\cdot\omega_k \tag{11}$$

对数减幅系数为

$$d_k=\frac{2\pi\left(a_k+\sin\frac{\nu_k}{2}\right)}{C_k\cdot\cos\frac{\nu_k}{2}} \tag{12}$$

2. 正弦激励下的强迫振动

设 (x_p, y_p) 点给以激励 $Q\sin pt$，强迫振动方程为

$$\rho h\ddot{W}+\mu\dot{W}+\nabla^2\nabla^2W_\nu=Q\sin pt\cdot\delta(x-x_p, y-y_p) \tag{13}$$

由 (8) 式，设：

$$W(x, y, t)=\sum_i W_i(x, y, t)=\sum_i \eta_i(t)\Phi_i(x, y) \tag{14}$$

将 (14) 式代入 (13) 式，两边乘上 $\Phi_j(x, y)=\sum_{k=1}^{\infty}A_k^jf_k(x, y)$，再对 x、y 分别从 0 到 a 和 0 到 b 积分，考虑到各阶振型正交的性质，最后得

$$\ddot{\eta}_j+2n_j\omega_j\dot{\eta}_j+\omega_j^2\eta_{j\nu}=Q\sin pt\cdot\sum_{k=1}^{\infty}A_k^jf_k(x_p, y_p)/\rho hV_j \tag{15}$$

其中：

$$V_j=\int_0^a\int_0^b\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}A_k^jA_l^jf_k(x, y)f_l(x, y)dxdy$$

对 (15) 式求解，略去因阻尼而衰减的自由振动部分的解，最后得第 i 阶强迫振动部分的解

$$\eta_i(t)=\frac{Q\sin(pt+\theta_i)\cdot\sum_{k=1}^{\infty}A_k^if_k(x, y)}{\rho hV_i\omega_i^2\sqrt{\left[\cos\nu_k-\left(\frac{p}{\omega_i}\right)^2\right]^2+\left(2n_i\frac{p}{\omega_i}+\sin\nu_i\right)^2}} \tag{16}$$

其中：

$$\theta_i=\mathrm{tg}^{-1}\frac{\sin\nu_i+2n_i\frac{p}{\omega_i}}{\left(\frac{p}{\omega_i}\right)^2-\cos\nu_i}$$

将 (16) 式代入 (8) 式，最后得：

$$W(x,\ y,\ t)=\sum_{j}\frac{Q\sin(pt+\theta_j)\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}A_k^jA_l^jf_k(x_p,\ y_p)f_l(x,\ y)}{\rho h\omega_j^2V_j\sqrt{\left[\cos\nu_j-\left(\frac{p}{\omega_j}\right)^2\right]^2+\left(2n_j\frac{p}{\omega_j}+\sin\nu_j\right)^2}} \tag{17}$$

阻尼对结构强迫振动的影响见图 2.

由(17)式,显然,在结构阻尼小的情况下,近似地取 $\frac{n_j}{\omega_j}=\sin\frac{\nu_j}{2}$,将结构阻尼换成等效粘滞阻尼进行计算是可行的,但在大的结构阻尼情况下,这样就会带来较大的误差,还是以本文的算法为宜.

三、随 机 响 应

1. 频率响应函数

设 $p(t)$ 为 $(x_p,\ y_p)$ 点的随机激励,此时 $W(x,\ y,\ t)$ 的振动微分方程为:

$$\rho h\ddot{W}+\mu\dot{W}+\nabla^2\nabla^2W_\nu=p(t)\delta(x-x_p,\ y-y_p) \tag{18}$$

取变换

$$\left.\begin{aligned}K_n&=\frac{1}{V_n}\iint W(x,\ y,\ t)\Phi_n(x,\ y)dxdy\\K_{n\nu}&=\frac{1}{V_n}\iint W_\nu(x,\ y,\ t)\Phi_n(x,\ y)dxdy\end{aligned}\right\} \tag{19}$$

显然,其逆变换为

$$\left.\begin{aligned}W(x,y,t)&=\sum_n W_n(x,y,t)=\sum_n K_n(t)\Phi_n(x,\ y)\\W_\nu(x,y,t)&=\sum_n W_{n\nu}(x,y,t)=\sum_n K_{n\nu}(t)\Phi_n(x,\ y)\end{aligned}\right\} \tag{20}$$

