

星系螺旋结构三维密度波的 流体动力学理论 (I)

——局部解及局部 Jeans 稳定性的讨论

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提要 本文是作者的盘状星系三维密度波工作的第一部分。文中给出了在厚度参数 σ 及准单色波参数 ϵ_* 的一级近似下的波传播方程及其局部近似解,并讨论了星盘厚度对局部 Jeans 稳定性判据的影响。这些结果是进一步建立整体模式解理论的准备。

一、引言

盘状星系螺旋结构的本质与起源是半个多世纪以来一直为天文学家及天体物理学家所注目的重大难题。近十五年来,林家翘等人所创建的密度波理论由于很好地经受了观测的检验已获得广泛的承认。特别从七十年代开始,林家翘等人为解决 Toomre (1969)提出的维持问题深入地探讨了螺旋结构的起源与维持的动力学机制(此即后人所谓的 Waser 机制),并求得了具有离散图象频率的增长型整体模式解。这些工作的开展使得密度波理论获得进一步的成功与完善。

但是,当前密度波理论还存在一些急待解决的问题。迄今的工作都把正常星系简化为一个位于中心的刚性球外加一无限薄盘,此外还假定波动过程只发生在盘内。这样,这种理论基本上是一种二维的平面盘理论,它的局限是:1)不能讨论星盘形状及厚度对螺旋结构的影响;2)在其波动方程中出现的函数 $Q(r)$ 不确定,而整体模式解却又对它的分布十分敏感;3)所得渐近解尚只适用于比较“紧卷”的螺旋结构。

为解决这些问题,作者将从流体动力学模型出发,提出一种三维星系密度波的模式解理论,并且不完全局限于“紧卷”情形,而是假定扰动态满足“准单色波近似条件”^[6],这个条件既可包括紧卷螺旋又可包括松卷螺旋。在此基础上,我们求得了盘状星系三维密度波的整体模式解及量子化条件,从而为星系的哈勃分类的动力学含义提供了进一步的理论说明。本文就是这项工作的第一部分。

二、星盘坐标系及无量纲方程组

我们采用流体动力学模型。假定星系是一种自引力的连续介质,其总质量为 M ,物

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质由于引力吸引以及离心力、“压力”的抗衡，绝大部分均分布在一个盘状区域内：

$$0 \leq r' \leq R, \quad -\delta(r') \leq z' \leq \delta(r') \quad \left(\varepsilon = \frac{\delta}{R} \ll 1 \right). \quad (2.1)$$

今选择一“星盘曲线正交坐标系” (r, θ, z) ，使得沿星盘边界，坐标 z 为常数（见

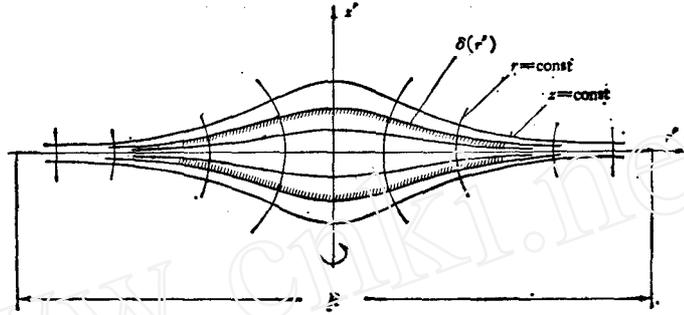


图 1 星盘曲线坐标系 (r, θ, z) (厚度参数 $\varepsilon = \frac{\delta}{R} \ll 1$)

图 1)。再以 $(M, G_0, R, \delta(r'))$ 为特征量（这里 G_0 为引力常数），将所有物理量无量纲化，便得特征速度值：

$$U_0 = \left(\frac{4\pi G_0 M}{R} \right)^{1/2} \quad (2.2)$$

这样，半径 $r' = Rr$ ；高度 $z' = \delta(r')z$ ；时间 $t' = \left(\frac{R}{U_0} \right)t$ ；体密度 $\rho' = \left(\frac{M}{R^3} \right)\rho$ ；投影面密度 $\sigma' = \left(\frac{M}{R^2} \right)\sigma$ ； r 方向速度分量 $u' = U_0 u$ ； θ 方向速度分量 $v' = U_0 v$ ； z 方向速度分量 $w' = U_0 w$ ； r 方向等效声速 $a_r'^2 = M_r^2 U_0^2$ ； θ 方向等效声速 $a_\theta'^2 = M_\theta^2 U_0^2$ ； z 方向等效声速 $a_z'^2 = M_z^2 U_0^2$ ；引力势 $\psi' = \left(\frac{4\pi G_0 M}{R} \right)\psi$ 。其中 M_r, M_θ, M_z 可称为各方向上的倒马赫数。

由于实际星盘扁平，边界形状十分平缓，盘内的坐标面 $z = \text{常数}$ 的曲率半径很大，所以我们引入的星盘坐标系与普通的柱面坐标系将十分接近。这样，在忽略拉米系数中的附加小量不计之后，可以想象在我们星盘坐标系下的流体力学基本方程组的形式应和柱面坐标系下的无量纲基本方程组相同。此即有：

$$\left. \begin{aligned} \varepsilon \left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial r} + \frac{1}{r} \frac{\partial(\rho v)}{\partial \theta} + \frac{\rho u}{r} \right) + \frac{\partial(\rho w)}{\partial z} &= 0 \\ \varepsilon \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} - \frac{v^2}{r} \right) + w \frac{\partial u}{\partial z} + \varepsilon \left(M_r^2 \frac{\partial \ln \rho}{\partial r} + \frac{\partial \psi}{\partial r} \right) &= 0 \\ \varepsilon \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} + \frac{uv}{r} \right) + w \frac{\partial v}{\partial z} + \varepsilon \left(\frac{M_\theta^2}{r} \frac{\partial \ln \rho}{\partial \theta} + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \right) &= 0 \\ \varepsilon \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial r} + \frac{v}{r} \frac{\partial w}{\partial \theta} \right) + w \frac{\partial w}{\partial z} + M_z^2 \frac{\partial \ln \rho}{\partial z} + \frac{\partial \psi}{\partial z} &= 0 \\ \varepsilon^2 \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} - \rho \right) + \frac{\partial^2 \psi}{\partial z^2} &= 0 \end{aligned} \right\} \quad (2.3)$$

