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各向異性的懸臂梁負担均佈載荷的彎曲問題*

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柱體是工程結構中常常遇到的結構單位. 按照載荷情況的不同,主要的柱體問題有下面三類:(1)柱體的扭轉問題;(2)懸臂梁在其懸空端負担一個横向載荷的問題;(3)懸臂梁負担均佈載荷的問題. 在各向同性的場合,前兩問題首先由聖維南建立一般性的理論. 第三個問題則首先由米恰耳[1]創立一般性的理論. 米恰耳將所設問題化爲一個平面形變問題. 在任意各向異性的場合,前兩問題已由 C. Γ. 列赫尼茨基[2,3]建立了一般性的理論. 本文的目的在建立任意各向異性的懸臂梁負担均佈載荷的一般理論. 我們利用 C. Γ. 列赫尼茨基的方法和結果,將所設問題化爲一個廣義平面形變問題. 後者的一般性的解法,亦已經由 C. Γ. 列赫尼茨基建立.

今考慮一個任意各向異性的懸臂梁負担均佈載荷的平衡問題. 取一直角坐標系 (x, y, z) 使 xy 面跟梁的懸空端相重,命 z 軸指向梁的內部,如圖 1 所示. 作用於梁上的體積載荷 X, Y 和表面載荷 X_n, Y_n 都只爲 x, y 的函數而跟坐標 z 無關. 假定作用於單位長度(沿 z 軸)內的外界載荷的合力 P 不等於零. (在實際問題中 P 等於零的情形極少遇到,同時如果 P 等於零,問題反可簡化許多.)於是適當地選擇坐標軸的原點和 y 軸的方向,我們可使 P 通過 z 軸,並和 y 軸同向,亦即我們可使

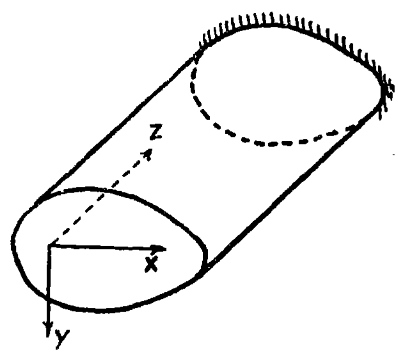

圖 1

$$\left.\begin{aligned} &\iint X\,dx\,dy + \int X_n\,ds = 0, \qquad \iint Y\,dx\,dy + \int Y_n\,ds = P, \\ &\iint (xY - yX)\,dx\,dy + \int (xY_n - yX_n)\,ds = 0. \end{aligned}\right\} \tag{1}$$

*1956 年 2 月 3 日收到.

式中的面積分遍及於截面的全部面積，線積分遍及於截面的全部週線.

梁中的應力和位移應適合下面的廣義胡克定律：

$$
\left.\begin{aligned}
\frac{\partial u}{\partial x} &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z + a_{14}\tau_{yz} + a_{15}\tau_{xz} + a_{16}\tau_{xy},\\
\frac{\partial v}{\partial y} &= a_{12}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z + a_{24}\tau_{yz} + a_{25}\tau_{xz} + a_{26}\tau_{xy},\\
\frac{\partial w}{\partial z} &= a_{13}\sigma_x + a_{23}\sigma_y + a_{33}\sigma_z + a_{34}\tau_{yz} + a_{35}\tau_{xz} + a_{36}\tau_{xy},\\
\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} &= a_{14}\sigma_x + a_{24}\sigma_y + a_{34}\sigma_z + a_{44}\tau_{yz} + a_{45}\tau_{xz} + a_{46}\tau_{xy},\\
\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} &= a_{15}\sigma_x + a_{25}\sigma_y + a_{35}\sigma_z + a_{45}\tau_{yz} + a_{55}\tau_{xz} + a_{56}\tau_{xy},\\
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} &= a_{16}\sigma_x + a_{26}\sigma_y + \sigma_{36}\sigma_z + a_{46}\tau_{yz} + a_{56}\tau_{xz} + a_{66}\tau_{xy},
\end{aligned}\right\} \tag{2}
$$

和平衡方程

$$
\left.\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0,\\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0,\\
&\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0.
\end{aligned}\right\} \tag{3}
$$

在梁的側面上有表面載荷 X_n, Y_n 作用，所以應力應適合下面的邊界條件（n 是梁側面的法線，以向外者爲正).

在側面上：

$$
\left.\begin{aligned}
&\sigma_x\cos(n,x) + \tau_{xy}\cos(n,y) = X_n,\\
&\tau_{xy}\cos(n,x) + \sigma_y\cos(n,y) = Y_n,\\
&\tau_{xz}\cos(n,x) + \tau_{yz}\cos(n,y) = 0.
\end{aligned}\right\} \tag{4}
$$

在梁的懸空端，我們不要求每點都不受外力作用. 我們根據聖維南原理，只要求作用於懸空端上的外力的合力和合力矩等於零. 這樣在梁的懸空端上應力只需要適合下面的邊界條件.