其中: $W_{n\nu}$ 表示对 W_n 滞后相位 $-\nu_n$

$K_{n\nu}$ 表示对 K_n 滞后相位 $-\nu_n$

将(20)式代入(18)式,两边乘上 $\Phi_n(x,\ y)=\sum_{k=1}^{\infty}A_k^nf_k(x,\ y)$,再对 x,y 分别从 0 到 a 和 0 到 b 积分,考虑到各阶振型正交的性质,得 $K_n(t)$ 的方程

$$\ddot{K}_n+2n_n\omega_n\dot{K}_n+\omega_n^2K_{n\nu}=p(t)\frac{\sum_{k=1}^{\infty}A_k^nf_k(x_p,\ y_p)}{\rho hV_n} \tag{21}$$

令:

$$\left.\begin{aligned}&p(t)=p_0e^{i\omega t}\\&K_n(t)=K_{n_0}e^{i\omega t}=H_{K_n}(\omega)\cdot p_0\cdot e^{i\omega t}\\&K_{n\nu}(t)=K_{n_0}e^{i(\omega t+\nu_n)}=H_{K_n}(\omega)\cdot p_0\cdot e^{i(\omega t+\nu_n)}\end{aligned}\right\} \tag{22}$$

代入(21)式,消去 p_0 后,得复频响应函数

$$H_{K_n}(\omega)=\frac{\sum_{k=1}^{\infty}A_k^n f_k(x_p, y_p)}{\rho h V_n[(\omega_n^2\cos\nu_n-\omega^2)+i(2n_n\omega+\omega_n\sin\nu_n)\omega_n]} \tag{23}$$

所以

$$\frac{K_{n_0}}{p_0}=|H_{K_n}(\omega)|$$

$$=\frac{\sum_{k=1}^{\infty}A_k^n f_k(x_p, y_p)}{\rho h V_n\omega_n^2}\cdot\frac{1}{\sqrt{\left[\cos\nu_n-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2n_n\frac{\omega}{\omega_n}+\sin\nu_n\right)^2}} \tag{24}$$

$K_n(t)$ 对 $p(t)$ 的滞后相位

$$\varphi_n=\mathrm{tg}^{-1}\left[\frac{\sin\nu_n+2n_n\frac{\omega}{\omega_n}}{\left(\frac{\omega}{\omega_n}\right)^2-\cos\nu_n}\right] \tag{25}$$

对稳定的随机振动系统而言，频率响应函数可代替传递函数.

2. 平稳随机振动的功率谱密度

设 $p(t)$ 为均值为零的平稳随机过程，自功率谱密度为 $S_{pp}(\omega)$.

$K_n(t)$ 的自功率谱密度

$$S_{K_nK_n}(\omega)=|H_{K_n}(\omega)|^2S_{pp}(\omega)$$

$$=\frac{\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}A_k^nA_l^n f_k(x_p, y_p)f_l(x_p, y_p)S_{pp}(\omega)}{\rho h V_n\omega_n^4\left\{\left[\cos\nu_n-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2n_n\frac{\omega}{\omega_n}+\sin\nu_n\right)^2\right\}} \tag{26}$$

显然，$W_n(x, y, t)$ 对应 $p(t)$ 的频率响应函数

$$|H_{W_n}(\omega, x, y)|=|H_{K_n}(\omega)|\cdot\Phi_n(x, y)$$

$$=\frac{\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}A_k^nA_l^n f_k(x_p, y_p)f_l(x, y)}{\rho h V_n\omega_n^2\sqrt{\left[\cos\nu_n-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(\frac{\omega}{\omega_n}\cdot 2n_n+\sin\nu_n\right)^2}} \tag{27}$$

所以 $W_n(x, y, t)$ 的功率谱密度为

$$S_{W_nW_n}(\omega, x, y)=|H_{W_n}(\omega, x, y)|^2\cdot S_{pp}(\omega)$$

$$=\frac{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}A_i^nA_j^nA_n^kA_l^n f_i(x, y)f_j(x, y)f_k(x_p, y_p)f_l(x_p, y_p)S_{pp}(\omega)}{\rho h V_n\omega_n^4\left\{\left[\cos\nu_n-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+\left(2n_n\frac{\omega}{\omega_n}+\sin\nu_n\right)^2\right\}} \tag{28}$$

