

About Generalized Scaling for Passive Scalars in Fully Developed Turbulence

G. He ^(1,2) and B. Dubrulle ^(1,3,*)

⁽¹⁾ CEA/DSM/DAPNIA/Service d'Astrophysique (**), CE Saclay, 91191 Gif sur Yvette, France

⁽²⁾ Laboratory for Nonlinear Mechanics of Continuous Media, Institute of Mechanics, Chinese Academy of Sciences, Beijing 100080, China

⁽³⁾ Observatoire Midi-Pyrénées (***), 14 avenue Belin, 31400 Toulouse, France

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Abstract. — In this paper we use a simple normal form approach of scale invariant fields to investigate scaling laws of passive scalars in turbulence. The coupling equations for velocity and passive scalar moments are scale covariant. Their solution shows that passive scalars in turbulence do not generically follow a general scaling observed for velocity field because of coupling effects.

1. Introduction

Recently, much attention has been paid to the scaling of velocity structure functions in fully developed turbulence. It is classically admitted that there is a range of scales, called the inertial range, where the structure functions of order n scale as power law:

$$S_n(\ell) \equiv \langle |\mathbf{u}(x + \ell) - \mathbf{u}(x)|^n \rangle \sim \ell^{\zeta(n)}, \quad (1)$$

where $\zeta(n)$ is some scaling exponent. Recent studies, however, have raised up doubts about the validity of such scaling: for example, it is shown in [1] that the true scaling could rather be $\exp(\zeta(n)a^{-1}\ell^a)$ where a is inversely proportional to the Reynolds number. Given such controversy, the discovery by [2] of a new form of scaling is very interesting. This general form of scaling, named General Scaling, extends down to the smallest resolvable scale. This new scaling involves the reduced structure functions $S_n/S_q^{n/q}$ and can be written:

$$\ln \frac{S_n}{S_q^{n/q}} \propto \ln \frac{S_3}{S_q^{3/q}}, \quad (2)$$

for any n, q , where the proportionality factor only depends on n and q .

(*) Author for correspondence (e-mail: bdubru@ariane.saclay.cea.fr)

(**) CNRS, URA 2052

(***) CNRS, URA 285

Note that if (2) is really valid for any n, q , and if the following limit exists:

$$S_\infty(\ell) = \lim_{n \rightarrow \infty} \frac{S_{n+1}}{S_n}, \quad (3)$$

then, the General Scaling property can also be written:

$$\ln \frac{S_n}{S_\infty} \propto \ln \frac{S_3}{S_\infty^3}, \quad (4)$$

where the proportionality factor only depends on n . The existence of the limit (3) is not always guaranteed: for example it does not exist if u is Gaussian. There is however good indication that such limit exists in turbulence (see *e.g.* [3, 4]). In such case, the function roughly characterizes the scaling properties of the most intense but rarest velocity increments (see *e.g.* [5, 6]).

For sake of simplicity, we then consider the General Scaling property under the form (4) in the sequel of the paper. All our results could be similarly derived using (2) but computations would be slightly more intricate.

The General Scaling of scaling has been observed in a large variety of numerical and experimental flows. By contrast, a similar form of scaling does not seem to hold for the structure function of a passive scalar transported by turbulence [2]. This indicates that General Scaling is representative of a fundamental property of velocity increments.

To our knowledge, the only theoretical explanations of General Scaling so far are based on scale invariance of the Navier–Stokes equations: Dubrulle [7] used a normal form approach to show that General Scaling is a generic outcome of scale invariance in the case of one random field; Dubrulle and Graner [8] used a Lagrangian formalism applied to a single random field to show that, in such case, General Scaling directly stems from the conservation of a general impulsion along the scale. In the case of the passive scalar, we must consider two coupled random fields. Moreover, it can be shown that the equations governing the passive scalar dynamics are also scale invariant (see *e.g.* [9]). One may wonder in this respect why passive scalar structure functions do not follow General Scaling. The goal of the present letter is to offer an explanation to this paradox: we show, using an approach similar to that developed in [7] that General Scaling is not a generic outcome of scale invariance in the case of the passive scalar. This difference from velocity increments can be traced down to the existence of a coupling with the velocity fields, which introduces a second characteristic scaling function.

