Nonlinear Dynamics **12:** 319–326, 1997.

Is There Chaotic Synchronization in Space Extend Systems?

LIU ZENGRONG
Department of Mathematics, Suzhou University, Suzhou 215006, P.R. China, and
LNM, Institute of Mechanics, Chinese Academy of Science, Beijing 100080, P.R. China

ZHANG WEI
Institute of Theoretical Physics, Academia Sinica, Beijing 100080, P.R. China

HUANG XIN
Department of Mathematics, Suzhou University, Suzhou 215006, P.R. China

(Received: 15 September 1996; accepted: 18 October 1996)

Abstract. Based on coupled map lattice (CML), the chaotic synchronous pattern in space extend systems is discussed. Making use of the criterion for the existence and the conditions of stability, we find an important difference between chaotic and nonchaotic movements in synchronization. A few numerical results are presented.

Key words: Space extended systems, Lyapunov exponents, chaotic synchronization.

1. Introduction

Synchronization is an important nonlinear phenomenon in the nonlinear dynamics. Pecora and Carroll have introduced the concept of synchronized chaos [1, 2] and the method of cascading synchronization [3]. This idea has been shown in many nonlinear systems, including the Lorenz equation, the Rössler system, hysteresis circuits, Chua's circuit, etc. [4–7]. Theoretically, the investigation of synchronized chaos itself is interesting because chaos seems to defeat synchronization of dynamical trajectories. In practice, there is a growing interest in synchronized chaos due to its potential application in secure communications [8].

Early works in synchronized chaos were concerned with two chaotic systems, [1] and [2] demonstrated that two chaotic systems can be synchronized if (a) some dynamical variables are used to link the two systems, and (b) the subsystems excluding the driving variables possess only negative Lyapunov exponents. Murali and Lakshmanan [9] demonstrated a method of achieving chaotic synchronization in identical nonlinear systems with one-way coupling without requiring that the system under study be divided into two subsystems. For the cases where nonchaotic subsystems cannot be found, an alternative method [10] based on the idea of controlling chaos has been suggested. It is shown that two nearly identical spatiotemporal systems can be synchronized by combining the Pecora–Carroll idea and the control method [11].

Recently, the investigation of synchronized chaos in space extend systems has been considered. Fujisaka and Yamada [12] and Hengy et al. [13] discussed the synchronized state in coupled oscillator systems. For a spatially one-dimensional, diffusively coupled system composed of identical, possibly chaotic oscillators, Fujisaka and Yamada derive general conditions for the stability of the synchronized state [12]. For the coupled oscillator array possess a shift-invariant symmetry, Hengy et al. give general conditions for the stability of synchronous chaotic behavior [13]. In these works, the conditions for the stability of the synchronized sate are addressed. Is there the condition for the stability of synchronized chaos in diffusive CML systems? Does the condition for the synchronized chaos break out when the elements of diffusive CML system is increasing? In fact, the synchronization problem can be divided into two distinct subproblems. The first problem is to reduce a high-dimensional system to a low-dimensional synchronization manifold, on which synchronized dynamics is constrained. The second problem is to determine its stability. If the Lyapunov exponents of the transverse variational equation is negative, the stability of synchronized chaos is ensured. For the diffusive CML system with N nonlinear elements, if the chaotic synchronization can be realized, the trajectories with positive Lyapunov exponent of all elements are the same, which is constrained on low-dimension manifold. By diffusive effect, the influence for variations transverse to the constrained manifold, which is imposed by the chaotic synchronized state, must be enhanced when N is increasing. Therefore, the conjecture that there is a threshold value of N^* is reasonable. When $N > N^*$, some Lyapunov exponents of the transverse variational equation may produce a change from negative to positive. This change demonstrates that synchronized chaotic trajectory does not persist.

In this paper, we show that chaotic synchronization does not exist in CML when lattice number $N \to \infty$. For a fixed chaotic state with positive Lyapunov exponent, there is the upper bound $N_{\max}$ of lattice number N. When $N < N_{\max}$, the CML can realize stable chaotic synchronization.

