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CAUCHY SINGULAR INTEGRAL EQUATION METHOD FOR TRANSIENT ANTIPLANE DYNAMIC PROBLEMS

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Abstract—In this paper, by use of the boundary integral equation method and the techniques of Green basic solution and singularity analysis, the dynamic problem of antiplane is investigated. The problem is reduced to solving a Cauchy singular integral equation in Laplace transform space. This equation is strictly proved to be equivalent to the dual integral equations obtained by Sih [Mechanics of Fracture, Vol. 4. Noordhoff, Leyden (1977)]. On this basis, the dynamic influence between two parallel cracks is also investigated. By use of the high precision numerical method for the singular integral equation and Laplace numerical inversion, the dynamic stress intensity factors of several typical problems are calculated in this paper. The related numerical results are compared to be consistent with those of Sih. It shows that the method of this paper is successful and can be used to solve more complicated problems. Copyright © 1996 Elsevier Science Ltd

Keywords—Fracture dynamics, antiplane problem, Cauchy singular integral equation, equivalence proof, numerical method.

1. INTRODUCTION

FOLLOWING THE development of fracture mechanics, much attention has been given in recent years to dynamic fracture mechanics problems. In this kind of problem, it is obviously an important topic to investigate the case of impact or transient loading wherein inertia forces can no longer be ignored. Most of the work in fracture dynamics involves solving the initial boundary value problems, and studying the nature of crack tip stress and displacement fields. Due to the high difficulties in mathematics, only limited solutions to dynamic fracture problems in close form have been obtained. The solutions are usually classified as steady state or transient. Since variate time (t) can not be simply separated from the equations of transient state, there is not a unified approach for transient problems at present. Also, because it is almost impossible to obtain analytical solutions for finite and irregular domains, there is a clear need for the development of powerful theories and effective numerical techniques. Finite difference method (FDM) and finite element method (FEM) have been applied with some success to solve dynamic problems of cracks. But some difficulties in using FDM and FEM for fracture dynamics have been pointed out. A comprehensive review is given by Kanninen [1].

In the transient crack problems, the most basic one is the antiplane problem. This problem was firstly considered by Sih and Chen [2]. They applied the traditional Laplace–Fourier integral transforms and carried out dual integral equations. The method was also used by them in type I and II Griffith basic problems. In the 1980s this traditional method was further applied to some other special problems [3–6]. The boundary integral equation method (BIEM) or BEM is a relatively recent theory and numerical method. However, care must be exercised in application of the BIEM to crack problems [7]. The ordinary boundary integral equations lead to a non-unique formulation of any crack problem if it is not symmetric with respect to the crack. For the problems of transient fracture dynamics, the BIEM was firstly used by Fan and Hahn [8], and Sladek and Sladek [9], where crack problems, which are symmetric with respect to crack, were considered. To solve more general problems, Chirino and Dominguez applied the subdomain technique where the domain has to be divided into subdomains by means of cuts along the crack [10]. Then the ordinary boundary integral equations can be applied to each subdomain. The compatibility and equilibrium conditions have to be satisfied along the new boundary between the subdomains. In the present paper, the antiplane problem of a crack subjected to an impact loading is investigated. In order to overcome the difficulties in using ordinary integral equations along the special boundary of the crack, some techniques of Green basic solution and singularity analysis are used in this paper. The problem is finally reduced to solving a Cauchy singular integral equation along the special boundary of the crack, where the dislocation density function between up and down surfaces of the crack is unknown. It means that the basic solution for a single crack of the dynamic antiplane problem is obtained. On this basis, there are no difficulties to solve multi-crack problems. As an example, the two parallel crack problem is also investigated and a set of Cauchy singular integral equations is easily obtained. It shows that the techniques of this paper are helpful to solve general problems. Since the equations of the present paper are all in the Laplace transform space about time. The numerical inversion method of Laplace transform must be applied to obtain the final results of dynamic stress intensity factors. In the last section of this paper, several typical examples are calculated and the stress intensity factors are obtained.

2. THE BASIC FORMULATION

In linear elastodynamics of the antiplane problem, the displacement field $w(y,t)$ must satisfy the equation of motion

$$\frac{\partial^2 w}{\partial y_1^2} + \frac{\partial^2 w}{\partial y_2^2} = \frac{1}{c_2^2}\frac{\partial^2 w}{\partial t^2}, \tag{1}$$

where $c_2 = \sqrt{\mu/\rho}$ is the velocity of the distortional wave, μ is the shear elastic modulus, ρ is the medium density. For the Griffith single crack problem the boundary and initial conditions can be given as

$$\sigma_{32}(y,t)|_{y_2=0} = q(y_1,t), \quad |y_1| < a \tag{2}$$

$$w(y,t)|_{t=0} = w_0(y), \quad \frac{\partial w(y,t)}{\partial t}\Big|_{t=0} = \dot{w}_0(y). \tag{3}$$

Suppose the initial conditions are homogeneous, i.e. $w_0(y) = \dot{w}_0(y) = 0$, and apply the Laplace transform to the partial differential eq. (1), the equation becomes

$$\frac{\partial^2 \bar{w}}{\partial y_1^2} + \frac{\partial^2 \bar{w}}{\partial y_2^2} = \frac{p^2}{c_2^2}\bar{w}, \tag{4}$$

where the bar denotes the Laplace transform and p is the transform parameter.