$W(x, y, t)$ 对应于 $p(t)$ 的频率特性为

$$|H_W(\omega, x, y)|=\sum_n|H_{W_n}(\omega, x, y)|$$

$$= \sum_n \frac{\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} A_k^n A_l^n f_k(x_p, y_p) f_l(x, y)}{\rho h V_n \omega_n^2 \sqrt{\left[\cos\nu_n - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2n_n \frac{\omega}{\omega_n} + \sin\nu_n\right)^2}} \tag{29}$$

如 $W_n(x, y, t)$ 可看作统计相互独立,则 $W(x, y, t)$ 的自功率谱密度为

$$S_{WW}(\omega, x, y) = \sum_n S_{W_nW_n}(\omega, x, y)$$

$$= \sum_n \frac{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} A_i^n A_j^n A_k^n A_l^n f_i(x, y) f_j(x, y) f_k(x_p, y_p) f_l(x_p, y_p) S_{pp}(\omega)}{\rho h V_n \omega_n^4 \left\{\left[\cos\nu_n - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2n_n \frac{\omega}{\omega_n} + \sin\nu_n\right)^2\right\}} \tag{30}$$

3. 输入为白噪声时的挠度均方值

当输入 $p(t)$ 的频谱为白噪声时,即 $S_{pp}(\omega) = S_p =$ 常数时，K_n 的均方值

$$E[K_n^2] = \int_{-\infty}^{\infty} S_{K_nK_n}(\omega) d\omega$$

$$= \frac{\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} A_k^n A_l^n f_k(x_p, y_p) f_l(x_p, y_p) S_p}{\rho h \omega_n^4 V_n} \times \int_{-\infty}^{\infty} \frac{d\omega}{\left[\cos\nu_n - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2n_n \frac{\omega}{\omega_n} + \sin\nu_n\right)^2}$$

$$= \frac{2\pi n_n}{(4\cos\nu_n \cdot n_n^2 - \sin^2\nu_n)} \cdot \frac{\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} A_k^n A_l^n f_k(x_p, y_p) f_l(x_p, y_p) \cdot S_p}{\rho h \omega_n^3 V_n} \tag{31}$$

同理，$W_n(x, y, t)$ 的均方值

$$E[W_n^2(x, y, t)] = \frac{2\pi n_n}{4\cos\nu_n \cdot n_n^2 - \sin^2\nu_n} \times \frac{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} A_i^n A_j^n A_k^n A_l^n f_i(x, y) f_j(x, y) f_k(x_p, y_p) f_l(x_p, y_p) S_p}{\rho h V_n \omega_n^3} \tag{32}$$

如 $W_n(x, y, t)$ 可看作统计相互独立,则挠度 $W(x, y, t)$ 的均方值

$$E[W^2(x, y, t)] = \frac{2\pi S_p}{\rho h} \sum_n \frac{\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} f_i(x, y) f_j(x, y) f_k(x_p, y_p) f_l(x_p, y_p)}{V_n \omega_n^3 (4\cos\nu_n \cdot n_n^2 - \sin^2\nu_n)} \tag{33}$$

同前所述,在求解强迫响应和随机响应时,并不需要取无穷项,而是取它的有限项.已

编制可用来求得各点正弦响应和随机响应的程序.

参　考　文　献

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THE RESPONSE OF A CLAMPED RECTANGULAR PLATE WITH VISCOUS DAMPING AND NON-LINEAR STRUCTURAL DAMPING

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Abstract

Firstly, the free vibration of a rectangular plate without damping was analysed by normal mode functions, $\Phi_j(x, y)=\sum_{i=1}^{\infty} A_{ij} f_i(x,y)$, which satisfy the clamped boundary conditions, where $f_i(x,y)$ is a normalized group of generalized Fourier function.

Afterwards, a non-linear structural damping model based on the phase difference ν between stress and strain was proposed. The differential equation of the plate with the viscous damping and structural damping was established. General solution of this equation for free vibration was obtained. The response of the system to deterministic and random excitation was calculated. A program was compiled for a plate with length-width ratio $a/b=2$. The results show that the natural frequencies and modes of lower orders converge rapidly, but more terms must be included to give satisfactory results for higher orders.