On the Topological Bifurcation of Flows around a Rotating Circular Cylinder*

Hu Guo-Hui, Sun De-Jun and Yin Xie-Yuan
(*Department of Modern Mechanics, University of Science and Technology of China, Hefei, Anhui 230026, LNM, Institute of Mechanics, Acdemic Sinica, Beijing 100080)*

Abstract

Flow fields around a rotating circular cylinder in a uniform stream are computed using a low dimensional Galerkin method. Results show that formation of a Fopple vortex pair behind a stationary circular cylinder is caused by the structural instability in the vicinity of the saddle located at the rear of the cylinder. For a rotating cylinder a bifurcation diagram with consideration of two parameters, Reynolds number Re and rotation parameter α, is built by a kinematic analysis for the steady flow fields.

KEY WORDS: topological bifurcation, Galerkin Method, circular cylinder

1 INTRODUCTION

During the past few decades, applications of dynamical system theory have attracted vast interest in various areas of fluid mechanics. Both dynamical and kinematic features of wakes behind bluff bodies, such as bifurcation, transport and mixing of fluid flows, are studied so as to improve our understanding of the nature of the wakes using dynamical system theory.

In a review regarding the topology of flow separation, Chapman[1] suggested that there are two independent levels of bifurcations, which correspond two forms of the evolution equations for the velocity field:

$$\begin{cases} \frac{d\mathbf{u}}{dt} = \mathbf{H}(\mathbf{u}, t, \lambda) \\ \mathbf{u} = \frac{d\mathbf{x}}{dt} = \mathbf{G}(\mathbf{x}, \lambda) \end{cases} \tag{1}$$

The first equation is the unsteady Navier-Stokes equation, where $\mathbf{u}$ is the velocity vector, $\mathbf{H}$ is a function that describes the time evolution of the velocity field, and λ is a parameter of the problem, such as Reynolds number. The second equation defines the pathlines of the flows, $\mathbf{x}$ represents the spatial coordinate vector. Generally, bifurcation of the first equation is called *dynamical bifurcation*, and the second, *structural bifurcation or topological bifurcation.*[1] Some recent studies suggest that a change of topology, often implying the appearance or annihilation of certain critical points in the streamline equation, may not be linked with any bifurcation of the hydrodynamics. This was supported by Tsitverblit[2] and Lopez[3] in their researches concerning the bifurcation structure of axisymmetric vortex breakdown.

For flows past a stationary cylinder we also observe the inconsistence of two kinds of bifurcations mentioned above. Several investigations[4−7] of linear global stability of the two dimensional flow around a circular cylinder

*Supported by the National Natural Science Foundation of China

at low Reynolds number Re, based on the radius of cylinder a and the incoming velocity U_∞, have showed that the first dynamical Hopf bifurcation corresponds the onset of the periodic flow, and the critical Reynolds number Re_{cr} is about 45. Below Re_{cr} one cannot find any other dynamical bifurcation point. Hu *et al.*[8] generalized Noack's LDGM[6] to the case of a rotating cylinder and obtained the Hopf transitional curve with two parameters, Re and α, where $\alpha = a\Omega/U_\infty$ and Ω represents the angular velocity of the rotation. Their results show that the rotation may delay the onset of vortex shedding.

However, we know that the streamlines of the steady flow undergoes a topological change at $Re = 5$, accompanied by the formation of the Fopple vortex pair. This fact was argued to be a bifurcation in the streamlines or kinematic equations[1,9]. Bakker[9] conducted detailed studies of topological bifurcations in flow patterns with the applications of the qualitative theory of ordinary differential equations in velocity vector fields. In this paper we will use generalized LDGM to investigate the topological bifurcations for flows past stationary and rotating cylinders respectively, and study the effects of two parameters Re and α on the topogies of flow fields.