3. CAUCHY INTEGRAL EQUATIONS FOR THE PROBLEMS

In this section the BIEM and some techniques are used to derive Cauchy integral equations for the problems. In ref. [11] one can find the basic solution for the governing eq. (4).

$$w^*(\eta - y,p) = \frac{1}{4i}H_0^{(2)}\left(\frac{-ipr}{c_2}\right) = \frac{1}{2\pi}K_0\left(\frac{pr}{c_2}\right), \tag{5}$$

where $H_0^{(2)}$ is the Hankel function and K_0 is the modified Bessel function. r is the distance from point η to point y; $r = \sqrt{(\eta_1 - y_1)^2 + (\eta_2 - y_2)^2}$.

Then the solution can be written as

$$\bar{w}(y,p) = \int_{\Gamma\pm}\left[w^*(\eta - y,p)\frac{\partial \bar{w}(\eta,p)}{\partial n_\eta} - \bar{w}(\eta,p)\frac{\partial w^*(\eta - y,p)}{\partial n_\eta}\right]\mathrm{d}\Gamma(\eta), \tag{6}$$

where Γ^+ and Γ^- are the up and down boundaries of the crack $(-a,a)$, n_η is the outward normal of boundary and the integral is about arc length.

Along the boundary of the crack, using the following conditions

$$n_\eta|_{\Gamma^+} = -n_\eta|_{\Gamma^-} = (0,-1), \quad d\Gamma(\eta)|_{\Gamma^\pm} = |d\eta_1|,$$
$$\frac{\partial \bar{w}}{\partial n_\eta}|_{\Gamma^+} = \frac{\partial \bar{w}}{\partial n_\eta}|_{\Gamma^-}, \quad \frac{\partial w^*}{\partial n_\eta}|_{\Gamma^+} = \frac{\partial w^*}{\partial n_\eta}|_{\Gamma^-} \tag{7}$$

solution (6) can easily be rewritten as

$$\bar{w}(y,p) = \int_{-a}^{a} \Delta\bar{w}(\eta_1,p)\frac{\partial w^*(\eta - y,p)}{\partial \eta_2} d\eta_1, \tag{8}$$

where $\Delta\bar{w} = \bar{w}|_{\Gamma^+} - \bar{w}|_{\Gamma^-}$ is the dislocation of the crack.

To get the corresponding integral equation along the crack $(-a,a)$, at first, we need to derive the formula of stress σ_{32}. According to the the equation for the basic solution

$$\frac{\partial^2 w^*}{\partial \eta_2^2} = -\frac{\partial^2 w^*}{\partial \eta_1^2} + \frac{p^2}{c_2^2}w^*, \ \eta \neq y \tag{9}$$

and the physical equation

$$\sigma_{32}(y,t) = \mu\frac{\partial w(y,t)}{\partial y_2}, \tag{10}$$

applying the Laplace transform to eq. (10) and using eqs (8) and (9), one can get

$$\bar{\sigma}_{32}(y,p) = -\mu\int_{-a}^{a}\frac{\partial w^*(\eta-y,p)}{\partial\eta_1}\Delta\bar{w}_{,1}(\eta_1,p)d\eta_1 - \frac{p^2}{c_2^2}\mu\int_{-a}^{a}\left[\int_{\eta_1}^{a} w^*(\eta_1^* - y,p)d\eta_1^*\right]\Delta\bar{w}_{,1}(\eta_1,p)d\eta_1, \tag{11}$$

where the closed conditions at crack tip $\Delta\bar{w}(\pm a,p)=0$ have been used and $\Delta\bar{w}_{,1}(\eta_1,p) = \partial\Delta\bar{w}_{,1}(\eta_1,p)/\partial\eta_1$ is the dislocation density function on the crack. From the basic solution (5), if we let $|\eta - y| \to 0$, the second integral kernel of eq. (11) is integrable and the first kernel is strong singular. To separate the singular dominant part from it, the following asymptotic relations are used in this paper

$$w^* - \frac{1}{2\pi}\ln\frac{1}{r} = 0(1), \quad -\frac{\partial w^*}{\partial\eta_1} - \frac{1}{2\pi}\frac{r_{,1}}{r} = 0(r\ln r), \tag{12}$$

where $r_{,1} = (\eta_1 - y_1)/r$, $r = |\eta - y|$.