2. Scale Invariant Moment Equations

The structure functions of velocity and temperature increments should in principle obey a number of physical constraints. For example, regularity conditions constrains the structure functions to follow a “regular behavior” $S_n(\ell) \sim \ell^n$ at small enough ℓ . Our goal is to derive a generic shape for the structure functions involving a minimal set of assumptions, and taking into account the scale symmetry. We adopt here the point of view developed in Dubrulle [7]. We therefore only sketch the main hypothesis and computations, referring the reader to [7] for the corresponding detailed discussion. We assume that the velocity and temperature increment structure functions obey a system of differential equations involving a scale invariant operator. Furthermore, we assume that there is a transition towards the regular behavior at a scale $\ell = \eta_n$ (resp. $\ell = \eta_{\theta, n}$) for velocity (resp. temperature) structure functions ⁽¹⁾. The generic

⁽¹⁾We note that Pumir [10] obtained numerically the following relation between these two scales $\eta_{\theta, n} = 0.59\eta_n \text{Pr}^{-1/2}$, for $1/8 < \text{Pr} < 1$ where Pr is the Prandtl number. The relation was obtained for Reynolds number (based on the Taylor scale) less than 70.

equation is then derived using only scale symmetry argument, in the spirit of normal form theory.

We introduce, for convenience, the log variables

$$\begin{aligned} X_n &= \ln\langle|\delta u_\ell|^n\rangle, \\ Y_n &= \ln\langle|\delta\theta_\ell|^n\rangle, \\ T &= \ln \ell. \end{aligned} \tag{5}$$

In the inertial range, we expect the velocity and temperature structure functions to be close to power laws (see further discussion after Eq. (11)), so that:

$$\begin{aligned} X_n &\sim \frac{\chi_n}{a_n} T, \\ Y_n &\sim \frac{\chi'_n}{a'_n} T, \end{aligned} \tag{6}$$

where χ_n/a_n and χ'_n/a'_n are scaling exponents. We thus set $A_n = dX_n/dT - \chi_n/a_n$ and $B_n = dY_n/dT - \chi'_n/a'_n$. Near the inertial range, these amplitudes are small. In the spirit of amplitude equation theory, we then seek for a differential equation for A_n and B_n as an expansion in A_n and B_n . The shape of the expansion is constrained by symmetry considerations and requirement of both global and local scale symmetry [7]. In the log coordinates, global scale symmetry amounts to a translation symmetry [6, 7]. The generic differential equation therefore only includes derivatives of X_n and Y_n with respect to T , *i.e.* the terms in A_n , B_n and their derivatives. The highest relevant derivative is determined by the number of boundary conditions. Like in [7], our initial assumptions are consistent with only two boundary conditions for each field: one set by the existence of a largest scale in the system, and one set by the requirement of transition towards the regular solution. The simplest coupled differential equations which fits our requirements are:

$$\begin{aligned} b_n \frac{dA_n}{dT} &= a_n A_n + d_n B_n + O(A_n^2, A_n B_n, B_n^2), \\ b'_n \frac{dB_n}{dT} &= a'_n B_n + c_n A_n + O(A_n^2, A_n B_n, B_n^2). \end{aligned} \tag{7}$$

We can further simplify equation (7) by remarking that Navier–Stokes and transport equations make the temperature field depend on the velocity field, but allow free evolution of the velocity field. This suggests $d_n = 0$. Finally, we can use the local scale symmetry [7] to rewrite the constants appearing in (7) as:

$$\begin{aligned} \chi_n &= \tilde{\chi}_n, & a_n &= \tilde{a}_n, & b_n &= \tilde{b}_n \ln(\ell_1/\ell_0), \\ \chi'_n &= \tilde{\chi}'_n, & a'_n &= \tilde{a}'_n, & b'_n &= \tilde{b}'_n \ln(\ell_1/\ell_0), & c_n &= \tilde{c}_n, \end{aligned} \tag{8}$$

where ℓ_0 and ℓ_1 are two fixed characteristic scales and the constants \tilde{a}_n, \dots are invariant under local scale transformations and only depend on the choice of ℓ_0 and ℓ_1 . Fixing $\ell_1 = L$, the largest scale in the system, and $\ell_0 = \eta_n$, and introducing the pseudo–Reynolds number R_n as:

$$R_n = \frac{L}{\eta_n},$$

dropping tildes, and coming back to the physical variable X_n and Y_n , we can finally write the simplest generic differential equation for velocity and temperature structure functions as:

$$\chi_n = a_n \frac{dX_n}{dT} + b_n \ln(R_n) \frac{d^2 X_n}{dT^2},$$

$$\chi'_n + \frac{c_n}{a_n} \chi_n = a'_n \frac{dY_n}{dT} + b'_n \ln(R_n) \frac{d^2 Y_n}{dT^2} + c_n \frac{dX_n}{dT}. \quad (9)$$