在懸空端上：

$$
\left.\begin{aligned}
&\iint \sigma_z\,dx\,dy = 0, \quad \iint (x\tau_{yz} - y\tau_{xz})\,dx\,dy = 0,\\
&\iint \tau_{xz}\,dx\,dy = 0, \quad \iint y\,\sigma_z\,dx\,dy = 0,\\
&\iint \tau_{yz}\,dx\,dy = 0, \quad \iint x\,\sigma_z\,dx\,dy = 0.
\end{aligned}\right\} \tag{5}
$$

這個邊界條件尙可加以推廣，使它不限制在懸空端上. 根據平衡條件，在任一截面 $z=z$ 上的横力是 $(-Pz,)$，撓矩是 $\left(-\frac{P}{2}z^2\right)$，所以在任一截面上，應力應適合下面的條件:

$$\left.\begin{aligned} &\iint \sigma_z\,dx\,dy=0, && \iint (x\,\tau_{yz}-y\,\tau_{xz})\,dx\,dy=0,\\ &\iint \tau_{xz}\,dx\,dy=0, && \iint y\,\sigma_z\,dx\,dy=-\frac{P}{2}z^2,\\ &\iint \tau_{yz}\,dx\,dy=-Pz, && \iint x\,\sigma_z\,dx\,dy=0. \end{aligned}\right\} \tag{6}$$

所以我們的問題是尋找應力和位移，使它們適合方程 (2)、(3) 和邊界條件 (4)、(6).

今試考慮下面的應力和位移狀態:

$$\left.\begin{aligned} &\sigma_x'=\frac{\partial\sigma_x}{\partial z}, && \sigma_y'=\frac{\partial\sigma_y}{\partial z}, && \sigma_z'=\frac{\partial\sigma_z}{\partial z},\\ &\tau_{yz}'=\frac{\partial\tau_{yz}}{\partial z}, && \tau_{xz}'=\frac{\partial\tau_{xz}}{\partial z}, && \tau_{xy}'=\frac{\partial\tau_{xy}}{\partial z},\\ &u'=\frac{\partial u}{\partial z}, && v'=\frac{\partial v}{\partial z}, && w'=\frac{\partial w}{\partial z}. \end{aligned}\right\} \tag{7}$$

將方程 (2) 和 (3) 以及邊界條件 (4) 和 (6) 各對 z 微分一次，然後將 (7) 式代入，可知 σ_x', σ_y', $\cdots$ w' 適合下面的方程和邊界條件:

$$\left.\begin{aligned} \frac{\partial u'}{\partial x}&=a_{11}\sigma_x'+a_{12}\sigma_y'+a_{13}\sigma_z'+a_{14}\tau_{yz}'+a_{15}\tau_{xz}'+a_{16}\tau_{xy}',\\ \frac{\partial v'}{\partial y}&=a_{12}\sigma_x'+a_{22}\sigma_y'+a_{23}\sigma_z'+a_{24}\tau_{yz}'+a_{25}\tau_{xz}'+a_{26}\tau_{xy}',\\ \frac{\partial w'}{\partial z}&=a_{13}\sigma_x'+a_{23}\sigma_y'+a_{33}\sigma_z'+a_{34}\tau_{yz}'+a_{35}\tau_{xz}'+a_{36}\tau_{xy}',\\ \frac{\partial w'}{\partial y}+\frac{\partial v'}{\partial z}&=a_{14}\sigma_x'+a_{24}\sigma_y'+a_{34}\sigma_z'+a_{44}\tau_{yz}'+a_{45}\tau_{xz}'+a_{46}\tau_{xy}',\\ \frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}&=a_{15}\sigma_x'+a_{25}\sigma_y'+a_{35}\sigma_z'+a_{45}\tau_{yz}'+a_{55}\tau_{xz}'+a_{56}\tau_{xy}',\\ \frac{\partial v'}{\partial x}+\frac{\partial u'}{\partial y}&=a_{16}\sigma_x'+a_{26}\sigma_y'+a_{36}\sigma_z'+a_{46}\tau_{yz}'+a_{56}\tau_{xz}'+a_{66}\tau_{xy}'. \end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned} \frac{\partial\sigma_x'}{\partial x}+\frac{\partial\tau_{xy}'}{\partial y}+\frac{\partial\tau_{xz}'}{\partial z}&=0,\\ \frac{\partial\tau_{xy}'}{\partial x}+\frac{\partial\sigma_y'}{\partial y}+\frac{\partial\tau_{yz}'}{\partial z}&=0,\\ \frac{\partial\tau_{xz}'}{\partial x}+\frac{\partial\tau_{yz}'}{\partial y}+\frac{\partial\sigma_z'}{\partial z}&=0. \end{aligned}\right\} \tag{9}$$