因为星盘的有限厚度 $\delta(r')$ 是由于 z 方向的弥散速度 a_z' 抵抗了引力坍塌之后得到

的,显然,当 $M_z \rightarrow 0$ 时,有 $\varepsilon \rightarrow 0$. 因此,可以得出 $M_z^2 = \varepsilon^N a_z^2(r)$. 再作一定分析后还可确定指数 $N = 1$. 事实上,我们考虑一个在无限薄盘上方运动的试验质点,其质量为 1,热速度为 a'_z . 假设该质点在 $z = 0$ 时势能为零;则当 $z = \delta(r')$ 时,其势能 $\phi \propto \frac{\delta(r')}{R} \phi_0$, 此外,试验质点的动能 $T = \frac{1}{2} a_z'^2$. 在平衡时,在一个“热”星盘的边界上应有: $T \approx \phi$. 由此便推出: $\delta(r') \propto R \frac{a_z'^2}{U_0^2} = RM_z^2$, 或者

$$M_z^2 = \varepsilon a_z^2(r). \quad (2.4)$$

假定物质运动可看作在基态上迭加一个不定常的扰动运动,从而

$$u = u_1 + u_0, \quad v = v_1 + v_0, \quad w = 0, \quad \rho = \rho_0 + \rho_1, \quad \phi = \phi_0 + \phi_1 \quad (2.5)$$

其中基态为:

$$u_0 = 0, \quad v_0 = r\Omega_0, \quad w_0 = 0, \quad \rho = \rho_0, \quad \phi = \phi_0. \quad (2.6)$$

把(2.4)(2.5)式代入方程组(2.3),便推出:

$$\left. \begin{aligned} & \left(\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial r} + \frac{v_0 + v_1}{r} \frac{\partial}{\partial \theta} \right) \varphi_1 + (1 + \varphi_1) \left[\frac{u_1}{r} \left(1 + \frac{\partial \ln \rho_0}{\partial \ln r} \right) \right. \\ & \quad \left. + \frac{\partial u_1}{\partial r} + \frac{1}{r} \frac{\partial v_1}{\partial \theta} \right] = 0 \\ & \left(\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial r} + \frac{v_0 + v_1}{r} \frac{\partial}{\partial \theta} \right) u_1 - \frac{v_1^2 + 2v_0v_1}{r} + \frac{M_r^2}{1 + \varphi_1} \frac{\partial \varphi_1}{\partial r} + \frac{\partial \phi_1}{\partial r} = 0 \\ & \left(\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial r} + \frac{v_0 + v_1}{r} \frac{\partial}{\partial \theta} \right) v_1 + u_1 \left(\frac{\partial v_0}{\partial r} + \frac{v_0 + v_1}{r} \right) + \frac{M_\theta^2}{r(1 + \varphi_1)} \frac{\partial \varphi_1}{\partial \theta} \\ & \quad + \frac{1}{r} \frac{\partial \phi_1}{\partial \theta} = 0 \\ & \varepsilon a_z^2 \frac{\partial \ln(1 + \varphi_1)}{\partial z} + \frac{\partial \phi_1}{\partial z} = 0 \end{aligned} \right\} \quad (2.7)$$

其中

$$\varphi_1 = \rho_1 / \rho_0 \quad (2.8)$$

在方程组(2.7)中,由于参数 $\varepsilon \ll 1$, 我们可将基态和扰动态分别按 ε 作级数展开,再逐次求解. 对于三维基态,我们在文献[5]中已经求得,故今只需处理扰动态. 令:

$$\left. \begin{aligned} \varepsilon \rho_1 &= \rho_1^{(0)} + \varepsilon \rho_1^{(1)} + \dots \\ \sigma_1 &= \sigma_1^{(0)} + \varepsilon \sigma_1^{(1)} + \dots \\ q_1 &= (u_1, v_1, w_1) = q_1^{(0)} + \varepsilon q_1^{(1)} + \dots \\ \varphi_1 &= \varphi_1^{(0)} + \varepsilon \varphi_1^{(1)} + \dots \end{aligned} \right\} \quad (2.9)$$

依次求各级近似解.为此,首先需确定沿 z 方向的扰动密度分布及对称面上的扰动引力势.