2. Theoretical Approach

We consider the model of coupled map lattices [14]

$$x_{n+1}(i) = (1-\varepsilon)f(x_n(i)) + \frac{\varepsilon}{2}\left[f(x_n(i-1)) + f(x_n(i+1))\right], \tag{1}$$

where n is a discrete time step, i ($i = 1, 2, \ldots, N =$ system size) is the index for lattice points and ε is a coupling constant. In this paper the period boundary condition is considered, namely $x_n(1) = x_n(N)$, $\forall n \in Z$.

Equation (1) is said to be in a chaotic synchronized state if the trajectories of all lattice points are the same (after a long time) and the Lyapunov exponent for the trajectory is positive. The chaotic synchronous pattern of Equation (1) shows that the structure of space is homogeneous at time n after a long time and the behavior of time evolution is chaotic in the sense of a positive Lyapunov exponent for any lattice points.

The existence of a chaotic synchronous pattern of Equation (1) is easy to see. If the parameter of f in Equation (1) is chosen so that the orbit $\{y_n\}_{n=0}^{\infty}$ of a one-dimensional map $y_{n+1} = f(y_n)$ is chaotic, and the initial values of system (1) are taken as

$$x_0(1) = x_0(2) = \ldots = x_0(N) = y_0, \tag{2}$$

then system (1) will be in a chaotic synchronized state.

In the following, the linear stability of the chaotic synchronous pattern will be investigated. For the initial values

$$\begin{aligned} x_0(1) &= y_0 + \delta x_0(1), \\ x_0(2) &= y_0 + \delta x_0(2), \\ &\cdots \\ x_0(N) &= y_0 + \delta x_0(N), \end{aligned} \tag{3}$$

where $\delta x_0(1), \delta x_0(2), \dots, \delta x_0(N)$ are the small deviations from y_0. Substituting Equation (3) into Equation (1) and keeping the linear terms, the linearized system is given by

$$\delta \mathbf{x}_{n+1} = f'(y_n) A \delta \mathbf{x}_n, \tag{4}$$

where $\delta \mathbf{x}_n$ is a column vector

$$\delta \mathbf{x}_n = (\delta x_n(1), \delta x_n(2), \dots, \delta x_n(N))^T$$

and A is a circulant matrix

$$A = \mathrm{Circ}\left(\frac{\varepsilon}{2}, 1-\varepsilon, \frac{\varepsilon}{2}, 0, \dots, 0\right). \tag{5}$$

From the theory of circulant matrix [15], we know that A has the following eigenvalues

$$\lambda_i = 1 - \varepsilon + \varepsilon \cos \frac{2\pi i}{N}. \tag{6}$$

The eigenvectors corresponding to the eigenvalues are denoted by $\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_{N-1}$, where $\boldsymbol{\varphi}_0 = (1, 1, \dots, 1)^T$. The set of these eigenvectors is a complete set of orthogonal basis of R^N, thus

$$\delta \mathbf{x}_n = \sum_{i=0}^{N-1} \delta x_n^i \boldsymbol{\varphi}_i. \tag{7}$$

In terms of Equation (7), Equation (5) is rewritten as

$$\delta \mathbf{x}_{n+1} = f'(y_n) \sum_{i=0}^{N-1} \lambda_i \delta x_n^i \boldsymbol{\varphi}_i. \tag{8}$$

All Lyapunov exponents of Equation (8) are

$$\begin{aligned} LE_1 &= \lim_{n\to\infty} \frac{1}{n} \ln |f'(y_0) f'(y_1) \dots f'(y_{n-1})|, \\ E_2 &= LE_1 + \ln |\lambda_1|, \\ &\dots \\ LE_N &= LE_1 + \ln |\lambda_{N-1}|. \end{aligned} \tag{9}$$

LE_1 is the Lyapunov exponent of the orbit of a one-dimensional map $x_{n+1} = f(x_n)$, therefore $LE_1 > 0$ by assumption. By choosing a suitable coupling constant ε, the relation $LE_1 + \ln \tilde{\lambda}$ (where $\tilde{\lambda} = \max |\tilde{\lambda}|$, $j = 1, 2, \dots, N-1$) can be satisfied, then $\lim_{n\to\infty} \delta x_n^i = 0$, ($i = 1, 2, \dots, N-1$), i.e. system (1) is in a chaotic synchronized state after a long time.