The solutions of this system of equation depends on the boundary conditions. We adopt a set of conditions analog to that chosen in Dubrulle [7]

$$\begin{aligned} \ln X_n(T=0) &= 0 \\ \frac{dX_n}{dT} \Big|_{T=-\ln R_n} &= n \\ \ln Y_n(T=0) &= 0 \\ \frac{dY_n}{dT} \Big|_{T=-\gamma_n \ln R_n} &= n. \end{aligned} \quad (10)$$

Here, we have fixed the scale origin at $\ell = L$ and introduce a pseudo-Prandtl number

$$\gamma_n = \frac{\ln(L/\eta_n^\theta)}{\ln(L/\eta_n)}.$$

The boundary conditions (11) guarantee the transition towards the regular solutions.

2.1. GENERIC SOLUTION OF MOMENT EQUATIONS. — We are only interested in generic solutions of the 2nd order differential equations: $a_n a'_n \neq 0$ and $b_n b'_n \neq 0$, since it exhibits a feature reminiscent of what is observed in turbulence. Observing that the set of coupling moment equations must not be degenerate, we suppose that $a_n b'_n - a'_n b_n \neq 0$. Then the solution of equation (9) is

$$\begin{aligned} X_n &= \frac{\chi_n}{a_n} T + \alpha_n e^{-\frac{a_n}{b_n \ln R_n} T} + \beta_n \\ Y_n &= \frac{\chi'_n}{a'_n} T + \alpha'_n e^{-\frac{a'_n}{b'_n \ln R_n} T} + \beta'_n + \frac{b_n c_n}{a_n b'_n - a'_n b_n} \alpha_n e^{-\frac{a_n}{b_n \ln R_n} T} \end{aligned} \quad (11)$$

where the constants α_n, β_n and α'_n, β'_n are determined by the boundary conditions (11):

$$\begin{aligned} \alpha_n &= -\beta_n = \left(\frac{\chi_n}{a_n} - n \right) \frac{b_n \ln R_n}{a_n} e^{-a_n/b_n} \\ \alpha'_n &= \left(\frac{\chi'_n}{a'_n} - n - \frac{a_n c_n}{(a_n b'_n - a'_n b_n) \ln R_n} \alpha_n e^{\frac{\gamma_n a_n}{b_n}} \right) \frac{b'_n \ln R_n}{a'_n} e^{-\frac{\gamma_n a'_n}{b'_n}} \\ \beta'_n &= -\alpha'_n - \frac{b_n c_n}{a_n b'_n - a'_n b_n} \alpha_n. \end{aligned} \quad (12)$$

2.2. PROPERTIES OF THE SOLUTION. — In (11), we find features already observed in [7]. The generic shape of the velocity and temperature structure functions is not power law (X_n and Y_n linear in T), but power-exponential law, as observed and discussed in [1, 11, 12]. The classical power law shape (or sum of power law shape depending on the way the limit is taken) is obtained in the limit $R_n \rightarrow \infty$, with T fixed or in the neighborhood of $T = 0$, with R_n fixed:

$$\begin{aligned} \frac{X_n}{T} &\equiv \zeta_n = \frac{\chi_n}{a_n} (1 - e^{-a_n/b_n}) + n e^{-a_n/b_n} \\ \frac{Y_n}{T} &\equiv \xi_n = \frac{\chi'_n}{a'_n} (1 - e^{-\frac{\gamma_n a'_n}{b'_n}}) - k_n \frac{\chi_n}{a_n} e^{-a_n/b_n} + n (e^{-\frac{\gamma_n a'_n}{b'_n}} + k_n e^{-a_n/b_n}) \end{aligned} \quad (13)$$

where

$$k_n = \frac{b_n c_n}{a_n b'_n - a'_n b_n} (1 - e^{(a_n/b_n - a'_n/b'_n)\gamma_n}). \tag{14}$$

Note that the coupling between the velocity and temperature field is lumped into the constant k_n .