Letting $y \to \Gamma^\pm$ and using the stress condition (2) on the crack, we can obtain the following singular integral equation

$$\frac{1}{\pi}\int_{-a}^{a}\left[\frac{p}{c_2}\frac{\eta_1 - y_1}{|\eta_1 - y_1|}K_1\left(\frac{p|\eta_1 - y_1|}{c_2}\right) - \frac{p^2}{c_2^2}\int_{\eta_1}^{a}K_0\left(\frac{p|\eta_1^* - y_1|}{c_2}\right)d\eta_1^*\right]\Delta\bar{w}_{,1}d\eta_1$$
$$= \frac{2}{\mu}\bar{q}(y_1,p) \quad -a < y_1 < a, \tag{13}$$

where $\bar{q}(y_1,p)$ is the Laplace transform of $q(y_1,t)$. Based on eq. (12), we have

$$\frac{p}{c_2}\frac{\eta_1 - y_1}{|\eta_1 - y_1|}K_1\left(\frac{p|\eta_1 - y_1|}{c_2}\right) = \frac{1}{\eta_1 - y_1} + 0(|\eta_1 - y_1|\ln|\eta_1 - y_1|), \tag{14}$$

so eq. (13) is a typical Cauchy singular integral equation. To solve it, the following single value condition of displacement is needed

$$\int_{-a}^{a} \bar{w}_{,1}(\eta_1, p)\mathrm{d}\eta_1 = 0 \,. \tag{15}$$

Now the dynamic Griffith antiplane problem is reduced to solving eqs (13) and (15) in Laplace transform space. Applying the Laplace transform inversion to the solution of eqs (13) and (15), we can get the result of the problem.

The above method can be used in the static problem in a similar way, where we only need to exchange the basic solution w^* for $1/2\pi \ln 1/r$ and remove the inertia part. Then the following Cauchy singular integral equation can be easily obtained

$$\frac{1}{\pi}\int_{-a}^{a} \frac{\Delta w_{,1}(\eta_1)}{\eta_1 - y_1}\mathrm{d}\eta_1 = \frac{2}{\mu} q(y_1), \quad -a < y_1 < a \,. \tag{16}$$

This is a very familiar equation to us. If we let $q(y_1) = -\tau_0$, then the stress intensity factor is given as

$$K_{\mathrm{I}} = \tau_0\sqrt{\pi\alpha} \,. \tag{17}$$

Since the Griffith single crack solution has been given by the formula of the dislocation density function (11), there are no difficulties to solve multi-crack problems if we apply the superposition principle. As an example, the two parallel crack problem is considered in the present paper.

Using the single crack solution (11) and the superposition principle, the stress $\bar{\sigma}_{32}$ of the problem shown in Fig. 1 can be expressed as

$$\bar{\sigma}_{32}(y, p) = \frac{\mu}{2\pi}\int_a^b\left[\frac{pr_{,1}}{c_2}K_1\left(\frac{pr}{c_2}\right) - \frac{p^2}{c_2^2}\int_{\eta_1}^{b}K_0\left(\frac{pr^*}{c_2}\right)\mathrm{d}\eta_1^*\right]\Delta\bar{w}_{,1}^{(1)}\mathrm{d}\eta_1$$

$$+\frac{\mu}{2\pi}\int_c^d\left[\frac{p\bar{r}_{,1}}{c_2}K_1\left(\frac{p\bar{r}}{c_2}\right) - \frac{p^2}{c_2^2}\int_{\eta_1}^{d}K_0\left(\frac{p\bar{r}^*}{c_2}\right)\mathrm{d}\eta_1^*\right]\Delta\bar{w}_{,1}^{(2)}\mathrm{d}\eta_1, \tag{18}$$

where $\bar{w}_{,1}^{(1)}$ and $\bar{w}_{,1}^{(2)}$ are the dislocation density functions of cracks (a,b) and (c,d).

$$\bar{r} = \sqrt{(\eta_1 - y_1)^2 + (l - y_2)^2} \quad \bar{r}_{,1} = (\eta_1 - y_1)/\bar{r},$$

$$\bar{r}^* = \sqrt{(\eta_1^* - y_1)^2 + (l - y_2)^2}, \quad r^* = \sqrt{(\eta_1^* - y_1)^2 + y_2^2}.$$

Letting $y \to \Gamma_1\,(a,b)$ and $y \to \Gamma_2\,(c,d)$ separately in eq. (18) and using the stress boundary conditions along the cracks, we can get a set of singular integral equations along two crack lines.

$$\frac{1}{\pi}\int_a^b\left[\frac{p}{c_2}\frac{\eta_1 - y_1}{|\eta_1 - y_1|}K_1\left(\frac{p|\eta_1 - y_1|}{c_2}\right) - \frac{p^2}{c_2^2}\int_{\eta_1}^{b}K_0\left(\frac{p|\eta_1^* - y_1|}{c_2}\right)\mathrm{d}\eta_1^*\right]\Delta\bar{w}_{,1}^{(1)}\mathrm{d}\eta_1$$

$$+\frac{1}{\pi}\int_c^d\left[\frac{p\bar{r}_{,1}}{c_2}K_1\left(\frac{p\bar{r}}{c_2}\right) - \frac{p^2}{c_2^2}\int_{\eta_1}^{d}K_0\left(\frac{p\bar{r}^*}{c_2}\right)\mathrm{d}\eta_1^*\right]\Delta\bar{w}_{,1}^{(2)}\mathrm{d}\eta_1 = \frac{2}{\mu}\bar{q}_1(y_1, p), \; a < y_1 < b, \; y_2 = 0$$