在側面上:

$$
\left.\begin{aligned}
&\sigma'_x \cos(n,x) + \tau'_{xy} \cos(n,y) = 0, \\
&\tau'_{xy} \cos(n,x) + \sigma'_y \cos(n,y) = 0, \\
&\tau'_{xz} \cos(n,x) + \tau'_{yz} \cos(n,y) = 0.
\end{aligned}\right\} \qquad (10)
$$

$$
\left.\begin{aligned}
&\iint \sigma'_z \, dx\, dy = 0, &&\iint (x\,\tau'_{yz} - y\,\tau'_{xz})\, dx\, dy = 0, \\
&\iint \tau'_{xz} \, dx\, dy = 0, &&\iint y\,\sigma'_z \, dx\, dy = -Pz, \\
&\iint \tau'_{yz} \, dx\, dy = -P, &&\iint x\,\sigma'_z \, dx\, dy = 0.
\end{aligned}\right\} \qquad (11)
$$

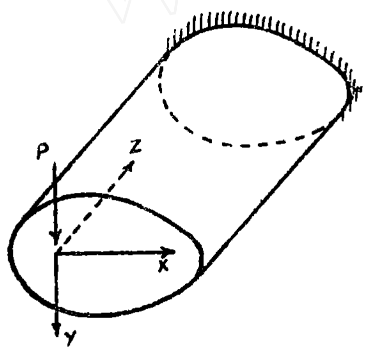

圖 2

方程 (8)、(9) 表示 $\sigma'_x, \cdots, w'$ 是適合連續條件和平衡條件的一組應力形變狀態,而邊界條件 (10)、(11) 則顯示出這組應力形變狀態乃是上述懸臂梁在其側面不受外力作用而在其懸空端負担一個橫向載荷 P 所產生的,如圖 2 所示.

懸臂梁在其懸空端負担一個橫向載荷的問題,業經 C. Γ. 列赫尼茨基詳加研究. 他建立了該問題的一般性的理論,並接着用這個理論求得了若干具體問題的解答. 本文不擬摘錄他所求得的結果,需要特別提出的是除 σ'_z 外,其他各應力張量的分量都爲與 z 無關的函數. 此後我們假定 $\sigma'_x, \cdots, w'$ 已利用 C. Γ. 列赫尼茨基的方法求得.

將 $\sigma'_x, \cdots, w'$ 各對 z 積分一次,得到 (文字右上角的 ◦ 號,表示該函數在 $z=0$ 時的數值)

$$
\left.\begin{aligned}
&\sigma_x = \int_0^z \sigma'_x \, dz + \sigma_x^{\circ}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&w = \int_0^z w' \, dz + w^{\circ}.
\end{aligned}\right\} \qquad (12)
$$

因此爲了求 $\sigma_x, \sigma_y, \cdots, w$, 必須先設法求出 $\sigma_x^{\circ}, \sigma_y^{\circ}, \cdots, w^{\circ}$. 將方程 (8)、(9) 和邊界條件 (10)、(11) 對 z 從零到 z 積分一次,並應用方程 (2)、(3) 和邊界條件 (4)、(6) 以消去積分後出現的 $\sigma_x, \cdots, w$ 等,便得到

$$\left.\begin{aligned}
\frac{\partial u^\circ}{\partial x} &= a_{11}\sigma_x^\circ + a_{12}\sigma_y^\circ + a_{13}\sigma_z^\circ + a_{14}\tau_{yz}^\circ + a_{15}\tau_{xz}^\circ + a_{16}\tau_{xy}^\circ,\\
\frac{\partial v^\circ}{\partial x} &= a_{12}\sigma_x^\circ + a_{22}\sigma_y^\circ + a_{23}\sigma_z^\circ + a_{24}\tau_{yz}^\circ + a_{25}\tau_{xz}^\circ + a_{26}\tau_{xy}^\circ,\\
(w')^\circ &= a_{13}\sigma_x^\circ + a_{23}\sigma_y^\circ + a_{33}\sigma_z^\circ + a_{34}\tau_{yz}^\circ + a_{35}\tau_{xz}^\circ + a_{36}\tau_{xy}^\circ,\\
\frac{\partial w^\circ}{\partial y} + (v')^\circ &= a_{14}\sigma_x^\circ + a_{24}\sigma_y^\circ + a_{34}\sigma_z^\circ + a_{44}\tau_{yz}^\circ + a_{45}\tau_{xz}^\circ + a_{46}\tau_{xy}^\circ,\\
(u')^\circ + \frac{\partial w^\circ}{\partial x} &= a_{15}\sigma_x^\circ + a_{25}\sigma_y^\circ + a_{35}\sigma_z^\circ + a_{45}\tau_{yz}^\circ + a_{55}\tau_{xz}^\circ + a_{56}\tau_{xy}^\circ,\\
\frac{\partial v^\circ}{\partial x} + \frac{\partial u^\circ}{\partial y} &= a_{16}\sigma_x^\circ + a_{26}\sigma_y^\circ + a_{36}\sigma_z^\circ + a_{46}\tau_{yz}^\circ + a_{56}\tau_{xz}^\circ + a_{66}\tau_{xy}^\circ.
\end{aligned}\right\} \quad (13)$$