From what has been discussed, the conditions of the existence for stable chaotic synchronous patterns are:

(a) the parameter of f in Equation (1) is a suitable one so that the trajectory of a one-dimensional map $x_{n+1} = f(x_n)$ is chaotic;
(b) coupled constant ε satisfies that:

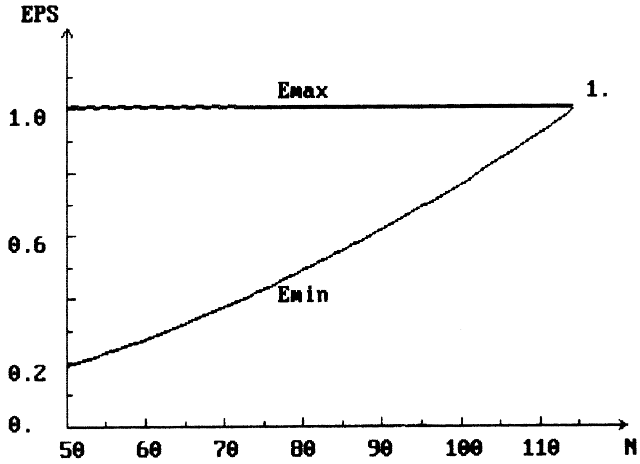

Figure 1. The relation between $\varepsilon_{\max}$, $\varepsilon_{\min}$ and N, $a = 1.401156$, $LE_1 = 1.5 * 10^{-3}$.

$$LE_1 + \ln\tilde{\lambda} = 0, \tag{10}$$

where $\tilde{\lambda} = \max|\lambda_j|$ $(j = 1, 2, \ldots, N-1)$.

In general, we can divide all kinds of movement into both chaotic and nonchaotic by Lyapunov exponents. If the movement has at least a positive Lyapunov exponent, it is known as chaotic; otherwise, it is known as nonchaotic. For nonchaotic synchronization in Equation (1), the existence and linear stability are certainly correct for all ε.

Now, f in Equation (1) is considered as

$$x_{n+1} = 1 - ax_n^2, \tag{11}$$

then LE_1 in Equation (9) can be written as

$$LE_1 = \lim_{n\to\infty} \frac{1}{n} \ln|2^n a^n y_0 y_1 \cdots y_{n-1}|, \tag{12}$$

where the series $[y]_{n=0}^{\infty}$ satisfies $y_{n+1} = 1 - ay_n^2$. If the state of Equation (11) is regular, then $LE_1 = 0$, i.e. the state of Equation (11) is nonchaotic.

Two cases are discussed, respectively.

(i) $LE_1 > 0$. From the condition $LE_1 + \ln\tilde{\lambda} < 0$ and $\lambda_i = 1 - \varepsilon + \varepsilon\cos(2\pi i/N)$ $(i = 1, 2, \ldots, N-1)$, we get

$$\frac{1-\exp(-LE_1)}{1-\cos(2\pi i/N)} < \varepsilon < \frac{1+\exp(-LE_1)}{1+\cos(2\pi i/N)} \qquad (i = 1, 2, \ldots, N-1), \tag{13}$$

i.e.

$$\frac{1-\exp(-LE_1)}{1-\cos(2\pi/N)} < \varepsilon < \frac{1+\exp(-LE_1)}{1+\cos(2\pi/N)} \quad \text{(when } N \text{ is an odd number)},$$

$$\frac{1-\exp(-LE_1)}{1-\cos(2\pi/N)} < \varepsilon < \frac{1+\exp(-LE_1)}{2} \quad \text{(when } N \text{ is an even number)},$$