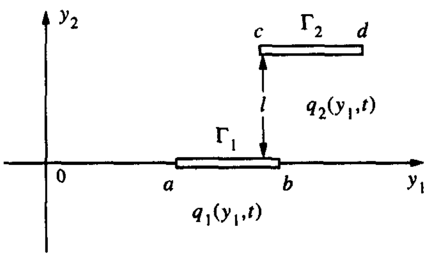

Fig. 1.

$$\frac{1}{\pi}\int_c^d\left[\frac{p}{c_2}\frac{\eta_1-y_1}{|\eta_1-y_1|}K_1\left(\frac{p|\eta_1-y_1|}{c_2}\right)-\frac{p^2}{c_2^2}\int_{\eta_1}^d K_0\left(\frac{p|\eta^*_1-y_1|}{c_2}\right)d\eta_1^*\right]\Delta\bar{w}^{(2)}_{,1}d\eta_1$$

$$+\frac{1}{\pi}\int_a^b\left[\frac{pr_{,1}}{c_2}K_1\left(\frac{pr}{c_2}\right)-\frac{p^2}{c_2^2}\int_{\eta_1}^b K_0\left(\frac{pr^*}{c_2}\right)d\eta_1^*\right]\Delta\bar{w}^{(1)}_{,1}d\eta_1=\frac{2}{\mu}\bar{q}_2(y_1,p),\ c<y_1<d,\ y_1=l. \quad (19)$$

Based on eqs (14) and (19) are also Cauchy singular integral equations. The complementary equations are

$$\int_a^b\Delta\bar{w}^{(1)}_{,1}(\eta_1,p)d\eta_1=\int_c^d\Delta\bar{w}^{(2)}_{,1}(\eta_1,p)d\eta_1=0\,. \quad (20)$$

The static equations for the problem can be obtained in a similar way.

$$\frac{1}{\pi}\int_a^b\frac{\Delta w^{(1)}_{,1}(\eta_1)}{\eta_1-y_1}d\eta_1+\frac{1}{\pi}\int_c^d\frac{\bar{r}_{,1}}{\bar{r}}\Delta w^{(2)}_{,1}(\eta_1)d\eta_1=\frac{2}{\mu}q_1(y_1)$$

$$a<y_1<b,\ y_2=0$$

$$\frac{1}{\pi}\int_a^b\frac{\Delta w^{(2)}_{,1}(\eta_1)}{\eta_1-y_1}d\eta_1+\frac{1}{\pi}\int_c^d\frac{r_{,1}}{r}\Delta w^{(1)}_{,1}(\eta_1)d\eta_1=\frac{2}{\mu}q_2(y_1)$$

$$c<y_1<d,\ y_2=l$$

$$\int_a^b\Delta w^{(1)}_{,1}(\eta_1)d\eta_1=\int_c^d\Delta w^{(2)}_{,1}(\eta_1)d\eta_1=0. \quad (21)$$

4. THE EQUIVALENCE PROOF BETWEEN THE TWO METHODS

In ref. [2] by use of Laplace–Fourier transforms, the Griffith antiplane single crack problem is reduced to solving a set of dual integral equations as

$$\int_0^\infty C(s,p)\cos(sy_1)ds=0,\ y_1>a$$

$$\int_0^\infty\sqrt{s^2+\left(\frac{p}{c_2}\right)^2}\,C(s,p)\cos(sy_1)ds=-\frac{\pi}{2\mu}\bar{q}(y_1,p),\ 0<y_1<a. \quad (22)$$

In the equations it has been supposed that the problem is symmetric about the y_2 axis, so the loading $q(y_1,t)$ satisfies condition $q(y_1,t)=q(-y_1,t)$. The displacement $\bar{w}(y,p)$ can be expressed by the unknown function $C(s,p)$

$$\bar{w}(y,\rho)=\frac{2}{\pi}\int_0^\infty C(s,p)e^{-\beta y_2}\cos(sy_1)ds\,, \quad (23)$$

where $\beta=-\sqrt{s^2+(p/c_2)^2}$. Using the differential relation and symmetry, one can easily obtain

$$C(s,p)=-\frac{1}{2s}\int_0^a\Delta\bar{w}_{,1}(\eta_1,p)\sin(s\eta_1)d\eta_1\,. \quad (24)$$

In the symmetric case, eq. (13) obtained in this paper can be rewritten as

$$\frac{1}{\pi}\int_0^a\left[\frac{p}{c_2}\frac{\eta_1-y_1}{|\eta_1-y_1|}K_1\left(\frac{p|\eta_1-y_1|}{c_2}\right)+\frac{p}{c_2}\frac{\eta_1+y_1}{|\eta_1+y_1|}K_1\left(\frac{p|\eta_1+y_1|}{c_2}\right)\right.$$

$$\left.-\frac{p^2}{c^2_2}\int_{\eta 1}^a K_0\left(\frac{p|\eta_1^*-y_1|}{c_2}\right)d\eta_1^*-\frac{p^2}{c^2_2}\int_{\eta_1}^{-a}K_0\left(\frac{p|\eta_1^*\times y_1|}{c_2}\right)d\eta_1^*\right]\Delta\bar{w}_{,1}(\eta_1,p)d\eta_1$$