$$\left.\begin{aligned}
&\frac{\partial\sigma_x^\circ}{\partial x} + \frac{\partial\sigma_y^\circ}{\partial y} + X + \tau_{xz}' = 0,\\
&\frac{\partial\tau_{xy}'}{\partial x} + \frac{\partial\sigma_y'}{\partial y} + Y + \tau_{yz}' = 0,\\
&\frac{\partial\tau_{xz}^\circ}{\partial x} + \frac{\partial\tau_{yz}^\circ}{\partial y} + (\sigma_z')^\circ = 0.
\end{aligned}\right\} \quad (14)$$

$$\left.\begin{aligned}
&\sigma_x^\circ\cos(n,x) + \tau_{xy}^\circ\cos(n,y) = X_n,\\
&\tau_{xy}^\circ\cos(n,x) + \sigma_y^\circ\cos(n,y) = Y_n,\\
&\tau_{xz}^\circ\cos(n,x) + \tau_{yz}^\circ\cos(n,y) = 0.
\end{aligned}\right\} \quad (15)$$

$$\left.\begin{aligned}
&\iint \sigma_z^\circ\,dx\,dy = 0, &&\iint (x\tau_{yz}^\circ - y\tau_{xz}^\circ)\,dx\,dy = 0,\\
&\iint y\sigma_z^\circ\,dx\,dy = 0, &&\iint \tau_{xz}^\circ\,dx\,dy = 0,\\
&\iint x\sigma_z^\circ\,dx\,dy = 0, &&\iint \tau_{yz}^\circ\,dx\,dy = 0.
\end{aligned}\right\} \quad (16)$$

由於方程（1）和（11），方程（14）中的體積力 $(X+\tau_{xz}')$，$(Y+\tau_{yz}')$，$(\sigma_z')^\circ$ 恰能與邊界條件（15）中的表面力 X_n，Y_n 維持平衡，因此從這組方程和邊界條件可以看出，σ_x°，σ_y°，…，w° 乃是一組廣義平面形變問題的應力和位移.

從（13）式的第三式可解得 σ_z° 如下：

$$\sigma_z^\circ = -\frac{1}{a_{33}}(a_{13}\sigma_x^\circ + a_{23}\sigma_y^\circ + a_{34}\tau_{yz}^\circ + a_{35}\tau_{xz}^\circ + a_{36}\tau_{xy}^\circ) + \frac{1}{a_{33}}(w')^\circ. \quad (17)$$

將此代入（13）式的其他各式，得到

$$\left.\begin{aligned}
\frac{\partial u^\circ}{\partial x} &= \beta_{11}\sigma_x^\circ + \beta_{12}\sigma_y^\circ + \beta_{14}\tau_{yz}^\circ + \beta_{15}\tau_{xz}^\circ + \beta_{16}\tau_{xy}^\circ + \frac{a_{13}}{a_{33}}(w')^\circ,\\
\frac{\partial v^\circ}{\partial y} &= \beta_{12}\sigma_x^\circ + \beta_{22}\sigma_y^\circ + \beta_{24}\tau_{yz}^\circ + \beta_{25}\tau_{xz}^\circ + \beta_{26}\tau_{xy}^\circ + \frac{a_{23}}{a_{33}}(w')^\circ,\\
\frac{\partial v^\circ}{\partial x} + \frac{\partial u^\circ}{\partial y} &= \beta_{16}\sigma_x^\circ + \beta_{26}\sigma_y^\circ + \beta_{46}\tau_{yz}^\circ + \beta_{56}\tau_{xz}^\circ + \beta_{66}\tau_{xy}^\circ + \frac{a_{36}}{a_{33}}(w')^\circ,\\
\frac{\partial w^\circ}{\partial x} &= \beta_{15}\sigma_x^\circ + \beta_{25}\sigma_y^\circ + \beta_{45}\tau_{yz}^\circ + \beta_{55}\tau_{xz}^\circ + \beta_{56}\tau_{xy}^\circ + \frac{a_{35}}{a_{33}}(w')^\circ - (u')^\circ,\\
\frac{\partial w^\circ}{\partial y} &= \beta_{14}\sigma_x^\circ + \beta_{24}\sigma_y^\circ + \beta_{44}\tau_{yz}^\circ + \beta_{45}\tau_{xz}^\circ + \beta_{46}\tau_{xy}^\circ + \frac{a_{34}}{a_{33}}(w')^\circ - (v')^\circ.
\end{aligned}\right\} \quad (18)$$

這裏

$$\beta_{ij} = a_{ij} - \frac{a_{i3}a_{j3}}{a_{33}}, \qquad (i, j = 1, 2, 4, 5, 6).$$

方程 (14)、(18) 在邊界條件 (15)、(16) 下的解答，可用 C. Γ. 列赫尼茨基的關於廣義平面形變問題的一般性的解法求得.