We see from equation (13) that the scale exponents are made from two contributions: χ_n/a_n and χ'_n/a'_n stemming from the solutions of moment equation with $b_n = b'_n = 0$ (no finite size effects) and the exponent power terms from the finite-size effect. This decomposition is reminiscent of the decomposition obtained by other general scale symmetry arguments [6, 9]:

$$\zeta_n = n\Delta_\infty(X) + \delta\zeta_n, \quad \xi_n = n\Delta_\infty(Y) + \delta\xi_n. \tag{15}$$

Here, $\delta\zeta_n$ and $\delta\xi_n$ are the contributions obtained from scale symmetry considerations. Their limits with n infinite, $C(X) = \lim_{n \rightarrow \infty} \delta\zeta_n$ and $C(Y) = \lim_{n \rightarrow \infty} \delta\xi_n$, can be interpreted as the codimensions of the most intermittent structures. $\Delta_\infty(X)$ and $\Delta_\infty(Y)$ are the scaling exponents of the maximum values of X_n and Y_n , which may depend on the experimental apparatus [8]. The factorizations (15) then suggest to interpret the constants as [7]

$$\begin{aligned} e^{-a_n/b_n} &= \Delta_\infty(X), \\ \frac{\chi_n}{a_n} &= \delta\zeta_n^*, \\ e^{-\gamma_n a'_n/b'_n} &= \Delta_\infty(Y|X) \\ \frac{\chi'_n}{a'_n} &= \delta\xi_n^*, \end{aligned} \tag{16}$$

where $\Delta_\infty(Y|X)$ is the “naive” scaling exponent of temperature moment without the velocity coupling, and $\delta\zeta_n^*$ and $\delta\xi_n^*$ are “bare” values of the scaling exponents, obtained in absence of finite size effects. In terms of these quantities, the scaling exponents can be written:

$$\begin{aligned} \zeta_n &= n\Delta_\infty(X) + \left[1 - \Delta_\infty(X)\right] \delta\zeta_n^*, \\ \xi_n &= n \left[\Delta_\infty(Y|X) + k_n \Delta_\infty(X) \right] + \left[1 - \Delta_\infty(Y|X)\right] \delta\xi_n^* - k_n \Delta_\infty(X) \delta\zeta_n^*. \end{aligned} \tag{17}$$

These expressions show that finite size effects act on two levels: first by introducing a linear part in the expression of scaling exponents; second, by modifying the value of the codimension of the most intermittent structures. To understand this, let us consider for example the log-Poisson case, where [9]

$$\begin{aligned} \delta\zeta_n^* &= C^*(1 - \beta_2^n), \\ \delta\xi_n^* &= (U_+^* - U_-^*)(1 - \beta_1^n) + U_-^*(1 - \beta_2^n), \end{aligned} \tag{18}$$

where β_1 and β_2 are two constants, C^* and U_+^* are the codimensions associated to most intermittent structures of the velocity field and temperature field, and U_-^* is a coupling constant. Finite size effects then induce a modification of these bare values into:

$$\begin{aligned} C^{\text{fs}} &= C^* \left[1 - \Delta_\infty(X)\right], \\ U_+^{\text{fs}} &= U_+^* \left[1 - \Delta_\infty(Y|X)\right] - k_n C^* \Delta_\infty(X), \\ U_-^{\text{fs}} &= U_-^* \left[1 - \Delta_\infty(Y|X)\right] - k_n C^* \Delta_\infty(X). \end{aligned} \tag{19}$$

Using this interpretation of the constants, we can finally investigate the existence of General Scaling property in the passive scalar situation. In the next Section, we explicitly consider the log–Poisson case, but similar results could be obtained with other statistics for $\delta\zeta$ and $\delta\xi$. The central hypothesis in fact lies in the “factorization” property (15). If we do not make this assumption, then General Scaling stands neither for velocity increments, nor for temperature increments. If we want to be sure that General Scaling does not occur for passive scalar even though it is valid for velocity, the factorization assumption is then natural.

2.3. GENERAL SCALING . — To investigate the General Scaling property, we introduce the maximal event functions δu_∞ for velocity and $\delta\theta_\infty$ for temperature increments. Again, we stress that this assumption is not essential, it just simplifies the computational burden. The maximal event functions follow:

$$\begin{aligned}\ln(\delta u_\infty) &= \lim_{n \rightarrow \infty} (X_{n+1} - X_n), \\ \ln(\delta\theta_\infty) &= \lim_{n \rightarrow \infty} (Y_{n+1} - Y_n).\end{aligned}\quad (20)$$