$$=\frac{2}{\mu}\bar{q}(y_1,p),\quad 0<y_1<a. \tag{25}$$

Now we will prove the equivalence between eq. (25) of the present paper and eq. (22) of ref. [2]. At first we substitute eq. (24) into the first equation of eq. (22), then the left-hand-side can be changed into

$$\int_0^\infty C(s,p)\cos(sy_1)ds=-\frac{1}{4}\int_0^a\Delta\bar{w}_{,1}(\eta_1,p)\left[\int_0^\infty\left(\frac{\sin s(y_1+\eta_1)}{s}-\frac{\sin s(y_1-\eta_1)}{s}\right)ds\right]d\eta_1. \tag{26}$$

Using the known relation

$$\int_0^\infty\frac{\sin(\alpha s)}{s}ds=\frac{\pi}{2},\ \alpha>0$$

and paying attention to condition $y_1 > a > \eta_1 > 0$ in eq. (26), one can easily observe that the first equation of eq. (22) is satisfied automatically. Next we need to prove the equivalence between eq. (25) and the second equation of eq. (22). So we substitute eq. (24) into the second equation one of eq. (22)

$$\frac{1}{\pi}\int_0^a\Delta\bar{w}_{,1}(\eta_1,p)\left[\int_0^\infty\frac{2}{s}\sqrt{s^2+\left(\frac{p}{c_2}\right)^2}\sin(s\eta_1)\cos(sy_1)ds\right]d\eta_1=\frac{2}{\pi}\bar{q}(y_1,p)$$

$$0<y_1<a. \tag{27}$$

Now we only need to prove that eqs (25) and (27) have the same integral kernel. Using the following integral variate substitution

$$s=\frac{p}{c_2}\mathrm{sh}\xi,\ ds=\frac{p}{c_2}\mathrm{ch}\xi d\xi\ , \tag{28}$$

the kernel of eq. (27) can be expressed as

$$\int_0^\infty\frac{2}{s}\sqrt{s^2+(p/c_2)^2}\sin(s\eta_1)\cos(sy_1)ds=$$

$$\frac{p}{c_2}\int_0^\infty\left[\frac{1}{\mathrm{sh}\xi}+\mathrm{sh}\xi\right]\left\{\sin\left[\frac{p(\eta_1-y_1)}{c_2}\mathrm{sh}\xi\right]+\sin\left[\frac{p(\eta_1+y_1)}{c_2}\mathrm{sh}\xi\right]\right\}d\xi. \tag{29}$$

Using the following integral formulae of the Bessel functions

$$K_1(Z)=\int_0^\infty\sin(Z\mathrm{sh}\xi)\mathrm{sh}\xi d\xi,\ K_0(Z)=\int_0^\infty\cos(Z\mathrm{sh}\xi)d\xi\ , \tag{30}$$

one can easily examine the following relations

$$\frac{p}{c_2}\int_0^\infty \sin\left[\frac{p}{c_2}(\eta_1 \pm y_1)\text{sh}\xi\right]\text{sh}\xi \text{d}\xi = \frac{p}{c_2}\frac{\eta_1 \pm y_1}{|\eta_1 \pm y_1|}K_1\left(\frac{p|\eta_1 \pm y_1|}{c_2}\right),$$

$$\frac{p}{c_2}\int_0^\infty \frac{1}{\text{sh}\xi}\sin\left[\frac{p(\eta_1 - y_1)}{c_2}\text{sh}\xi\right]\text{d}\xi + \frac{p}{c_2}\int_0^\infty \frac{1}{\text{sh}\xi}\sin\left[\frac{p(\eta_1 + y_1)}{c_2}\text{sh}\xi\right]\text{d}\xi\text{d}\eta_1^*$$

$$= -\frac{p^2}{c_2^2}\int_{\eta_1}^a K_0\left(\frac{p|\eta_1^* - y_1|}{c_2}\right)\text{d}\eta_1^* - \frac{p^2}{c_2^2}\int_{\eta_1}^{-a} K_0\left(\frac{p|\eta_1^* + y_1|}{c_2}\right). \tag{31}$$

Substituting eq. (31) into eq. (29), we have

$$\int_0^\infty \frac{2}{s}\sqrt{s^2 + \left(\frac{p}{c_2}\right)^2}\sin(s\eta_1)\cos(sy_1)\text{d}s =$$

$$\frac{p}{c_2}\frac{\eta_1 - y_1}{|\eta_1 - y_1|}K_1\left(\frac{p|\eta_1 - y_1|}{c_2}\right) + \frac{p}{c_2}\frac{\eta_1 + y_1}{|\eta_1 + y_1|}K_1\left(\frac{p|\eta + y_1|}{c_2}\right)\text{d}\eta_1^*$$

$$-\frac{p^2}{c_2^2}\int_{\eta_1}^a K_0\left(\frac{p|\eta_1^* - y_1|}{c_2}\right)\text{d}\eta_1^* - \frac{p^2}{c_2^2}\int_{\eta_1}^{-a} K_0\left(\frac{p|\eta_1^* + y_1|}{c_2}\right), \tag{32}$$

where the left-hand-side is the kernel of eq. (27) and the right-hand-side is the kernel of eq. (25). Now we have proved that the method of this paper is correct and reliable.

