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A new method of studying the dynamical behaviour of the sine-Gordon equation

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Abstract

We try to connect the theory of infinite dimensional dynamical systems and nonlinear dynamical methods. The sine-Gordon equation is used to illustrate our method of discussing the dynamical behaviour of infinite dimensional systems. The results agree with those of Bishop and Flesch [SIAM J. Math. Anal. 21 (1990) 1511].

1. Introduction

One of the most important and interesting subjects in the field of nonlinear science is the dynamical analysis of space–time systems. The study of this subject is developing along two directions. On the one hand, the theory, established by Temam and his co-workers, of the existence of unique global compact attractors and inertial manifolds in dissipative PDEs is an important step in one direction (see Refs. [2,3]). On the other hand, pattern dynamics was remarkably developed based on nonlinear dynamical methods (see Ref. [4]). In Ref. [5], we suggested to connect these two subjects and that a new method of discussing the dynamical behaviour of infinite dimensional systems could be developed. We hope that this method is not only reliable in theory but also applicable and computable in practice.

In this Letter, the sine-Gordon equation is used to illustrate a new method suggested by the authors. Firstly, a new concept is put forward, namely the generalized asymptotic inertial manifold (GAIM), and it is proven prove that all solutions of the equation enter a sufficiently thin layer of this manifold after a long time, therefore the dynamical behaviour on the GAIM correctly reflects that of the sine-Gordon equation. Then we derive explicitly the form on the GAIM. Lastly, we study the dynamics of the explicit ODE in detail, and compare our results with those obtained by numerical simulation in Ref. [1]. We find that they are identical in both qualitative and quantitative aspects, so we conclude the method proposed is reliable and useful for discussing the dynamical behaviour of infinite dimensional systems.

2. GAIM

We consider the one dimensional sine-Gordon equation with periodic boundary conditions,

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$$
\begin{aligned}
&u_{tt} + \alpha u_t - \triangle u + \sin u = \Gamma f(t),\\
&x \in [-L/2, L/2] = \Omega, \quad t \in \mathrm{R}^+\\
&u(-L/2, t) = u(L/2, t),\\
&u_x(-L/2, t) = u_x(L/2, t),\\
&u(-x, t) = u(x, t),\\
&u(x, 0) = u_0(x),\ u_t(x, 0) = u_1(x),
\end{aligned} \tag{2.1}
$$

where $\alpha > 0$ and $\Gamma > 0$ are constants, and $f(t)$ is a sufficiently smooth function. The maximum norms of $f(t)$ with its derivatives of any order are smaller than one. In Ref. [2], it was proved that (2.1) has a unique compact attractor with finite Hausdorff dimension and finite fractal dimension in the subspace

$$\tilde{\mathrm{X}}_0 = \left\{ u \in \mathrm{X}_0, \int_\Omega u(x, t)\, \mathrm{d}x = 0 \right\}$$

of $\mathrm{X}_0 = \mathrm{H}^1_{\mathrm{per}}(\Omega) \times \mathrm{L}^2(\Omega)$ when $f(t) = 0$. The attractor only consists of non-zero modes, therefore the dynamics on the attractor does not correctly reflect the dynamics of (2.1) on the whole space X_0.

The eigenvalues and the eigenvectors of $A = -\triangle$ are

$$
\begin{aligned}
&\lambda_j = (2\pi j/L), \quad j = 0, 1, 2, \dots,\\
&Aw = \lambda_j w.
\end{aligned}
$$

A complete orthogonal basis consists of all the eigenvectors. Let

$$m(t) = \frac{1}{L}\int_\Omega u(x, t)\, \mathrm{d}x$$

for $u \in \mathrm{X}_0$. Then (2.1) can be written as

$$
\begin{aligned}
&\frac{\mathrm{d}^2 m}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}m}{\mathrm{d}t} + \frac{1}{L}\int_\Omega \sin u\, \mathrm{d}x = \Gamma f(t),\\
&m(0) = \frac{1}{L}\int_\Omega u_0(x)\, \mathrm{d}x = m(u_0),\\
&\frac{\mathrm{d}m(0)}{\mathrm{d}t} = \frac{1}{L}\int_\Omega u_1(x)\, \mathrm{d}x = m(u_1)
\end{aligned} \tag{2.2}
$$

and

$$
\begin{aligned}
&\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}v}{\mathrm{d}t} + Av + g(v+m) = 0,\\
&v(x, 0) = u_0(x) - m(u_0),\\
&v_t(x, 0) = u_1(x) - m(u_1),
\end{aligned} \tag{2.3}
$$

where $v(x, t) = u(x, t) - m(t)$, and

$$g(v+m) = \sin u - \frac{1}{L}\int_\Omega \sin u\, \mathrm{d}x.$$

For any fixed N, P is a projection operator from X_0 to span$\{w_1, w_2, \dots w_N\}$. $Q = I - P$, where I is the identity operator. Let $\lambda = \lambda_N$, $\Lambda = \lambda_{N+1}$, $\delta = \lambda_1/\lambda_{N+1}$, $p = Pu$, $q = Qu$. (2.3) can be divided into two parts,

$$
\begin{aligned}
&\frac{\mathrm{d}^2 p}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}p}{\mathrm{d}t} + Ap + Pg(v+m) = 0,\\
&\frac{\mathrm{d}^2 q}{\mathrm{d}r^2} + \alpha\frac{\mathrm{d}q}{\mathrm{d}t} + Aq + Qg(v+m) = 0,
\end{aligned} \tag{2.4}
$$

where $v = p + q$.