三、扰动密度沿 z 方向的分布及对称面上的扰动引力势

设扰动密度具有下述形式:

$$\rho_1 = \hat{\rho}_1(r, z) H_m(\zeta) e^{i(\omega t - m\theta)} \quad (3.1)$$

这里

$$H_m(\zeta) = \begin{cases} H_m^{(1)}(\zeta) & \text{对应 } \varepsilon = +1, \text{ 为第一类 Hankel 函数} \\ H_m^{(2)}(\zeta) & \text{对应 } \varepsilon = -1 \text{ 为第二类 Hankel 函数} \end{cases} \quad (3.2)$$

而 $\zeta = \zeta(r, \varepsilon)$ 为某一复函数, $\omega = \omega(\varepsilon)$ 一般为复数. 函数 $\rho_1(r, z) = \hat{\rho}_1(r, z, \varepsilon)$ 也应依赖 ε , 不过当 $z = 0$ 时, 由于 $\rho_1 = \hat{\rho}_1(r, 0, \varepsilon)H_m(\zeta)e^{i(\omega t - m\theta)}$, 而因子 $H_m(\zeta)$ 已包含了变量 r 与 ε 的影响, 所以可以选定 $\varepsilon \hat{\rho}_1(r, 0, \varepsilon) = \hat{\rho}_{10}^{(0)}$ = 常数(与 ε 无关). 再设

$$\zeta = \zeta_* + \int_0^r k dr, \quad k = \frac{d\zeta}{dr} \quad (3.3)$$

其中复常数 ζ_* 这样选取, 使得在 $r = r_m$ 时, $\zeta = kr$ 成立. 由 (3.3) 式推出, 一般地只要 $\zeta_* \neq 0$, (3.1) 式在 $r = 0$ 点并无奇性. 现除展开式(2.9)外, 还有:

$$\left. \begin{aligned} \varepsilon \hat{\rho}_1(r, z) &= \hat{\rho}_1^{(0)} + \varepsilon \hat{\rho}_1^{(1)} + \dots \\ \varepsilon \hat{\rho}_1(r, 0) &= \hat{\rho}_{10}^{(0)}; \text{ 或 } \hat{\rho}_1^{(N)}(r, 0) = 0 \quad (N > 0) \\ \zeta &= \zeta^{(0)} + \varepsilon \zeta^{(1)} + \dots \\ k &= k^{(0)} + \varepsilon k^{(1)} + \dots \\ \omega &= \omega^{(0)} + \varepsilon \omega^{(1)} + \dots \end{aligned} \right\} \quad (3.4)$$

此外, 对于扰动投影面密度 $\sigma_1 = \hat{\sigma}_1 H_m(\zeta)e^{i(\omega t - m\theta)}$ 有:

$$\hat{\sigma}_1 = \int_{-\infty}^{+\infty} \varepsilon \hat{\rho}_1 dz = \hat{\sigma}_1^{(0)} + \varepsilon \hat{\sigma}_1^{(1)} + \dots; \quad \hat{\sigma}_1^{(N)} = \int_{-\infty}^{+\infty} \hat{\rho}_1^{(N)} dz. \quad (3.5)$$

为求出在自洽引力场中扰动密度沿 z 方向的分布, 我们必须把泊松方程 (2.7), 与运动方程 (2.7), 联立求解. 对于泊松方程, 在文献[6]中我们引入了小参数 ε_* , 这里姑且称之为准单色波参数, 其定义为:

$$\varepsilon_* = \frac{1}{k} \frac{d \ln k}{dr}. \quad (3.6)$$

同时又把扰动引力势表为: $\psi_1 = \hat{\psi}_1(r, z, \varepsilon)H_m(\zeta)e^{i(\omega t - m\theta)}$ (3.7)

从而将泊松方程化为

$$\frac{\partial^2 \hat{\psi}_1}{\partial z^2} - \varepsilon^2 k^2 (1 + \Delta + \chi) \hat{\psi}_1 = \varepsilon^2 \hat{\rho}_1 \quad (3.8)$$

其中 $\Delta = G \left(\frac{1}{\zeta} - \frac{1}{kr} \right) - m^2 \left(\frac{1}{\zeta^2} - \frac{1}{k^2 r^2} \right) - \frac{G}{k^2} \frac{dk}{dr} = O(\varepsilon_*)$ (3.9)

$$\chi = \beta^2 + 2G\beta + \frac{1}{k} \frac{d\beta}{dr} + \frac{\beta}{k^2} \frac{dk}{dr}, \quad \beta = \frac{1}{k} \frac{\partial \ln \hat{\psi}_1}{\partial r} = O(\varepsilon_*, \varepsilon_0) \quad (3.10)$$

这里 $\varepsilon_0 = \varepsilon(0), G = \frac{H'_m(\zeta)}{H_m(\zeta)}$

由此解出了其关于 $\varepsilon_0, \varepsilon_*$ 的一级近似解. 现将结果列举如下:

$$\hat{\psi}_1 = -\frac{\hat{\sigma}_1^{(0)}}{2k} (1 + \psi_\varepsilon + \psi_{\varepsilon_*} + \dots); \quad (3.11)$$

$$\left\{ \begin{aligned} \psi_\varepsilon &= \varepsilon \frac{\hat{\sigma}_1^{(1)}}{\hat{\sigma}_1^{(0)}} - \frac{F_0(z)}{\hat{\sigma}_1^{(0)}} \varepsilon k + \frac{2}{\hat{\sigma}_1^{(0)}} I_{(0)}^{(0)}(z) G \frac{d\varepsilon}{dr}, \\ \psi_{\varepsilon_*} &= \hat{\sigma}_1^{(0)} I_{(0)}^{(0)}(z) \left(\frac{2G}{k^2} \frac{dk}{dr} - \Delta \right) \end{aligned} \right. \quad (3.12)$$

$$I_0^{(0)}(z) = -2(W_{*1}^{(0)}\hat{\rho}_{10}^{(0)} - \hat{g}_1^{(0)}z),$$

$$F_0(z) = 2 \left[W_{*1}^{(0)}\hat{\rho}_{10}^{(0)} + z \int_0^z \hat{\rho}_1^{(0)} dz_1 - \int_0^z z_1 \hat{\rho}_1^{(0)} dz_1 \right] \quad (3.13)$$

$$\begin{cases} \phi_1|_{z=0} = -\frac{\hat{g}_1^{(0)}}{2k} (1 + \phi_{\varepsilon_0} + \phi_{\varepsilon_{*0}} + \dots) H_m(\zeta) e^{i(\omega t - m\theta)} \\ \left. \frac{\partial \phi_1}{\partial r} \right|_{z=0} = (G + \beta)k\phi_1|_{z=0} \end{cases} \quad (3.14)$$