5. THE NUMERICAL METHODS

The analytical solutions to the integral equations of this paper are not available and it is therefore necessary to solve the equations numerically. The equations will be reduced to a system of linear algebraic equations by the numerical method of singular integral equation [1]. As an example, eqs (13) and (15) can be discretised as

$$\frac{a}{m}\sum_{j=1}^{m} K(\eta_j, y_k, p)F(\eta_j, p) = \frac{2}{\mu}\bar{q}(y_k, p), \quad \sum_{j=1}^{m} F(\eta_j, p) = 0\,, \tag{33}$$

where $k = 1, 2, \cdots, m - 1$.

$$K(\eta_j, y_k, p) = \frac{p}{c_2}\frac{\eta_j - y_k}{|\eta_j - y_k|}K_1\left(\frac{p|\eta_j - y_k|}{c_2}\right) - \frac{p^2}{c_2^2}\int_{\eta_j}^a K_0\left(\frac{p|\eta_1^* - y_k|}{c_2}\right)\text{d}\eta_1^* \tag{34}$$

and η_j, y_k are the zeros of the first and second kinds of Cheyshev polynomials in $(-a, a)$.

$$\eta_j = a\cos\frac{\pi(2j-1)}{2m}\,, \; j = 1, 2, \cdots, m \tag{35}$$

$$y_k = a\cos\frac{\pi k}{m}\,, \; k = 1, 2, \cdots, m - 1.$$

The dislocation density $\Delta\bar{w}_{,1}$ is related to the function F by

$$F(\eta_j, p) = \frac{1}{a}\sqrt{(a + \eta_j)(a - \eta_j)}\Delta\bar{w}_{,1}(\eta_j, p)\,. \tag{36}$$

The stress intensity factors in Laplace transform space are determined by

$$\bar{K}_{\rm I}(a,p) = -\lim_{y_1\to a}\frac{\mu}{2}\sqrt{2\pi(a-y_1)}\Delta\bar{w}_{,1}(y_1,p),$$

$$\bar{K}_{\rm I}(-a,p) = \lim_{y_1\to -a}\frac{\mu}{2}\sqrt{2\pi(a+y_1)}\Delta\bar{w}_{,1}(y_1,p). \tag{37}$$

Using eqs (36) and (37) can be rewritten as

$$\bar{K}_{\rm I}(\pm a,p) = \mp\frac{\mu}{2}\sqrt{\pi a}F(\pm a,p), \tag{38}$$

where $F(a,p)$ and $F(-a,p)$ are computed by the following interpolation formulae

$$F(a,p) = \frac{1}{m}\sum_{j=1}^{m}(-1)^{j+1}\mathrm{ctg}\frac{(2j-1)\pi}{4m}\cdot F(\eta_j,p)$$

$$F(-a,p) = \frac{(-1)^m}{m}\sum_{j=1}^{m}(-1)^j\mathrm{tg}\frac{(2j-1)\pi}{4m}\cdot F(\eta_j,p). \tag{39}$$

In order to obtain the solution as a function of time we have to take the inversion of the Laplace transform, but this can only be done numerically. There are several numerical methods of Laplace inversion. In the present paper we choose the method of Miller and Guy suggested in refs [2, 8]. This method is based on the terms of Jacobi polynomials and has been proved to be best for the impact loading problems. Due to the the limited space, the details of the method are omitted. Here we only list the formulae.

Select a set of points on the positive real axis for the Laplace parameter p

$$p_k = (\beta_0 + k)\delta,\ k = 1, 2, \cdots N,\ \beta_0 > -1,\ \delta > 0,$$

then the dynamic stress intensity factors can be approximately computed by

$$K_{\rm I}(\pm a,t) = \sum_{n=0}^{N} C_n P^{(0,\beta_0)}{}_n(2e^{-\delta t} - 1), \tag{40}$$

where $P_n^{(0,\beta 0)}(x)$ is a Jacobi polynomial defined as

$$P^{(0,\beta_0)}{}_n(x) = \frac{(-1)^{n-1}}{2^{n-1}(n-1)!}(1+x)^{-\beta_0}\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}[(1-x)^{n-1}(1+x)^{n-1+\beta_0}]. \tag{41}$$