2 THE LOW DIMENSIONAL GALERKIN METHOD

The streamfunction can be approximated by a finite expansion in Hilbert space:

$$\Psi = \Psi^0(r,\theta) + \sum_{i,j} a_{ij}(t) R_i(r) \Theta_j(\theta) \qquad i = 0, \cdots, K; j = -L, \cdots, L \tag{2}$$

where Ψ^0 is the basic mode which satisfies the boundary conditions and is written as:

$$\Psi^0(r,\theta) = (r - \frac{1}{r})(1 - e^{-[(r-1)/\delta_{bm}]}) \sin\theta + \alpha \ln r \tag{3}$$

where $\delta_{bm} = 4/\sqrt{Re}$[6] , $R_i(r)$ and $\Theta_j(\theta)$ are the radial and azimuthal modes of a complete orthonormal system in the Hilbert space respectively and are chosen to satisfy the homogeneous boundary conditions. Then the Galerkin projection of the Navier-Stokes equation with streamfunction expansion yields a set of autonomous ordinary differential equations(ODEs):

$$\frac{da_{ij}}{dt} = c_{ij} + \sum_{kl} l_{ij,kl} a_{kl} + \sum_{klmn} q_{ij,klmn} a_{kl} a_{mn} \tag{4}$$

The ODEs are regarded as a (K+1)(2L+1) dimension nonlinear dynamical system with two parameters, i.e., Re and α. The Newton-Raphson iteration was used to obtain the equilibrium solutions of the ODEs (4).

As we know, the equilibrium $a_{ij}^{(s)}$ of the ODEs corresponds to a steady solution of N-S equation. The global stability characteristics of the ODEs are determined by the eigenvalues $\lambda = \pi(\sigma + iSt)$ of the linearized Jacobian matrix in the neighborhood of $a_{ij}^{(s)}$, where $\pi\sigma$ is denots the disturbance growth rate and $St = 2af/U_\infty$, where f is frequency of vortex shedding. The eigenvalues of this matrix are computed by the QR algorithm. It is well known that the Hopf bifurcation occurs if there is a pair of complex eigenvalues crossing the imaginary axis transversely.[6–8]

3 TOLOGICAL BIFURCATIONS OF FLOWS

3.1 Flows over a stationary cylinder

The streamline equations for the steady flow field in a polar coordinate system are given as:

$$\begin{cases} \dfrac{dr}{dt} = V_r(r,\theta) = \dfrac{1}{r}\dfrac{\partial\Psi}{\partial\theta} \\ \dfrac{d\theta}{dt} = \dfrac{1}{r}V_\theta(r,\theta) = -\dfrac{1}{r}\dfrac{\partial\Psi}{\partial r} \end{cases} \tag{5}$$

The velocity vector field can be achieved by the expression of streamfunction (2). Here we are primarily interested in the topological feature of the singular points in the two dimensional velocity vector field (5). The velocity field can be expanded in Taylor series in the neighborhood of the singular point and its topologies can be given by the leading terms of the Taylor expansion.

To clarify the formation of the Fopple vortex pair, we concentrate on the flow over a circular cylinder as Re increases through 5. Below $Re = 5$ the flow is attached to the surface of the cylinder with two half-saddles, one in the front and one in the rear. Just above 5 the flow is separated at the rear, and the rear half-saddle splits to three half-saddles, a full saddle and two centers in the flow. Due to the no-slip condition, all the points on the surface is regarded as the singular points of (5), *i.e.*, a singular circle, and this fact leads to the generation of a high order singularity on the wall of the cylinder. Equation (5) can be expanded as a Taylor series in the neighborhood of a point $S(r_0, \theta_0)$ on the surface, with using the equation of continuity and no-slip condition:

$$\begin{cases} \dfrac{d\xi}{dt} = \xi^2(b_1 + b_2\xi + b_3\eta + b_4\xi^2 + b_5\xi\eta + b_6\eta^2) + hot \\ \dfrac{d\eta}{dt} = \xi(a_1 + a_2\xi + a_3\eta + a_4\xi^2 + a_5\xi\eta + a_6\eta^2 + a_7\xi^3 + a_8\xi^2\eta + a_9\xi\eta^2 + a_{10}\eta^3) + hot \end{cases} \tag{6}$$

in which $\xi = r - r_0$, $\eta = \theta - \theta_0$, hot means *high order terms*. The coefficients can be obtained using streamfunction expansion, such as $b_1 = \frac{1}{2}\frac{\partial^2 V_r}{\partial r^2}$.