For constructing the GAIM of (2.1), the following assumption is needed,

$$Aq + Qg(p+m) = 0. \tag{2.5}$$

The solution of (2.5) is

$$q_0 = \Phi_0(p+m) = -A^{-1}Qg(p+m). \tag{2.6}$$

Φ_0 defines a smooth function in X_0. In other words, $\mathscr{M} = \{p+m, q_0\}$ gives a smooth manifold in X_0.

The main result is

Theorem. For sufficiently large t, all the solutions of (2.1) eventually enter a thin layer around $\mathscr{M}$ with width $\delta^{3/2}$.

Remark. The GAIM differs from the asymptotic inertial manifold. It is an unbounded manifold in X_0, and includes the zero mode with eigenvalue $\lambda_0 = 0$, therefore the ODE on $\mathscr{M}$ correctly reflects the dynamical behaviour of (2.1) in the whole space X_0.

The proof of this theorem will be published elsewhere (see Ref. [6]).

3. ODE on the GAIM

According to (2.2), (2.4) and (2.6), the ODE on the GAIM of (2.1) must satisfy the relations

$$\frac{\mathrm{d}^2 m}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}m}{\mathrm{d}t} + \frac{1}{L}\int_\Omega \sin(m+p+q_0)\,\mathrm{d}x = \Gamma f(t), \tag{3.1a}$$

$$\frac{\mathrm{d}^2 p}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}p}{\mathrm{d}t} + Ap + Pg(m+p+q_0) = 0, \tag{3.1b}$$

$$Aq_0 + Qg(m+p) = 0. \tag{3.1c}$$

For a fixed sufficiently large N, $m+p$ is taken as

$$c(t) + \sum_{n=1}^{N} b(t)\cos(2\pi nx/L). \tag{3.2}$$

Let us substitute (3.2) into (3.1c). Then an expression for q_0 can be obtained. Furthermore, by substitution into (3.1a), (3.1b) the equations for c and b can be derived. By a tedious calculation, the ODE on GAIM is eventually obtained,

$$\frac{\mathrm{d}^2 c}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}c}{\mathrm{d}t} + c - \frac{1}{3!}\left(c^3 + \frac{3cb_1^2}{2} + \ldots + \frac{3cb_N^2}{2}\right) + g_0(c, b_1, b_2, \ldots, b_N) = \Gamma f(t), \tag{3.3a}$$

$$\frac{\mathrm{d}^2 b_1}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}b_1}{\mathrm{d}t} + b_1 + \left(\frac{2\pi}{L}\right)^2 b_1 - \frac{1}{3!}\left(3c^2 b_1 + \frac{3b_1^3}{4} + \frac{3b_1 b_2^2}{2} + \cdots + \frac{3b_1 b_N^2}{2}\right) + g_1(c, b_1, b_2, \ldots, b_N) = 0, \quad \ldots, \tag{3.3b}$$

$$\frac{\mathrm{d}^2 b_N}{\mathrm{d}t^2} + \alpha\frac{\mathrm{d}b_N}{\mathrm{d}t} + b_N + \left(\frac{2\pi N}{L}\right)^2 b_N - \frac{1}{3!}\left(3c^2 b_N + \frac{3b_N^3}{4} + \frac{3b_N b_1^2}{2} + \cdots + \frac{3b_N b_{N-1}^2}{2}\right) + g_N(c, b_1, b_2, \ldots, b_N) = 0, \tag{3.3c}$$

where $g_i(c, b_1, b_2, \ldots, b_N)$ $(i = 0, 1, 2, \ldots N)$ are functions and their Taylor expansions include quintic and higher-order terms in the variables, and $g_0(b_1 = 0) = 0$, $g_j(b_{j-1} = 0) = 0$ $(j = 1, 2, \ldots N-1)$.

Any solution of (3.3) is considered as a curve in R^{2N+2} with basis functions $(c, c', b_1, b_1', b_2, b_2', \ldots, b_N, b_N')$. R is a subspace spanned by some of the basis functions given above. If the initial conditions of (3.3) are located on R, the corresponding unique solution of (3.3) is still on R for all t, then R is called an invariant subspace of solution. The following invariant subspaces of solution for the system (3.3) exist,

$\mathrm{R}_1 = (c, c', 0, 0, \ldots, 0, 0)$,
$\mathrm{R}_2 = (c, c', b_1, b_1', 0, 0, \ldots, 0, 0)$, $\ldots$,
$\mathrm{R}_{N+1} = (c, c', b_1, b_1', \ldots, b_N, b_N')$.