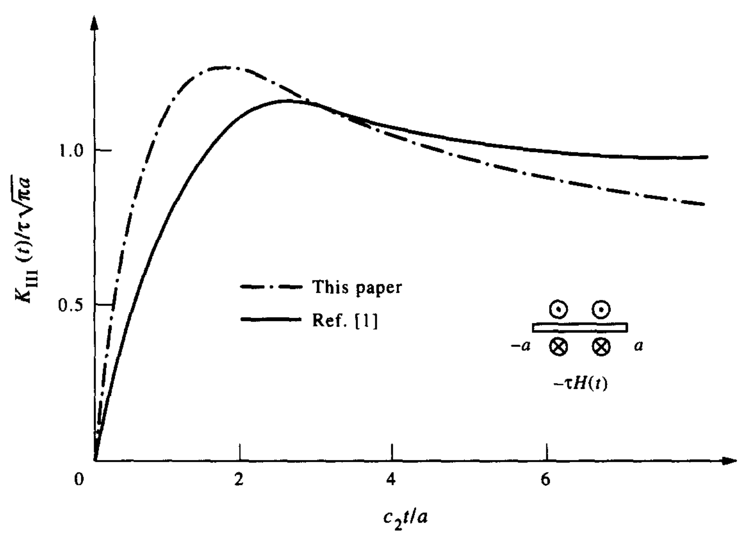

Fig. 2.

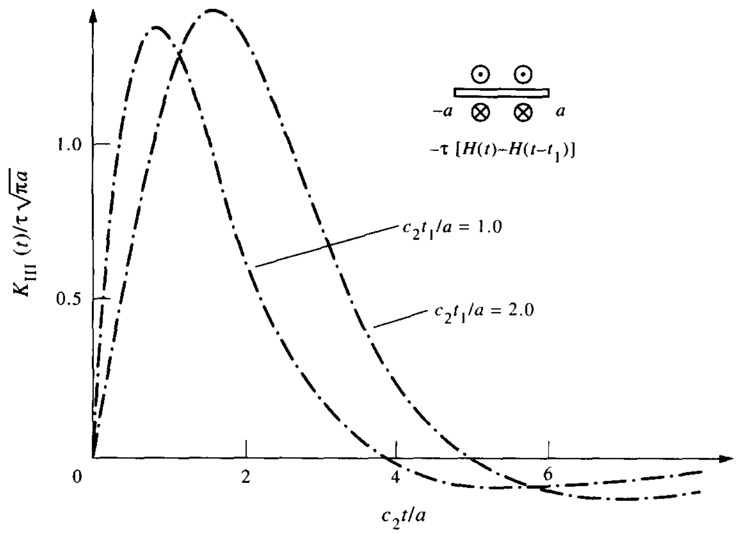

Fig. 3.

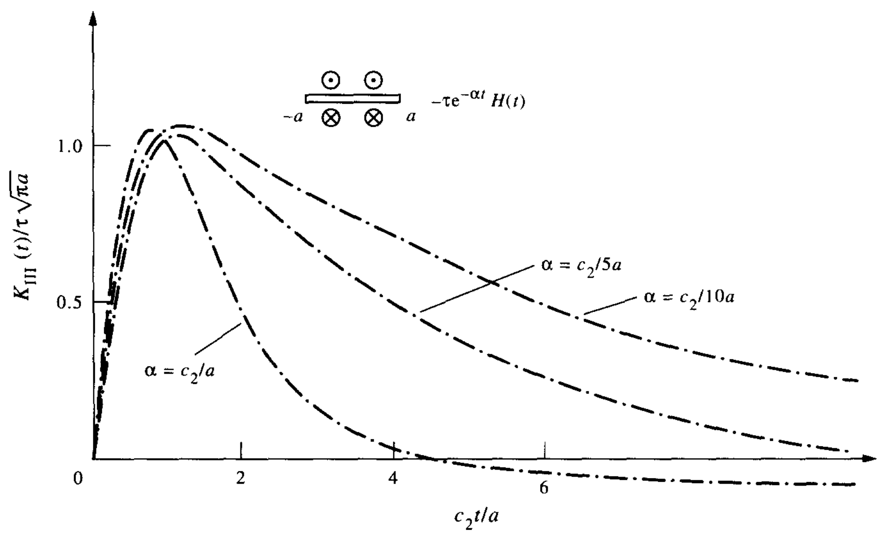

Fig. 4.

The unknown coefficients C_n are determined by the linear equations as

$$\sum_{n=1}^{L} \frac{(L-1)(L-2)\cdots(L-n+1)}{(L+\beta_0)(L+\beta_0+1)\cdots(L+\beta_0+n-1)} C_n = \delta \bar{K}_I[\pm a,(\beta_0+L)\delta]$$
$$L = 1, 2 \cdots N, \tag{42}$$

where $\bar{K}_I[\pm a,(\beta_0 + L)\delta]$ are the numerical solutions of stress intensity factor in Laplace transform space.