It is obvious that shear stress at the rear half-saddle $S_0 : r_0 = 1, \theta_0 = 0$ is zero when $Re < 5$, so there is a branch streamline though S_0. In order to get rid of the high order singularity in the vicinity of S_0, a new variable t^* is introduced[9,10]:

$$\frac{1}{\xi}\frac{d}{dt} = \frac{d}{dt^*} \tag{7}$$

Considering the symmetry with respect to the axis of $\theta = 0$ and the transformation (7), (6) can be written as:

$$\begin{cases} \dfrac{d\xi}{dt^*} = b_1\xi + b_2\xi^2 + b_4\xi^3 + b_6\xi\eta^2 + hot \\ \dfrac{d\eta}{dt^*} = a_3\eta + a_5\xi\eta + a_8\xi^2\eta + a_{10}\eta^3 + hot \end{cases} \tag{8}$$

All coefficients in (8) are functions of the Reynolds number Re and can be solved explicitly, and $a_3 = -b_1/2$. Therefore the point $S_0 : \xi = 0, \eta = 0$ is evidently a saddle of the system (8) if $a_3 \neq 0$. Suppose there exists a Reynolds number Re^0_{tb} satisfying $a_3 = 0$, then it is defined as *topological bifurcation point*. When $Re = Re^0_{tb}$, the topological bifurcation appears and suggests the structural instability in the vicinity of the rear half-saddle. The values of a_3 in the neighborhood of $Re = 5$ are computed(Fig 1, for $K = 6, L = 4$). Figure 1 shows a_3 varies with Re linearly. When the Reynolds number is about 5.07, a_3 crosses the zero-axis from negative to positive. This finding provides evidence that the formation of the Fopple vortex pair is caused by the structural instability in the vicinity of the rear half-saddle. In order to confirm the change of topology is not associated with the dynamical bifurcation, the eigenvalue spectrum for $Re = 5$ with $K = 6$ and $L = 4$ is depicted in Figure 2. It can be seen that all eigenvalues are located at the left of the complex plane and far from the imaginary axis, which means no dynamical bifurcation occurs.

To test the robustness of Re_{tb}^0 with the number of Galerkin mode, we computed the Re_{tb}^0 when $K = 8$ and $L = 6$. The obtained result, $Re_{tb}^0 = 4.6$, is reasonable considering LDGM as just a qualitative method.

3.2 Flows over a rotating cylinder

It is anticipated that the approach outlined in the study for the stationary cylinder could be applied to the wakes behind a rotating (clockwise) circular cylinder. Using the streamfunction (2) and the streamline equation (5), the singular points in the flow fields are solved by Newton-Raphson iteration. In contrast to the flows past the stationary cylinder, no singular point, i.e. zero-velocity point, is found on the surface of the cylinder on account of the no-slip condition. It can be expected that closed streamlines around the cylinder will always exist for any $\alpha \neq 0$, so the effects of rotation is to change substantially the topology near the surface of the cylinder.

The procedure we conducted to pursue the topology of the wakes past a rotating cylinder is similar to the one above mentioned. Expanding the velocity field in the neighborhood of singular point $S(r_0, \theta_0)$ with a Taylor series results in:

$$\begin{cases} \frac{d\xi}{dt} = a_1\xi + a_2\eta + a_{11}\xi^2 + a_{12}\xi\eta + a_{22}\eta^2 + hot \\ \frac{d\eta}{dt} = b_1\xi + b_2\eta + b_{11}\xi^2 + b_{12}\xi\eta + b_{22}\eta^2 + hot \end{cases} \tag{9}$$

Consequently the topology in the vicinity of S must be determined by the characteristics of the linearized matrix of (9), $\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$.