We now observe that if the pseudo–Reynolds numbers R_n , Prandtl number γ_n or the coupling constant k_n depend on n , the General Scaling property is not satisfied for the velocity increments (nor for the temperature increments). We then assume that $R_n \sim R$, $k_n = k$ and $\gamma_n \sim \gamma$ (see [2] for a discussion on these assumptions). From the equations (11), we compute

$$\begin{aligned}\ln \frac{\langle |\delta u_\ell|^n \rangle}{|\delta u_\infty|^n} &= A(T)\delta\zeta_n \\ \ln \frac{\langle |\delta\theta_\ell|^n \rangle}{|\delta\theta_\infty|^n} &= B(T)\delta\zeta_n + D(T)\delta\xi_n\end{aligned}\quad (21)$$

where $A(T)$, $B(T)$ and $D(T)$ are functions of T . They can be explicitly computed, for example,

$$\begin{aligned}A(T) &= T - \Delta_\infty(X) \frac{\ln R}{\ln \Delta_\infty(X)} (\Delta_\infty^{T/\ln R}(X) - 1). \\ B(T) &= k\Delta_\infty(X) \left[\ln R \left[\frac{\gamma\Delta_\infty(Y|X)}{\Delta_\infty^\gamma(X) \ln \Delta_\infty(Y|X)} (\Delta_\infty^{T/\gamma \ln R}(Y|X) - 1) - \frac{\Delta_\infty^{T/\ln R}(X) - 1}{\ln \Delta_\infty(X)} \right] \right. \\ &\quad \left. + D(T) \right] (1 - \Delta_\infty(X))^{-1} \\ D(T) &= \left[T - \frac{\gamma\Delta_\infty(Y|X) \ln R}{\ln \Delta_\infty(Y|X)} (\Delta_\infty^{T/\gamma \ln R}(Y|X) - 1) \right] (1 - \Delta_\infty(Y|X))^{-1}.\end{aligned}\quad (22)$$

We solve the equations (21) with $n = 3$ for $A(T)$ and $D(T)$, and then substitute them into the equation (21). We get, turning back to explicit notations:

$$\begin{aligned}\ln \frac{\langle |\delta u_\ell|^n \rangle}{|\delta u_\infty|^n} &= \frac{\delta\zeta_n}{\delta\zeta_3} \ln \frac{\langle |\delta u_\ell|^3 \rangle}{|\delta u_\infty|^3}, \\ \ln \frac{\langle |\delta\theta_\ell|^n \rangle}{|\delta\theta_\infty|^n} &= B(T) \left(\delta\zeta_n - \frac{\delta\xi_n}{\delta\xi_3} \delta\zeta_3 \right) + \frac{\delta\xi_n}{\delta\xi_3} \ln \frac{\langle |\delta\theta_\ell|^3 \rangle}{|\delta\theta_\infty|^3}.\end{aligned}\quad (23)$$

This shows that while velocity increments obey General Scaling, temperature increments do not follow this property, due to the coupling with the velocity field. Deviations with respect to General Scaling depend on the ratio $B(T)/D(T)$: when this ratio is very large (resp. very

small), the temperature increments follow approximate General Scaling with exponent $\delta\zeta_n/\delta\zeta_3$ (resp. $\delta\xi_n/\delta\xi_3$). In situations where D dominates at $T \gg T_0$ and at $T \ll T_0$, while B dominates at $T \sim T_0$ (where T_0 is a characteristic log-scale), one can get the situation observed experimentally by Ruiz *et al.* [13]: when plotting the reduced temperature structure function of order n as a function of the reduced structure function of order 3, one observes two parallel lines (of slope $\delta\xi_n/\delta\xi_3$) for $T \ll T_0$ and $T \gg T_0$, connected together by a transition region.

3. Summary

Using scale symmetry arguments, we were able to write the generic simplest coupled equations relating moments of velocity and temperature increments. Their solutions reproduce many observed features of velocity and passive scalar in turbulence. The velocity and temperature share such typical properties as: (i) transition from an exponential power law to a power law with increasing Reynolds number; (ii) transition between a regular scaling at small scale and to anomalous scaling at larger scale, although the transition scale may be different for the two fields.

However, they are also characterized by a striking and important difference: velocity structure functions obey a general scaling [2] property in any situation. Such property is not observed *in general* for temperature increments, because of coupling effects quantified by a constant k_n . This result is in agreement with a recent study of scaling properties of velocity and temperature scaling laws [2].

Since we were interested in generic results, we only considered in the present paper the simplest case of linear coupling. In a next step, nonlinearity should be considered. Nonlinear coupling may be introduced in two different ways: forcing and parameter excitation. For the convection in turbulence, the latter deserves more attention. Finally, we note that in the present formulation, moments are not coupled with higher order moments since linear coupling between them can be removed by matrix transformation. By contrast, nonlinear coupling with higher moment could not be simplified such easily, and would lead to a more difficult treatment of the equations. We however believe that the results obtained in the linear case are generic, and should also be obtained in such nonlinear case.

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