Table 1. SIF for static loading $q_1 = -\tau$, $q_2 = -\tau$

d/a	0.2	0.4	1.0	2.0	4.0	∞
K_I/τ	1.33566	1.39151	1.63836	1.72496	1.75906	1.77245

Table 2. SIF for static loading $q_1 = -\tau$, $q_2 = \tau$

d/a	0.2	0.4	1.0	2.0	4.0	∞
K_I/τ	3.64233	2.82858	1.93407	1.82252	1.78583	1.77245

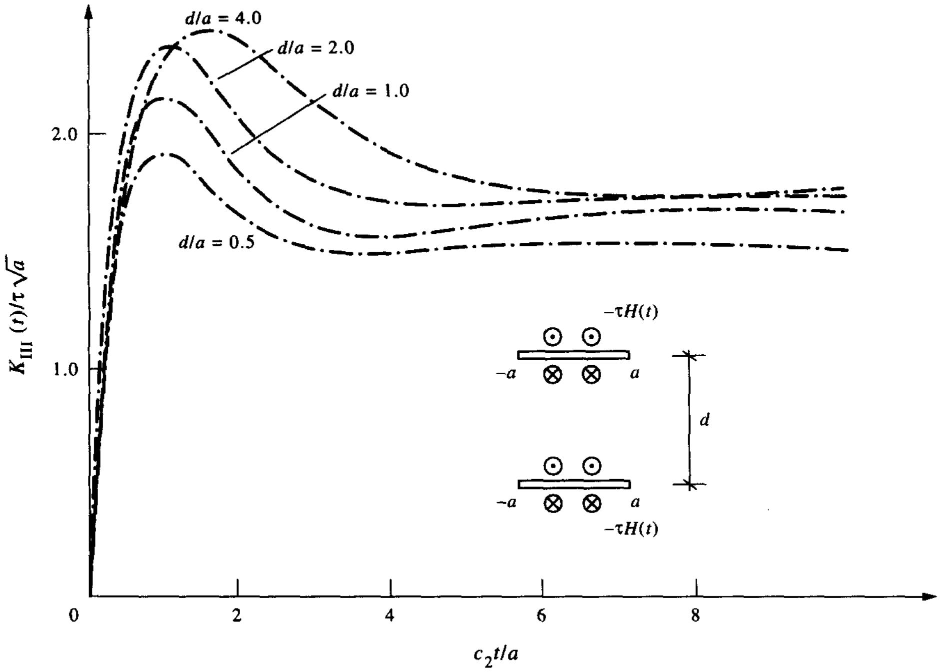

Fig. 5.

6. SEVERAL NUMERICAL EXAMPLES

To show reliability of the method in this paper, some typical examples are calculated and their dynamic stress intensity factors are obtained in this section. The following numerical values are assigned to the constants describing the medium

$$\nu = 0.29,\ \rho = 7800\ \text{kg/m}^3,\ \mu = 8 \times 10^{10}\ \text{Pa}.$$

Example 1. *Single crack subjected to impact loadings*

In this example, the dynamic stress intensity factors are computed for three kinds of impact loading. In the case of step impact loading (Fig. 2), the result in this paper is consistent with that

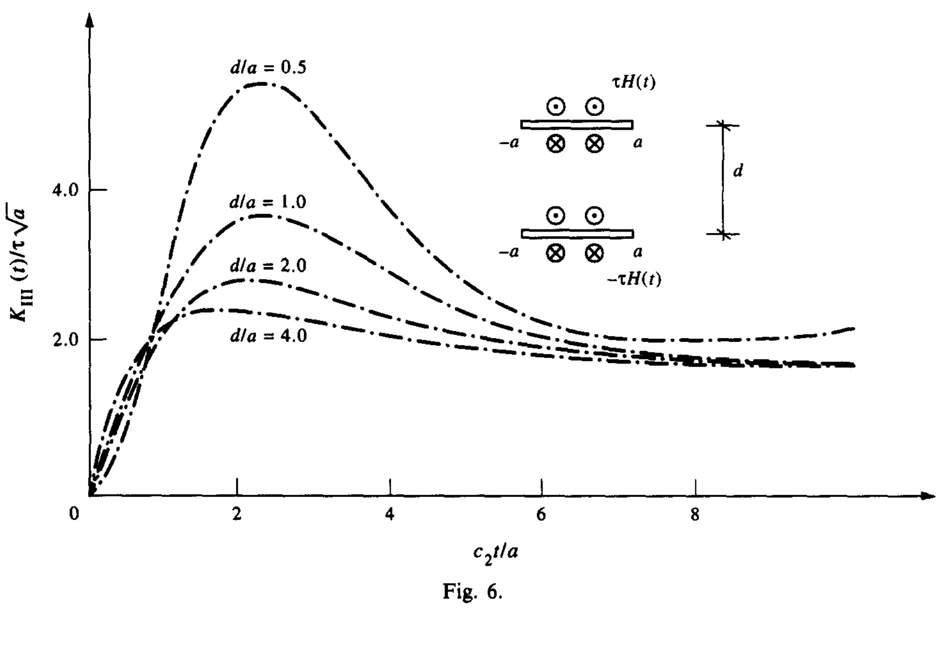

Fig. 6.

from ref. [2]. In the cases of interval and cushion impact loadings (Figs 3 and 4) the dynamic K factors tend to zero when time (t) tends to infinity.

Example 2. Double parallel cracks subjected to impact loadings

Here the two cracks have the same horizontal coordinate ($-a,a$) and their vertical distance is $d(\neq 0)$.

In the first case, the loadings on two cracks have the same direction. The static K factors are given in Table 1 and the dynamic K factors are shown in Fig. 5.

In the second case, the loadings on two cracks have the opposite direction. The static K factors are given in Table 2 and the dynamic K factors are shown in Fig. 6.

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