For a fixed a, i.e. a fixed LE_1, the relationship between ε and N is clear from Figure 1. Without loss of generality, N is taken as an odd number; the condition for the existence of ε is obtained by

$$\frac{1-\exp(-LE_1)}{1-\cos(2\pi/N)} < \frac{1+\exp(-LE_1)}{1+\cos(2\pi/N)} ,$$

i.e.

$$\cos\frac{2\pi}{N} < \exp(-LE_1). \tag{14}$$

For a fixed LE_1, $N_{\max}$ is given by

$$N_{\max} = \frac{2\pi}{\arccos(\exp -LE_1)} . \tag{15}$$

Thus, for a given system with a fixed LE_1, there is an interval $\Delta\varepsilon = [\varepsilon_{\min}, \varepsilon_{\max}]$ in which the chaotic synchronous state exists if the system size $N < N_{\max}$, where

$$\varepsilon_{\min} = \frac{1-\exp(LE_1)}{1-\cos(2\pi/N)} \quad \text{and} \quad \varepsilon_{\max} = \frac{1+\exp(LE_1)}{1+\cos(2\pi/N)} .$$

From the above analysis, we can see that the bigger LE_1 is, the smaller $N_{\max}$ is. That is to say, the more chaotic a system is, the more difficult it is for a system to realize synchronization. In fact, if $N > N_{\max}$, the system cannot realize synchronization.

REMARK. (a) From Figure 1, we can see that $\Delta\varepsilon$ decreases when N increases, if $N < N_{\max}$. That is to say, the probability of appearance of a chaotic synchronous pattern in space-extend system decreases quickly when the system size increases. (b) Our criterion is a sufficient condition. Numerical simulation shows that the chaotic synchronized state usually appears for random initial conditions if the criterion is satisfied. But for a given system $N = 6$, $a = 1.60$ and $LE_1 = 0.373$, ε must be 0.623–0.844 for satisfying the criterion. If $\varepsilon = 0.6$ (the criterion is not satisfied) and initial values $x_0(1) = 0.3 + 10^{-10}$, $x_0(2) = 0.3 + 3 * 10^{-10}$, $x_0(3) = 0.3 + 8 * 10^{-10}$, $x_0(4) = 0.3 + 2 * 10^{-10}$, $x_0(5) = 0.3 + 5 * 10^{-10}$ and $x_0(6) = 0.3 + 7 * 10^{-10}$, the system arrives at the chaotic synchronous state after 1500 steps. This example tells us that the chaotic synchronous state may still set up even if the criterion is not satisfied. Our numerical experiments demonstrate that the probability of the appearance of synchronization is very very small when the criterion is not satisfied.

(ii) $LE_1 \leq 0, \forall N \in Z^+, LE_i < 0$ ($i = 1, 2, \ldots, N-1$), i.e. the nonchaotic synchronized state is linear stable. For a given system (1) with any large N, the system can realize nonchaotic synchronization.

From above discussion, we find an important difference between chaotic and nonchaotic states in synchronization. Nonchaotic synchronization in space-extend systems can be realized for any large systems, but chaotic synchronization in space-extend systems is limited. For a given system (1), every lattice points of the system in the same chaotic state as the coupled effect does not exist, the system size decides whether it could realize synchronization. There is an upper bound of system size $N_{\max}$. When $N < N_{\max}$, the system can realize chaotic synchronization.