For $Re_{tb}^0 < Re < Re_{cr}$, several closed streamline regions, called *standing vortices*, may emerge in the flows. Our results suggest that, as Ingham[11] supposed, the effects of rotation appear to annihilate the standing vortices. A topological bifurcation diagram was built to clarify the transition processes of the topologies, which is illustrated in Figure 3. The topologies can be classified to three regions (**A**, **B**, **C**) based on the number of the standing vortices for the steady wakes with a fixed Reynolds number Re. For very small α in region **A**, the pair of Fopple vortices only detaches from the surface of the cylinder slightly, and two standing vortices stand asymmetrically with different sizes. Following the increase of α the size of the two vortices decreases. A typical picture of the streamlines is shown in Figure 4(a) when $Re = 20$ and $\alpha = 0.02$. Then we turn to region **B** for α up to a critical value α_1, which means the topological bifurcation point(Fig. 4(b)). Figure 5 is drawn to show the sketch of the flow pattern when the first topological bifurcation occurs. In this case, there are four singular points in the velocity vector field (14), two of them are the saddles S_1 and S_2, the center C_0, and the cusp point P_c, which is formed by folding a saddle and a center together. Then the vortex rotating clockwise disappears, and only one vortex rotating counterclockwise stands in the wakes. With further increase of α, the saddle S_2 and the center C_0 are folded as a cusp point again when α reaches another topological bifurcation point α_2, see Fig. 4(c). The topological bifurcation point α_2 is computed to be 0.017 with $K = 6$ and $L = 4$ when $Re = 10$, and 0.013 with $K = 12, L = 10$. When $Re = 20$ with $K = 12, L = 10$, α_2 is 0.111. The computation implies that all of the standing vortices will vanish if $\alpha > \alpha_2$, which points the transition to region **C**(Fig. 4(d)).

4 CONCLUSIONS

Summarily, the view of the topological bifurcation is used to examine the structures of the flows for either a stationary or a rotating circular cylinder. When the rotation is absent, the global stability analysis excludes any connection between the formation of the Fopple vortex pair and the dynamical bifurcation. Moreover, it is confirmed that the flow undergoes a topological bifurcation when the Reynolds number is about 5, which indicates the structural instability in the vicinity of the rear half-saddle and the occurrence of the vortex pair. For the wakes behind the rotating circular cylinder, the obtained topological bifurcation diagram illustrates three regions based on the number of the standing vortices in the flows. Results support Ingham's assumption[11] that the effects of the rotation tend to annihilate the standing vortex.

References

[1] Chapman G T, Yates L A. Topology of flow separtion on three-dimension bodies. Applied Mech. Rev., 1991, 44: 329.

[2] Tsitverblit N. Vortex breakdown in a cylinder container in the light of continuation of a steady solution. Fluid Dyn. Res., 1993, 11: 19.

[3] Lopez J M. On the bifurcation structure of axisymmetric vortex breakdown in a constricted pipe. Phys. Fluids, 1994, 6(11): 3683.

[4] Zebib A. Stability of viscous flow past a circular cylinder. J. Eng. Math., 1987, 21: 155.

[5] Jackson C P. A finite-element study of the onset of vortex shedding in flow past variously shaped bodies. J. Fluid Mech., 1987, 182: 23.

[6] Noack B R, Eckelmann H. On chaos in wakes. Physica D, 1992, 56: 151.

[7] Sun D-J. The stability analyses and control of wakes past a circular cylinder: [Ph. D. dissertation]. Hefei: USTC, 1995. (in Chinese)

[8] Hu G-H, Sun D-J, Yin X-Y, Tong B-G. Hopf bifurcation in wakes behind a rotating and translating circular cylinder. Phys. Fluids, 1996, 8(7): 1972-1974.

[9] Bakker P G. Bifurcation in flow patterns. Kluwer Academic Publishers. 1991.

[10] Wu J-Z, Ma H-Y, Zhou M-D. Introduction to vortex dynamics. Beijing: Advanced Education Press. 1993. (in Chinese)

[11] Ingham D B. Steady flow past a rotating cylinder. Compt. Fluids, 1983, 11(4): 351.