命

$$\Sigma_x = \int (X + \tau'_{xz})\,dx, \quad \Sigma_y = \int (Y + \tau'_{yz})\,dy, \quad T = \int (\sigma'_z)^\circ\,dx. \qquad (19)$$

於是適合 (14) 式的各應力分量可用兩個應力函數 F, ψ 表示如下:

$$\left.\begin{aligned}
&\sigma_x^\circ = \frac{\partial^2 F}{\partial y^2} - \Sigma_x, \quad \sigma_y^\circ = \frac{\partial^2 F}{\partial x^2} - \Sigma_y, \quad \tau_{xy}^\circ = -\frac{\partial^2 F}{\partial x\,\partial y},\\
&\tau_{xz}^\circ = \frac{\partial \psi}{\partial y} - T, \qquad \tau_{yz}^\circ = -\frac{\partial \psi}{\partial x}.
\end{aligned}\right\} \quad (20)$$

將此代入 (18) 式，然後消去 u°, v°, w°, 得 F 和 ψ 所適合的方程如下:

$$\begin{aligned}
L_4 F + L_3 \psi = &\left(\beta_{12}\frac{\partial^2}{\partial x^2} - \beta_{16}\frac{\partial^2}{\partial x\,\partial y} + \beta_{11}\frac{\partial^2}{\partial y^2}\right)\Sigma_x + \\
&+ \left(\beta_{22}\frac{\partial^2}{\partial x^2} - \beta_{26}\frac{\partial^2}{\partial x\,\partial y} + \beta_{12}\frac{\partial^2}{\partial y^2}\right)\Sigma_y + \\
&+ \left(\beta_{25}\frac{\partial^2}{\partial x^2} - \beta_{56}\frac{\partial^2}{\partial x\,\partial y} + \beta_{15}\frac{\partial^2}{\partial y^2}\right)T - \\
&- \left(\frac{a_{23}}{a_{33}}\frac{\partial^2}{\partial x^2} - \frac{a_{36}}{a_{33}}\frac{\partial^2}{\partial x\,\partial y} + \frac{a_{13}}{a_{33}}\frac{\partial^2}{\partial y^2}\right)(w')^\circ, \qquad (21\text{-a})
\end{aligned}$$

$$\begin{aligned}
L_3 F + L_2 \psi = &-\left(\beta_{14}\frac{\partial}{\partial x} - \beta_{15}\frac{\partial}{\partial y}\right)\Sigma_x - \left(\beta_{24}\frac{\partial}{\partial x} - \beta_{25}\frac{\partial}{\partial y}\right)\Sigma_y - \\
&- \left(\beta_{45}\frac{\partial}{\partial x} - \beta_{55}\frac{\partial}{\partial y}\right)T + \left(\frac{a_{34}}{a_{33}}\frac{\partial}{\partial x} - \frac{a_{35}}{a_{33}}\frac{\partial}{\partial y}\right)(w')^\circ - \\
&- \frac{\partial}{\partial x}(v')^\circ + \frac{\partial}{\partial y}(u')^\circ. \qquad (21\text{-b})
\end{aligned}$$

其中 L_2, L_3, L_4 是 C. Γ. 列赫尼茨基的二級、三級、四級微分算子

$$\left.\begin{aligned} L_2 &= \beta_{44}\frac{\partial^2}{\partial x^2} - 2\beta_{45}\frac{\partial^2}{\partial x\,\partial y} + \beta_{55}\frac{\partial^2}{\partial y^2}, \\ L_3 &= -\beta_{24}\frac{\partial^3}{\partial x^3} + (\beta_{25}+\beta_{46})\frac{\partial^3}{\partial x^2\,\partial y} - (\beta_{14}+\beta_{56})\frac{\partial^3}{\partial x\,\partial y^2} + \beta_{15}\frac{\partial^3}{\partial y^3}, \\ L_4 &= \beta_{22}\frac{\partial^4}{\partial x^4} - 2\beta_{26}\frac{\partial^4}{\partial x^3\,\partial y} + (2\beta_{12}+\beta_{66})\frac{\partial^4}{\partial x^2\,\partial y^2} - 2\beta_{16}\frac{\partial^4}{\partial x\,\partial y^3} + \beta_{11}\frac{\partial^4}{\partial y^4}. \end{aligned}\right\} \tag{22}$$