对于小扰动 $|\phi_1| = |\rho_1/\rho_0| \ll 1$, 方程 (2.7)₄ 可以线性化, 从而化为:

$$\varepsilon a_z^2 \frac{d^2 \hat{\varphi}_1}{dz^2} + \frac{d^2 \hat{\varphi}_1}{dz^2} = 0 \quad (3.15)$$

这里

$$\hat{\varphi}_1 = \hat{\rho}_1/\rho_0 = \hat{\varphi}_1^{(0)} + \varepsilon \hat{\varphi}_1^{(1)} + \dots \quad (3.16)$$

$$\varepsilon \rho_0 = \varepsilon \rho_0^{(0)} + \varepsilon \rho_0^{(1)} + \dots \quad (3.17)$$

$$\hat{\varphi}_1^{(0)} = \hat{\rho}_1^{(0)}/\rho_0^{(0)}; \quad \hat{\varphi}_1^{(1)} = \frac{\hat{\rho}_1^{(1)}}{\rho_0^{(0)}} - \left(\frac{\rho_0^{(1)}}{\rho_0^{(0)}} \right) \hat{\varphi}_1^{(0)} + \dots \quad (3.18)$$

而基态 ρ_0 的各级近似解已有结果(参看文献[6]):

$$\left. \begin{aligned} \rho_0^{(0)}(r, z) &= \rho_{00}^{(0)} \varphi_0^{(0)}(x), \quad \rho_{00}^{(0)} = \rho_0^{(0)}(r, 0), \quad x = \frac{z}{z_{*0}} \\ \varphi_0^{(0)}(x) &= \operatorname{sech}^2 x, \quad z_{*0} = \sqrt{\frac{2}{\alpha}}, \quad \alpha = \frac{\rho_{00}^{(0)}}{a_z^2} = \text{const} \\ \sigma_0^{(0)}(r) &= \sigma_0(r) = 2z_{*0} \rho_{00}^{(0)} \end{aligned} \right\} \quad (3.19)$$

$$\left. \begin{aligned} \left(\frac{\rho_0^{(1)}}{\rho_0^{(0)}} \right) &= \frac{2C(r)}{\rho_{00}^{(0)}} Y(x) = 4z_{*0} \frac{C(r)}{\sigma_0(r)} Y(x) \\ Y(x) &= Y(0)(1 - x \operatorname{th} x) + \left[\operatorname{th} x \int_0^x (1 - x_1 \operatorname{th} x_1) dx_1 - (1 - x \operatorname{th} x) \int_0^x \operatorname{th} x_1 dx_1 \right] \\ Y(0) &= -0.6932 \end{aligned} \right\} \quad (3.20)$$

将方程 (3.8), (3.15) 合并, 即得:

$$\varepsilon a_z^2 \frac{d^2 \hat{\varphi}_1}{dz^2} + \varepsilon^2 \hat{\rho}_1 + \varepsilon^2 (1 + \Delta + \chi) k^2 \hat{\varphi}_1 = 0. \quad (3.21)$$

利用以上结果及前面各展开式, 不难推出此方程的对于 ε 的各级近似方程, 其结果如下:

1) 零级近似 方程与边条件为:

$$\frac{\partial^2 \hat{\varphi}_1^{(0)}}{\partial x^2} + 2\varphi_0^{(0)}(x) \hat{\varphi}_1^{(0)} = 0 \quad (3.22)$$

$$\hat{\varphi}_1^{(0)}|_{x=0} = \hat{g}_1^{(0)} = \hat{\rho}_{10}^{(0)}/\rho_{00}^{(0)} \quad (3.23)$$

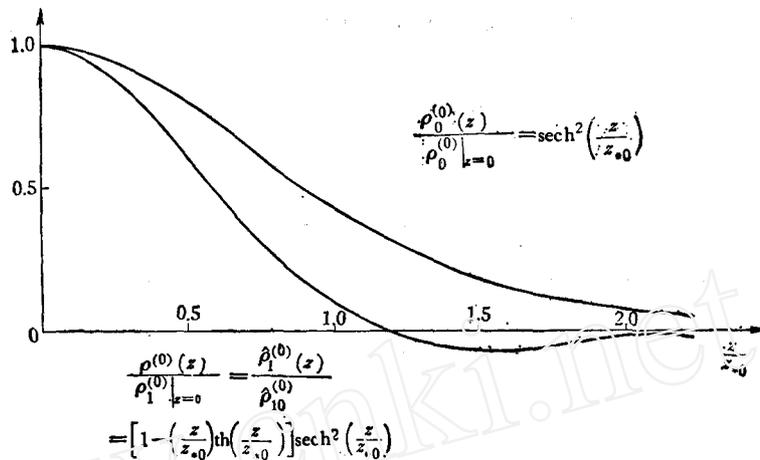
$$\left. \frac{\partial \hat{\varphi}_1^{(0)}}{\partial x} \right|_{x=0} = 0 \quad (3.24)$$