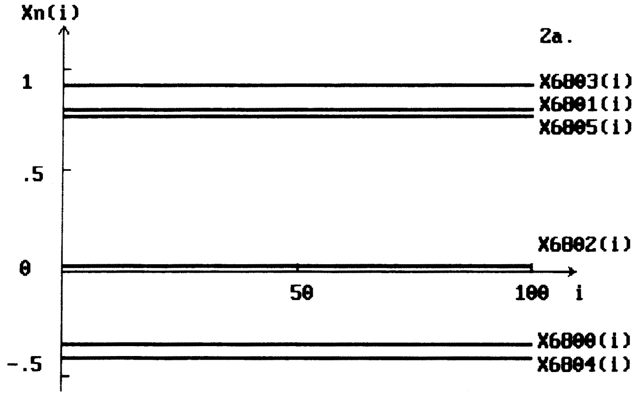

Figure 2a. Amplitude of lattice points from time $n = 6800$ to $n = 6805$. $a = 1.4011552$, $\varepsilon = 0.85$, $N = 100$. i: lattice point.

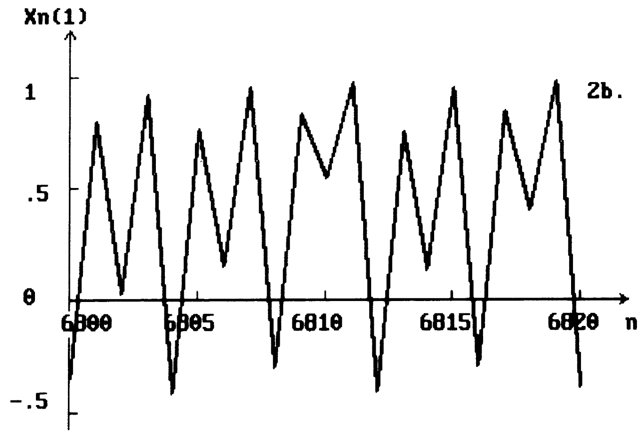

Figure 2b. Time evolution of one of lattice points. $a = 1.4011552$, $\varepsilon = 0.85$, $N = 100$. n: time from 6800 to 6820.

3. Numerical Results

In this section, two examples are given. The criterion for these two examples is satisfied. As expected, our numerical results also show that the systems tend to the chaotic synchronized state.

(a) $N = 100$, $a = 1.4011552$, $\varepsilon = 0.85$ and initial values $x_0(i) = 0.3$ ($i = 1, 2, \ldots, 99$), $x_0(100) = 0.3 + 10^{-10}$. Numerical results are shown in Figure 2. In Figure 2, (a) is an amplitude of lattices, and (b) is time evolution of one lattice points.

(b) $N = 30$, $a = 1.4012$, $\varepsilon = 0.8$ and initial values $x_0(i) = 0.5 + c_0^i * 10^{-10}$, where c_0^i ($i = 1, 2, \ldots, 99$) are random numbers. Figure 3 gives our numerical results. In Figure 3, (a) and (b) are the same as in Figure 2.

Acknowledgement

This work was supported by the National Science Foundation of China.

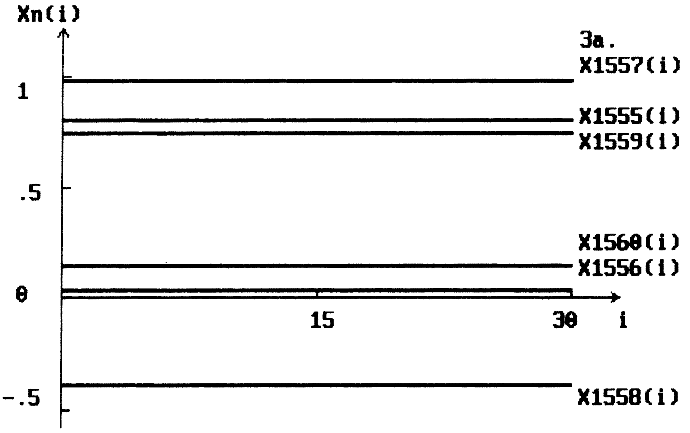

Figure 3a. Amplitude of lattice points from time $n = 1556$ to $n = 1560$. $a = 1.4012$, $\varepsilon = 0.8$, $N = 30$. i: lattice point.