設 F_0 和 ψ_0 爲方程 (21) 的任一特解，則根據 C. Г. 列赫尼茨基的結果，各應力分量可表示如下

$$\left.\begin{aligned} \sigma_x^\circ &= 2\,\mathrm{Re}\left\{\mu_1^2\Phi_1'(z_1) + \mu_2^2\Phi_2'(z_2) + \mu_3^2\lambda_3\Phi_3'(z_3)\right\} + \frac{\partial^2 F_0}{\partial y^2} - \Sigma_x, \\ \sigma_y^\circ &= 2\,\mathrm{Re}\left\{\Phi_1'(z_1) + \Phi_2'(z_2) + \lambda_3\Phi_3'(z_3)\right\} + \frac{\partial^2 F_0}{\partial x^2} - \Sigma_y, \\ \tau_{xy}^\circ &= -2\,\mathrm{Re}\left\{\mu_1\Phi_1'(z_1) + \mu_2\Phi_2'(z_2) + \mu_3\lambda_3\Phi_3'(z_3)\right\} - \frac{\partial^2 F_0}{\partial x\,\partial y}, \\ \tau_{xz}^\circ &= 2\,\mathrm{Re}\left\{\mu_1\lambda_1\Phi_1'(z_1) + \mu_2\lambda_2\Phi_2'(z_2) + \mu_3\Phi_3'(z_3)\right\} + \frac{\partial\psi_0}{\partial y} - T, \\ \tau_{yz}^\circ &= -2\,\mathrm{Re}\left\{\lambda_1\Phi_1'(z_1) + \lambda_2\Phi_2'(z_2) + \Phi_3'(z_3)\right\} - \frac{\partial\psi_0}{\partial x}. \end{aligned}\right\} \tag{23}$$

設 s 是邊界曲線的長度． 命

$$\left.\begin{aligned} f_1(s) &= -\int_0^s\left[Y_n + \Sigma_y\cos(n,y)\right]ds, \\ f_2(s) &= \int_0^s\left[X_n + \Sigma_x\cos(n,x)\right]ds, \\ f_3(s) &= \int_0^s T\,dy. \end{aligned}\right\} \tag{24}$$

則 Φ_1, Φ_2, Φ_3 應適合下面的邊界條件 (c_1, c_2, c_3 爲常數)：

$$\left.\begin{aligned} 2\,\mathrm{Re}\left[\Phi_1 + \Phi_2 + \lambda_3\Phi_3\right] &= f_1(s) - \frac{\partial F_0}{\partial x} + c_1, \\ 2\,\mathrm{Re}\left[\mu_1\Phi_1 + \mu_2\Phi_2 + \mu_3\lambda_3\Phi_3\right] &= f_2(s) - \frac{\partial F_0}{\partial y} + c_2, \\ 2\,\mathrm{Re}\left[\lambda_1\Phi_1 + \lambda_2\Phi_2 + \Phi_3\right] &= f_3(s) - \psi_0 + c_3. \end{aligned}\right\} \tag{25}$$

從這組邊界條件求得 $\Phi_1(z_1)$, $\Phi_2(z_2)$, $\Phi_3(z_3)$ 之後，便可由公式 (23) 計算 σ_x°, $\sigma_y^\circ, \cdots, w^\circ$． 求得了 $\sigma_x', \sigma_y', \cdots$, 和 $\sigma_x^\circ, \sigma_y^\circ, \cdots$, 等兩組應力，便可根據 (12) 式簡便地求得梁中眞正的應力．

所以總起來說，懸臂梁負担均佈載荷的彎曲問題，可分解爲前後相繼的兩個問題．第一個問題的目的在求 $\sigma'_x, \sigma'_y, \cdots, w'$，乃相當於該懸臂梁在其懸空端負担橫向載荷的問題． 第二個問題的目的在求 $\sigma^\circ_x, \sigma^\circ_y, \cdots, \tau^\circ_{xy}$，乃相當於一個廣義平面形變問題．

假如在梁內任一點都有一彈性對稱面跟梁的軸垂直，那末問題便可簡化許多． 因爲這時

$$\left.\begin{aligned} &a_{14} = a_{15} = a_{24} = a_{25} = a_{34} = a_{35} = a_{46} = a_{56} = 0\,, \\ &\beta_{14} = \beta_{15} = \beta_{24} = \beta_{25} = \beta_{46} = \beta_{56} = 0\,, \end{aligned}\right\} \quad (26)$$

經過一個簡單的推算後可以證明

$$\left.\begin{aligned} &\sigma'_x = 0\,,\quad \sigma'_y = 0\,,\quad \tau'_{xy} = 0\,,\quad \sigma'_z = -\frac{P}{I}\,y\,z\,, \\ &(u')^\circ = -\omega_2\,,\quad (v')^\circ = \omega_1\,,\quad (w')^\circ = a_{33}(Ax + By + C)\,; \end{aligned}\right\} \quad (27)$$

$$\left.\begin{aligned} &\tau^\circ_{xz} = 0\,,\quad \tau^\circ_{yz} = 0\,,\quad \Phi_3(z_3) \equiv 0\,, \\ &\sigma^\circ_z = -\frac{1}{a_{33}}\,(a_{13}\,\sigma^\circ_x + a_{23}\,\sigma^\circ_y + a_{36}\,\tau^\circ_{xy}) + Ax + By + C\,, \\ &w^\circ = \omega_1\,y - \omega_2\,x + w_0\,. \end{aligned}\right\} \quad (28)$$