方程(3.22)的基本解为:

$$P_1(x) = (1 - x \operatorname{th} x); \quad P_2(x) = \operatorname{th} x \quad (3.25)$$

满足条件(3.23), (3.24)的解为:

$$\hat{\varphi}_1^{(0)} = P_1(x) \hat{g}_1^{(0)} \quad (3.26)$$

图 2 相对密度沿 z 方向的分布

相对扰动密度的分布曲线 (见图 2)。

2) 一级近似 类似地得方程:

$$\frac{\partial^2 \hat{\varphi}_1^{(1)}}{\partial x^2} + 2\varphi_0^{(0)}(x)\hat{\varphi}_1^{(1)} = [B_0(r) + B_1(r)T(x)]\hat{g}_1^{(0)} \quad (3.27)$$

其中

$$\left. \begin{aligned} B_0(r) &= 2kz_{*1}; & B_1(r) &= -8z_{*0} \frac{C(r)}{\sigma_0(r)}; \\ T(x) &= \varphi_0^{(0)}(x)P_1(x)Y(x) \end{aligned} \right\} \quad (3.28)$$

由于

$$\hat{\rho}_1^{(1)} = \rho_{00}^{(0)}\varphi_0^{(0)}(x)\hat{\varphi}_1^{(1)} + 2C(r)Y(x)\varphi_0^{(0)}(x)P_1(x)\hat{g}_1^{(1)} \quad (3.29)$$

以及 $\hat{\rho}_{10}^{(1)} = 0$, $\frac{\partial \hat{\rho}_1^{(1)}}{\partial x} \Big|_{x=0} = 0$, 可得边条件:

$$\hat{\varphi}_1^{(1)} \Big|_{x=0} = \frac{1}{2} Y(0)B_1(r)\hat{g}_1^{(0)} \quad (3.30)$$

$$\frac{\partial \hat{\varphi}_1^{(1)}}{\partial x} \Big|_{x=0} = 0 \quad (3.31)$$

方程 (3.27) 的通解不难得出, 是

$$\begin{aligned} \hat{\varphi}_1^{(1)} &= C_1P_1(x) + C_2P_2(x) \\ &+ P_2(x) \int_0^x P(x_1)(B_0 + B_1T)dx_1 - P_1(x) \int_0^x P_2(x_1)(B_0 + B_1T)dx_1 \end{aligned} \quad (3.32)$$

由边条件 (3.30), (3.31) 推出:

$$C_1 = 0; \quad C_2 = \frac{1}{2} Y(0)B_1(r)\hat{g}_1^{(0)} \quad (3.33)$$

由以上结果及 (3.29) 式我们算出:

$$\left. \begin{aligned}
 \hat{\sigma}_1^{(0)} &= 2z_{*1}^{(0)}\hat{\rho}_{10}^{(0)} = z_{*0}\hat{\rho}_{10}^{(0)}; \quad W_{*1}^{(0)} = (\ln 4)z_{*0}^2, \quad z_{*1}^{(0)} = \frac{1}{2}z_{*0} \\
 \hat{\sigma}_1^{(1)} &= 2z_{*1}^{(1)}\hat{\rho}_{10}^{(0)} = 2z_{*0}^2 \left[0.3466k - 0.4525 \frac{C(r)}{\sigma_0(r)} \right] \hat{\rho}_{10}^{(0)} \\
 &\quad \text{(其中函数 } C(r) \text{ 的意义见文献[5])} \\
 \Phi_0(x) &= P_2(x) \int_0^x P_1(x_1) dx_1 - P_1(x) \int_0^x P_2(x_1) dx_1 \\
 \Phi_1(x) &= P_2(x) \int_0^x P_1(x_1) T(x_1) dx_1 - P_1(x) \int_0^x P_2(x_1) T(x_1) dx_1 \\
 &\quad + \frac{1}{2} Y(0) P_2(x)
 \end{aligned} \right\} \quad (3.34)$$

并且有:

$$\rho_1 = \hat{\rho}_{10}^{(0)} \left\{ \frac{1}{\varepsilon} P_2(x) + \left[B_0(r) \Phi_0(x) + B_1(r) \Phi_1(x) - \frac{1}{2} B_1(r) P_1(x) Y(x) \right] \right\} \\
 \cdot \varphi_0^{(0)}(x) H_m(\zeta) e^{i(\omega t - m\theta)} + \dots \quad (3.35)$$

可见,只要由对称面上的运动方程解出函数 $\zeta = \zeta(r)$, 密度波的三维结构就完全确定了.

四、对称面上波的传播方程

为得到星盘对称面上波的传播方程,我们将方程组(2.7)₁₋₃线性化,并令 $z = 0$, 从而得

$$\left. \begin{aligned}
 \left(\frac{\partial}{\partial t} + \Omega_0 \frac{\partial}{\partial \theta} \right) g_1 + u_1 \left(\frac{1}{r} + \frac{\partial \ln \rho_{00}}{\partial r} \right) + \frac{\partial u_1}{\partial r} + \frac{1}{r} \frac{\partial v_1}{\partial \theta} &= 0 \\
 \left(\frac{\partial}{\partial t} + \Omega_0 \frac{\partial}{\partial \theta} \right) u_1 - 2\Omega_0 v_1 + M_r^2 \frac{\partial g_1}{\partial r} + \frac{\partial \phi_1}{\partial r} &= 0 \\
 \left(\frac{\partial}{\partial t} + \Omega_0 \frac{\partial}{\partial \theta} \right) v_1 + u_1 \left(\frac{dv_0}{dr} + \frac{v_0}{r} \right) + \frac{M_\theta^2}{r} \frac{\partial g_1}{\partial \theta} + \frac{1}{r} \frac{\partial \phi_1}{\partial \theta} &= 0
 \end{aligned} \right\} \quad (4.1)$$