From the numerical results in Ref. [1], we find that the variation of the space structure with variation of the parameter Γ can be obtained by the results of the invariant subspaces of the solution.

Remark. If α and Γ are small, and $f(t) = \cos(wt)$, (2.1) can be written as

$$u_{tt} + \epsilon\alpha u_t - \triangle u + \sin u = \epsilon\Gamma\cos(wt),$$

where ϵ is a small parameter. For $N = 1$, $m+p$ is taken as

$$u = 2\sqrt{\epsilon\tilde{\omega}}\left[c(T)\mathrm{e}^{\mathrm{i}\omega t} + b_1(T)\mathrm{e}^{\mathrm{i}\omega t}\cos(kX) + \mathrm{O}(\epsilon^{1/2})\right],$$

where $w = 1 - \epsilon\tilde{\omega}$. $X = \sqrt{2\epsilon\tilde{\omega}}X$, $T = \epsilon\tilde{\omega}t$. Substituting this into (3.3), the first order equation coincides with the truncated model system in Ref. [1]. Therefore (3.3) is considered as a good model for the dynamical investigation of (2.1).

4. Numerical simulation

In Ref. [1], the following sine-Gordon equation was considered,

$$u_{tt} + \epsilon\alpha u_t - \triangle u + \sin u = \epsilon\Gamma\cos(wt),$$
$$u(-L/2, t) = u(L/2, t),$$
$$u_x(L/2, t) = u_x(-L/2, t),$$
$$u(x, t) = u(-x, t), \tag{4.1}$$

where $w = 0.87$, $L = 12$, $\epsilon\alpha = 0.04$. As the parameter $\epsilon\Gamma$ varies, the results obtained by the numerical method are as shown in Fig. 1.

$\epsilon\Gamma$		0.052		0.059		0.07		0.15	
space	K_0		$\mathrm{K}_0 \oplus \mathrm{K}_1$		K_0		$\mathrm{K}_0 \oplus \mathrm{K}_1$		$\mathrm{K}_0 \oplus \mathrm{K}_1 \oplus \mathrm{K}_2$
time	periodic with frequency						chaotic		

Fig. 1.

In Fig. 1, K_0, K_1 and K_2 respectively represent the zero mode with eigenvalue $\lambda_0 = 0$, the first mode with eigenvalue $\lambda_1 = 2\pi/L$ and the second mode with eigenvalue $\lambda_2 = 4\pi/L$ in phase space. In Section 3, the reliability of the space structure is shown by discussing the invariant subspaces of the solutions of the ODE on the GAIM. In this section, we further consider (3.3) by numerical simulation. Because the space structure in a wide range of parameters only consists of zero and first modes, and $R_2 = (c, c', b_1, b'_1, 0, 0, \ldots, 0, 0)$ is also an invariant subspace of solutions of (3.3), (3.3a) and (3.3b) are taken for a numerical calculation, namely, let $b_j = b'_j = 0$ ($j = 2, 3, \ldots, N$). In the numerical calculation, all the values of the parameters in (3.3) are taken as in Ref. [1], and $\epsilon\Gamma$ is considered as a variable parameter. The results of the numerical simulation are summarized in Table 1.

All the numerical results given show that the dynamical behaviour of (4.1) can be expressed qualitatively and quantitatively by the ODE on the GAIM of (4.1). This confirms that the method suggested is reasonable and useful for the investigation of infinite dynamical systems.

Table 1

Parameter	Initial conditions (c, c', b_1, b'_1)	Space	Time
0.03	(0.1550, 0.0000, 1.4150, 0.0000)	K_0	periodic
0.052	(1.5498, 0.0000, 1.4138, 0.0000)	K_0	periodic
0.055	(1.5486, 0.0000, 1.4131, 0.0000)	$K_0 \oplus K_1$	periodic
0.059	(1.5200, 0.0000, 1.4000, 0.0000)	$K_0 \oplus K_1$	periodic
0.062	(1.5200, 0.0000, 1.3200, 0.0000)	K_0	periodic
0.065	(1.2500, 0.0000, 1.3200, 0.0000)	K_0	periodic
0.07	(1.2500, 0.0000, 1.3200, 0.0000)	$K_0 \oplus K_1$	chaotic
0.1	(1.1000, 0.0000, 0.9000, 0.0000)	$K_0 \oplus K_1$	chaotic

In this Letter, our interest is focused on the reliablity of the method we put forward. Most of the results obtained have been compared with the results of Ref. [1], so that the reliability of the new concept and the method can be shown. Further results for (3.3) by other nonlinear methods will be published elsewhere.

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