式中 A, B, C 和 ω_1, ω_2, w_0 是未定常數． 其他餘下的應力分量 $\sigma^\circ_x, \sigma^\circ_y, \tau^\circ_{xy}$ 可用兩個函數 $\Phi_1(z_1), \Phi_2(z_2)$ 表示如下：

$$\left.\begin{aligned} &\sigma^\circ_x = 2\,\mathrm{Re}\left\{\mu_1^2\,\Phi'_1(z_1) + \mu_2^2\,\Phi'_2(z_2)\right\} + \frac{\partial^2 F_0}{\partial y^2} - \Sigma_x\,, \\ &\sigma^\circ_y = 2\,\mathrm{Re}\left\{\Phi'_1(z_1) + \Phi'_2(z_2)\right\} + \frac{\partial^2 F_0}{\partial x^2} - \Sigma_y\,, \\ &\tau^\circ_{xy} = -2\,\mathrm{Re}\left\{\mu_1\,\Phi'_1(z_1) + \mu_2\,\Phi'_2(z_2)\right\} - \frac{\partial^2 F_0}{\partial x\,\partial y}\,. \end{aligned}\right\} \quad (29)$$

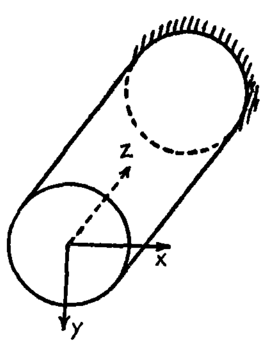

圖 3

而 $\Phi_1(z_1), \Phi_2(z_2)$ 適合下面的邊界條件：

$$\left.\begin{aligned} &2\,\mathrm{Re}\,[\Phi_1 + \Phi_2] = f_1(s) - \frac{\partial F_0}{\partial x} + c_1\,, \\ &2\,\mathrm{Re}\,[\mu_1\,\Phi_1 + \mu_2\,\Phi_2] = f_2(s) - \frac{\partial F_0}{\partial y} + c_2\,. \end{aligned}\right\} \quad (30)$$

現在我們來計算一個具體的問題以說明上述一般性理論的應用． 設欲求一圓柱形懸臂梁負担其本身重量而產生的應力． 假定梁截面的半徑爲 1（這個假定顯然不限制問題的一般性），單位體積的重量爲 W． 於是作用於單位長度內載荷的合力是

$$P = \pi W. \tag{31}$$

爲了簡便起見，假定梁內任一點均有一彈性對稱面跟梁的軸垂直．根據 C. Г. 列赫尼茨基的結果，得到

$$\left.\begin{aligned}\tau'_{xz} &= 2Bxy + C(x^2 + 3y^2 - 1),\\ \tau'_{yz} &= -B(3x^2 + y^2 - 1) - 2Cxy + 2W(x^2 + y^2 - 1),\end{aligned}\right\} \tag{32}$$

其中

$$\left.\begin{aligned}B &= 2\cdot\frac{(a_{44}-2a_{13})(a_{44}+3a_{55})-2a_{45}(a_{36}+a_{45})}{(3a_{44}+a_{55})(a_{44}+3a_{55})-4a_{45}^2},\\ C &= 2\cdot\frac{2a_{45}(a_{44}-2a_{13})-(a_{36}+a_{45})(3a_{44}+a_{35})}{(3a_{44}+a_{55})(a_{44}+3a_{55})-4a_{45}^2}.\end{aligned}\right\} \tag{33}$$

因此

$$\left.\begin{aligned}\Sigma_x &= \int\tau'_{xz}\,dx = B(x^2y+y^3-y)+C\left(\frac{x^3}{3}+3xy^2-x\right),\\ \Sigma_y &= \int\tau'_{yz}\,dy = -B\left(3x^2y+\frac{y^3}{3}-y\right)-C(xy^2+x^3-x)+2W\left(x^2y+\frac{y^3}{3}-\frac{y}{2}\right).\end{aligned}\right\} \tag{34}$$

F 適合下面的方程

$$\beta_{22}\frac{\partial^4F}{\partial x^4} - \beta_{26}\frac{\partial^4F}{\partial x^3\partial y} + (2\beta_{16}+\beta_{66})\frac{\partial^4F}{\partial x^2\partial y^2} - \beta_{16}\frac{\partial^4F}{\partial x\partial y^3} + \beta_{11}\frac{\partial^4F}{\partial y^4} = $$
$$= \beta_{22}px + \beta_{11}qy, \tag{35}$$

其中

$$\left.\begin{aligned}p &= \frac{1}{\beta_{22}}\left\{6\beta_{11}C + 2(3\beta_{26}-\beta_{16})B - 4\beta_{26}W\right\},\\ q &= \frac{1}{\beta_{11}}\left\{2(\beta_{26}-3\beta_{16})C - 6\beta_{22}B + 4(\beta_{12}+\beta_{22})W\right\}.\end{aligned}\right\} \tag{36}$$

所以 F 的一個特解是

$$F_0 = \frac{p}{5!}\left(x^5 - \frac{5}{3}x^3 + \frac{5}{8}x\right) + \frac{q}{5!}\left(y^5 - \frac{5}{3}y^3 + \frac{5}{8}y\right). \tag{37}$$