这里

$$g_1 = \varphi_1|_{z=0} = \frac{\hat{\rho}_{10}}{\rho_{00}} H_m(\zeta) e^{i(\omega t - m\theta)} \quad (4.2)$$

再设

$$\left. \begin{aligned}
 g_1 &= \tilde{g}_1 H_m(\zeta) e^{i(\omega t - m\theta)}; \quad \phi_1 = \tilde{\phi}_1 H_m(\zeta) e^{i(\omega t - m\theta)} \\
 u_1 &= \tilde{u}_1 H_m(\zeta) e^{i(\omega t - m\theta)}; \quad v_1 = \tilde{v}_1 H_m(\zeta) e^{i(\omega t - m\theta)}
 \end{aligned} \right\} \quad (4.3)$$

$$\left. \begin{aligned}
 \text{其中 } \tilde{g}_1 &= \frac{\hat{\rho}_{10}}{\rho_{00}} = \frac{\hat{\rho}_{10}^{(0)}}{\varepsilon \rho_{00}}; \quad \tilde{\phi}_1 = \hat{\phi}_1(r, 0) = -\frac{\hat{\sigma}_1^{(0)}}{2k} (1 + \phi_{\varepsilon_0} + \phi_{\varepsilon_{*0}}) \\
 \text{而 } \phi_{\varepsilon_0} &= -D_0(r)k - D_1(r), \quad \left\{ \begin{aligned} D_0(r) &= 4.1588 z_{*1}^{(0)} \varepsilon(r) \\ D_1(r) &= 1.8100 z_{*1}^{(0)} \varepsilon \frac{C(r)}{\sigma_0(r)} + (8 \ln 4) z_{*1}^{(0)} G \frac{d\varepsilon}{dr} \end{aligned} \right. \quad (4.4)
 \end{aligned} \right\}$$

$$\left. \begin{aligned} \frac{\partial}{\partial r} \ln g_1 &= A_1 = G(\zeta)k - \frac{d \ln(\varepsilon \rho_{00})}{dr} \\ \frac{\partial}{\partial r} \ln \phi_1 &= B_1 = G(\zeta)k + \beta k \\ \frac{\partial}{\partial r} \ln u_1 &= G(\zeta) + \frac{d \ln \tilde{u}_1}{dr}; \quad \frac{\partial}{\partial \theta} \ln q' = -im, \quad \frac{\partial}{\partial z} \ln q' = i\omega \end{aligned} \right\} \quad (4.5)$$

由此方程组(4.1)便化作

$$\left. \begin{aligned} i\kappa\nu\tilde{g}_1 + \tilde{u}_1 \left[\frac{1}{r} + \frac{d \ln(\varepsilon \rho_{00})}{dr} + Gk + \frac{d \ln \tilde{u}_1}{dr} \right] - \frac{im}{r} \tilde{v}_1 &= 0 \\ -i\kappa\nu\tilde{u}_1 + 2\Omega_0\tilde{v}_1 &= M_r^2 A_1 \tilde{g}_1 + B_1 \tilde{\phi}_1 \\ \frac{\kappa^2}{2\Omega_0} \tilde{u}_1 + i\kappa\nu\tilde{v}_1 &= \frac{im}{r} M_\theta^2 \tilde{g}_1 + \frac{im}{r} \tilde{\phi}_1 \end{aligned} \right\} \quad (4.6)$$

这里和通常一样,采用定义:

$$\nu = \frac{\omega - m\Omega_0}{\kappa}, \quad \kappa^2 = 4\Omega_0^2 \left(1 + \frac{1}{2} \frac{d \ln \Omega_0}{d \ln r} \right) \quad (4.7)$$

由(4.6)₂₋₃解出 \tilde{v}_1, \tilde{u}_1 , 然后代入(4.6)₁, 便得:

$$\frac{dU}{dr} + \left[Gk + \frac{d}{dr} \ln \frac{\varepsilon \rho_{00} r}{\kappa^2(1-\nu^2)} \right] U + \frac{m}{r} V = \kappa^3 \nu (1-\nu^2) \tilde{g}_1 \quad (4.8)$$

其中

$$\left\{ \begin{aligned} U &= \kappa\nu(A_1 M_r^2 \tilde{g}_1 + B_1 \tilde{\phi}_1) - \frac{2m\Omega_0}{r} (M_\theta^2 \tilde{g}_1 + \tilde{\phi}_1) \\ V &= \frac{\kappa^2}{2\Omega_0} (A_1 M_r^2 \tilde{g}_1 + B_1 \tilde{\phi}_1) - \frac{m}{r} \kappa\nu (M_\theta^2 \tilde{g}_1 + \tilde{\phi}_1) \end{aligned} \right. \quad (4.9)$$

将(4.8)式化简,便得

$$\begin{aligned} G \frac{dk}{dr} + Fk^2 + GkR_{10} + R_{00} - k_0^2(1-\nu^2) \\ + \left(\frac{\varepsilon \rho_{00}}{M_r^2} \right) \left[\frac{d(B_1 \tilde{\phi}_1)}{dr} + B_1 \tilde{\phi}_1 (Gk + R_2) - R_{30} \tilde{\phi}_1 \right] \\ = \frac{1}{\kappa\nu} \left(\frac{2m\Omega_0}{r} \right) \left\{ \frac{\varepsilon \rho_{00}}{M_r^2} \left[\frac{d\tilde{\phi}_1}{dr} + \tilde{\phi}_1 (R_{3*} - \beta k) \right] - GkR_{1*} - R_{0*} \right\} \end{aligned} \quad (4.10)$$