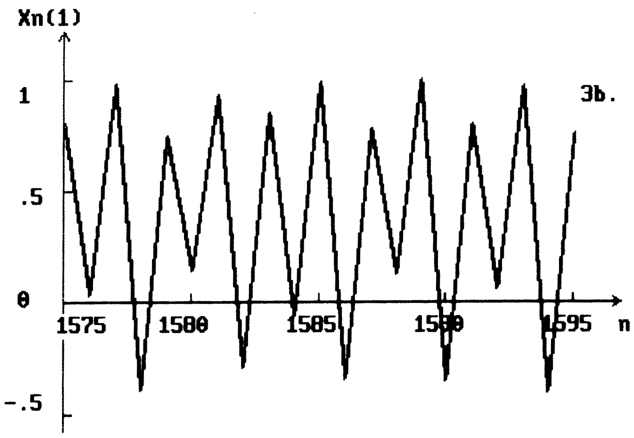

Figure 3b. Time evolution of one of lattice points. $a = 1.4012$, $\varepsilon = 0.8$, $N = 30$. n: time from 1555 to 1575.

References

1. Pecora, L. M. and Carroll, T. L., 'Synchronization in chaotic systems', *Physical Review Letters* **64**, 1990, 821–824.
2. Pecora, L. M. and Carroll, T. L., 'Driving systems with chaotic signals', *Physical Review A* **44**, 1991, 2374–2383.
3. Carroll, T. L. and Pecora, L. M., 'A circuit for studying the synchronization of chaotic systems', *International Journal of Bifurcation and Chaos* **2**, 1992, 659–667.
4. Volkovskii, A. R. and Rulkov, N. F., 'Experimental study of the bifurcations at the threshold for stochastic looking', *Soviet Technical Physics Letters* **15**, 1989, 249–251.
5. Rul'kov, N. F., Volkovskii, A. R., Rodriguez-Iozano, A., Rio, E. D., and Velarde, M. G., 'Mutual synchronization of chaotic self-oscillators with dissipative coupling', *International Journal of Bifurcation and Chaos* **2**, 1992, 669–676.
6. Chua, L. O., Kocarev, L., Eckert, K., and Itoh, M., 'Experimental chaos synchronization in Chua's circuit', *International Journal of Bifurcation and Chaos* **2**, 1992, 705–708.
7. Rulkov N. F. and Volkovskii, A. R., 'Chaos in electronic circuits', in *Proceedings of the International Society of Optical Engineering*, San Diego, CA, L. M. Pecora (ed.), Spin Bellingham, WA, 1993, pp. 132–140.
8. Cuomo, K. M. and Oppenheim, A. V., 'Circuit implementation of synchronized chaos with applications to communications', *Physical Review Letters* **71**, 1993, 65–68.

9. Murali, K. and Lakshmanan, M., 'Drive-response scenario of chaos synchronization in identical nonlinear systems', *Physical Review E* **49**, 1994, 4882–4885.
10. Lai, Y. C. and Grebogi, C., 'Synchronization of chaotic trajectories using control', *Physical Review E* **47**, 1993, 2357–2360.
11. Cai, Y.-C. and Grebogi, C., 'Synchronization of spatiotemporal chaotic systems by feedback control', *Physical Review E* **50**, 1994, 1894–1899.
12. Fujisaka, H. and Yamada, T., 'Stability theory of synchronized motion in coupled-oscillator systems', *Progress of Theoretical Physics* **69**, 1983, 32–47.
13. Hengy, J. F., Carroll, T. L., and Pecora, L. M., 'Synchronous chaos in coupled oscillator systems', *Physical Review E* **50**, 1994, 1874–1885.
14. Kanekok, K., 'Pattern dynamics in spatiotemporal chaos', *Physica D* **34**, 1989, 1–41.
15. Davis, P. J., *Circulant Matrices*, Wiley, New York, 1979.