在截面的邊界上，x, y 可用參數 θ（極角）表示如下：

$$x = \cos\theta,\quad y = \sin\theta. \tag{38}$$

因此邊界上的弧長是

$$ds = d\theta,$$

於是

$$\left.\begin{aligned}f_1(\theta) &= -\int_0^\theta \Sigma_y \sin\theta\,d\theta = \frac{1}{6}\left(B+\frac{W}{2}\right)\left(\sin 2\theta - \frac{1}{2}\sin 4\theta\right),\\ f_2(\theta) &= \int_0^\theta \Sigma_x \cos\theta\,d\theta = -\frac{C}{6}\left(\sin 2\theta + \frac{1}{2}\sin 4\theta\right).\end{aligned}\right\} \tag{39}$$

因此 $\Phi_k(z_k)$ 的邊界條件化爲

$$\left.\begin{aligned} 2\,\mathrm{Re}\,[\Phi_1+\Phi_2] &= \frac{1}{6}\left(B+\frac{W}{2}\right)\left(\sin 2\theta-\frac{1}{2}\sin 4\theta\right)-\frac{1}{8}\cdot\frac{p}{4!}\cos 4\theta, \\ 2\,\mathrm{Re}\,[\mu_1\Phi_1+\mu_2\Phi_2] &= -\frac{C}{6}\left(\sin 2\theta+\frac{1}{2}\sin 4\theta\right)-\frac{1}{8}\cdot\frac{q}{4!}\cos 4\theta, \end{aligned}\right\} \tag{40}$$

設

$$\begin{aligned} \Phi_k = A_k\{(z_k-\sqrt{z_k^2-1-\mu_k^2})^2+(z_k+\sqrt{z_k^2-1-\mu_k^2})^2\}+ \\ +B_k\{(z_k-\sqrt{z_k^2-1-\mu_k^2})^4+(z_k-\sqrt{z_k^2-1-\mu_k^2})^4\}. \end{aligned} \tag{41}$$

因爲截面的邊界上

$$z_k=\cos\theta+\mu_k\sin\theta,$$

所以在截面的邊界上 Φ_k 化爲

$$\begin{aligned} \Phi_k = A_k\{[(1-i\mu_k)^2+(1+i\mu_k)^2]\cos 2\theta+i\,[(1-i\mu_k)^2-(1+i\mu_k)^2]\sin 2\theta\}+ \\ +B_k\{[(1-i\mu_k)^4+(1+i\mu_k)^4]\cos 4\theta+i\,[(1-i\mu_k)^4-(1+i\mu_k)^4]\sin 4\theta\}. \end{aligned}$$

將此代入邊界條件 (40),然後比較兩端同類項的係數,得到

$$\left.\begin{aligned} 2\,\mathrm{Re}\sum A_k\Big[(1-i\mu_k)^2+(1+i\mu_k)^2\Big] &= 0, \\ 2\,\mathrm{Re}\sum \mu_k A_k\Big[(1-i\mu_k)^2+(1+i\mu_k)^2\Big] &= 0, \\ 2\,\mathrm{Re}\sum iA_k\Big[(1-i\mu_k)^2-(1+i\mu_k)^2\Big] &= \frac{1}{6}\left(B+\frac{W}{2}\right), \\ 2\,\mathrm{Re}\sum i\mu_k A_k\Big[(1-i\mu_k)^2-(1+i\mu_k)^2\Big] &= -\frac{C}{6}, \end{aligned}\right\} \tag{42}$$

$$\left.\begin{aligned} 2\,\mathrm{Re}\sum B_k\Big[(1-i\mu_k)^4+(1+i\mu_k)^4\Big] &= -\frac{1}{8}\cdot\frac{p}{4!}, \\ 2\,\mathrm{Re}\sum \mu_k B_k\Big[(1-i\mu_k)^4+(1+i\mu_k)^4\Big] &= -\frac{1}{8}\cdot\frac{q}{4!}, \\ 2\,\mathrm{Re}\sum iB_k\Big[(1-i\mu_k)^4-(1+i\mu_k)^4\Big] &= -\frac{1}{12}\left(B+\frac{W}{2}\right), \\ 2\,\mathrm{Re}\sum i\mu_k B_k\Big[(1-i\mu_k)^4-(1+i\mu_k)^4\Big] &= -\frac{C}{12}. \end{aligned}\right\} \tag{43}$$

從方程 (42)、(43) 可解得 A_1, $\bar{A}_1$, A_2, $\bar{A}_2$ 和 B_1, $\bar{B}_1$, B_2, $\bar{B}_2$.

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ON THE THEORY OF UNIFORMLY LOADED ANISOTROPIC CANTILEVER BEAMS

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ABSTRACT

In this paper, J. H. Michell's theory[1] of uniformly loaded beams is extended to the case of anisotropic beams. The problem is reduced to two consecutive problems of bending under terminal load and of generalized plane deformation in the sense of S. G. Lehnitzky[2, 3].