其中

$$\left. \begin{aligned} Q &= \frac{2\kappa M_r}{\varepsilon \rho_{00} z_{*1}^{(0)}}, \quad \frac{\varepsilon \rho_{00} z_{*1}^{(0)}}{M_r^2} = \frac{2k_0}{Q}, \quad e = \frac{M_r}{M_\theta}, \quad k_0 = \frac{\kappa}{M_r}, \\ G &= H'_m(\zeta)/H_m(\zeta); \quad F = G' + G^2, \quad \varepsilon \rho_{00} = \rho_{00}^{(0)} + \varepsilon \rho_{00}^{(1)} + \dots \\ R_{00} &= -\frac{d^2 \ln \varepsilon \rho_{00}}{dr^2} - \frac{d \ln \varepsilon \rho_{00}}{dr} \frac{d}{dr} \ln \frac{M_r r}{\kappa^2(1-\nu^2)} - \left(\frac{m}{er} \right)^2 + \frac{4m\Omega_0}{e^2 r} \frac{\nu'}{\kappa(1-\nu^2)} \\ R_{0*} &= \frac{d \ln \varepsilon \rho_{00}}{dr} + \frac{1}{e^2} \frac{d}{dr} \ln \left(\frac{M_\theta^2 \Omega_0}{\kappa^2} \right) \\ R_{10} &= \frac{d}{dr} \ln \frac{M_r^2 r}{\varepsilon \rho_{00} \kappa^2(1-\nu^2)}; \quad R_{1*} = \left(1 - \frac{1}{e^2} \right); \quad R_{3*} = \frac{d}{dr} \ln \frac{\varepsilon \rho_{00} \Omega_0}{\kappa^2} \\ R_2 &= \frac{d}{dr} \ln \frac{\varepsilon \rho_{00} r}{\kappa^2(1-\nu^2)}, \quad R_{30} = \left(\frac{m}{r} \right)^2 + \frac{4m\Omega_0}{r} \frac{\nu'}{\kappa(1-\nu^2)} \end{aligned} \right\} \quad (4.11)$$

由(4.4)式,可知

$$B_1 \tilde{\phi}_1 = -z_{*0}^{(0)}(G + \beta) \cdot [1 - D_0(r)k - D_1(r) + \phi_{\varepsilon_{*0}}] \quad (4.12)$$

因此,我们进一步得到:

$$\begin{aligned} G \left(1 + \frac{2k_0}{Q} D_0 \right) \frac{dk}{dr} + F \left(1 + \frac{2k_0}{Q} D_0 \right) k^2 + Gk \left[\frac{2k_0}{Q} D_0' - \frac{2k_0}{Q} \frac{F}{G} (1 - D_1) + R_{10} \right] \\ - k_0^2 (1 - \nu^2) + R_{00} + \frac{2k_0}{Q} \left[G(D_1' + \phi_{\varepsilon_{*0}}') - GR_2(1 + \phi_{\varepsilon_0} + \phi_{\varepsilon_{*0}}) \right. \\ \left. + \frac{R_{30}}{k} (1 + \phi_{\varepsilon_0} + \phi_{\varepsilon_{*0}}) \right] - \frac{2k_0}{Q} \left[\frac{d}{dr} \beta (1 + \phi_{\varepsilon_0} + \phi_{\varepsilon_{*0}}) + G\beta k (1 + \phi_{\varepsilon_0} + \phi_{\varepsilon_{*0}}) \right. \\ \left. + R_2 \beta (1 + \phi_{\varepsilon_0} + \phi_{\varepsilon_{*0}}) \right] = \frac{-1}{\kappa \nu} \left(\frac{2mQ_0}{r} \right) \cdot \left\{ \frac{2k_0}{Q} \left(-\frac{1}{k^2} \frac{dk}{dr} + \frac{R_{3*}}{k} \right) \right. \\ \left. + GkR_{1*} + R_{0*} + \frac{2k_0}{Q} \left[\frac{d}{dr} \left(\frac{\phi_{\varepsilon_0} + \phi_{\varepsilon_{*0}}}{k} \right) + R_{2*} \frac{\phi_{\varepsilon_0} + \phi_{\varepsilon_{*0}}}{k} \right. \right. \\ \left. \left. - \beta (1 + \phi_{\varepsilon_0} + \phi_{\varepsilon_{*0}}) \right] \right\} \quad (4.13) \end{aligned}$$

这就是在参数 $\varepsilon, \varepsilon_*$ 一级近似下的对称面上波的传播方程。当 $\varepsilon \rightarrow 0, \varepsilon_* \rightarrow 0$ 时,它便退化为文献[4]中的方程(3.3)(这里符号略有改动)。

五、局部解及局部 Jeans 稳定性的讨论

一般地,在盘状星系中有 $k_0 \gg 1$ 。我们可首先讨论方程(4.13)的局部近似解。若忽略共转圈共振效应不计,并设 $\varepsilon, \varepsilon_* \rightarrow 0$; 对于短波扰动 ($|\zeta| \gg 1$) 作渐近展开式:

$$\left. \begin{aligned} F &= -1 - \frac{i\varepsilon}{\zeta} + \frac{2m^2 + 1}{2\zeta^2} + \dots \\ G &= i\varepsilon - \frac{1}{2\zeta} + i\varepsilon \frac{(1/4) - m^2}{2\zeta^2} + \dots \end{aligned} \right\} \quad (5.1)$$

取其第一项,这样若只保留方程(4.13)中的主要项,便得到平面星盘密度波的局部色散关系:

$$\nu^2 = \hat{x}^2 + \frac{2}{Q^{(0)}} \hat{x} + 1 \quad (5.2)$$

这里

$$Q^{(0)} = Q|_{\varepsilon \rightarrow 0}, \quad \hat{x} = k/k_0 \quad (5.3)$$

由此推出局部 Jeans 稳定性判据为:

$$Q^{(0)} \geq 1 \quad (5.4)$$

这是我们在文献[4]中已获得的结果。如果仍保留“准单色波近似”假设 ($\varepsilon_* \rightarrow 0$), 但考虑有限厚星盘 ($\varepsilon > 0$) 上的短波扰动,同样可推出其局部色散关系

$$\begin{aligned} \nu^2 = (1 + \mu) \hat{x}^2 - \frac{2}{Q} (1 + D_0' - D_1) \hat{x} + 1, \\ \left(\mu = \frac{2k_0}{Q} D_0 \right) \end{aligned} \quad (5.5)$$

对应的 Jeans 稳定性判据是:

$$Q^2 + 2k_0 D_0 Q \geq (1 + D'_0 - D_1)^2$$

或

$$Q \geq Q_* = -k_0 D_0 + \sqrt{k_0^2 D_0^2 + (1 + D'_0 - D_1)^2} \approx 1 - (k_0 D_0 + D_1 - D'_0) + \dots \quad (5.6)$$

这里 $k_0 D_0 + D_1 = \psi_*$, 恰好是由盘厚度引起的对称面上引力势的变化, 容易想象其值必为正. 所以当盘厚度的变化不剧烈 (从而 $D'_0(r)$ 影响甚微) 时, $Q_* \ll 1$. 这就说明, 厚度效应将引起局部 Jeans 稳定性的 Q 临界值减少. 此即当 $M_r, \sigma_0(r)$ (或 $Q, \sigma_0(r)$) 固定不变时, 盘越厚, 运动越稳定. 或者说, 一个处于临界 Jeans 稳定的无限薄盘, 若保持 (M_r, σ_0) (或 $Q, \sigma_0(r)$) 值不变, 而令盘厚逐渐增加, 则星盘仍将稳定; 反之则不然. 一个处于临界局部 Jeans 稳定的有限厚盘进一步坍缩后, 其对称面上必将出现局部引力不稳定性, 从而使某一种波数的短波扰动迅速增长起来, 这种小尺度波动最终会转化为湍流, 因而提高对称面上的弥散速度 M_r , 随之提高 Q 值. 这样经过充分长时间之后, Jeans 不稳定将被消耗完, 最后星盘又重新处于临界 Jeans 稳定状态. 因此看来, 当 $\varepsilon \rightarrow 0$ 时, 似乎应有 $Q^{(0)} \equiv 1$. 不过这只是当弥散速度仅仅是由于局部 Jeans 不稳定性引起时的情形. 而实际上, 除去 Jeans 不稳定性之外, 在星盘内还有许多其它的激发不稳定性的机制 (包括微观的及宏观的). 所以实际上我们会得到 $Q^{(0)} = 1 + \eta$ ($\eta > 0$), 其中 η 即代表这些物理机制的影响. 以银河系为例, 在太阳附近, 若取观测值 $a'_r = 36$ 公里/秒, 则推出 $\eta \approx 0.2$.

这里强调一点, 本文中对于 ε 零级近似下的色散关系及稳定性判据 (5.2), (5.4) 与林家翘等人对平面星盘所作出的相应结果是有区别的. 应当注意, 这里 Q 值的定义与林家翘等人所使用的 Q 值定义 (姑且记作 Q_c) 是不同的. 这里

$$Q^{(0)} = \frac{2\kappa M_r}{\rho_0^{(0)} z_{*1}^{(0)}} = \frac{4\kappa M_r}{\sigma_0^{(0)}} \left(\frac{z_{*0}}{z_{*1}^{(0)}} \right) = Q_c \left(\frac{z_{*0}}{z_{*1}^{(0)}} \right) = 2Q_c \quad (5.7)$$

出现因子 $(z_{*0}/z_{*1}^{(0)})$ 的原因是基态, 扰动态体密度沿 z 方向有不同的分布. 从而有

$$\left(\frac{\rho_1^{(0)}}{\rho_0^{(0)}} \right) = \left(\frac{z_{*0}}{z_{*1}^{(0)}} \right) \frac{\sigma_1^{(0)}}{\sigma_0^{(0)}} \neq \frac{\sigma_1^{(0)}}{\sigma_0^{(0)}}$$

因子 $(z_{*0}/z_{*1}^{(0)})$ 的出现把 Jeans 稳定性所要求的最小弥散速度值从林家翘的计算值降低了一半.

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HYDRODYNAMICAL THEORY OF THREE-DIMENSIONAL DENSITY WAVE FOR SPIRAL STRUCTURE OF GALAXIES (I)

—LOCAL SOLUTION AND DISCUSS OF LOCAL JEAN'S STABILITY

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Abstract

The present paper is the first part of the author's works on three-dimensional density waves theory of disk-shaped galaxies of finite thickness. In this paper the basic equation governing wave propagation over symmetric plane of the galaxy, to the first order approximation in the thickness parameter ε and the "quasi-monochromatic wave" parameter ε_* , is derived. Its local approximate solutions are given, and influence of the thickness of disk on the local Jeans stabilities is discussed. These results are a preparation for developing a theory to obtain global mode solutions.