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Discussion on the Geometric Structure of Strange Attractor*

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We discuss the transversal heteroclinic cycle formed by hyperbolic periodic points of diffeomorphism on the differential manifold. We point out that every possible kind of transversal heteroclinic cycle has the Smalehorse property and the unstable manifolds of hyperbolic periodic points have the closure relation mutually. Therefore the strange attractor may be the closure of unstable manifolds of a countable number of hyperbolic periodic points. The Hénon mapping is used as an example to show that the conclusion is reasonable.

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It is well-known that chaotic state can appear in nonlinear dynamical system under certain parametric conditions. Numerically, we usually use Lyapunov exponent, topological entropy and fractal dimension as characteristic measurments to decide whether a system is in the chaotic state. Theoretically, we usually analyze whether a system is in chaotic state from the point of view of topological transitivity, Li-Yorke's definition, Smale horseshoe transformation and transversal homoclinic point. According to the large amount of accumulated materials, we may conjecture that the strange attractor appearing numerically may possess more plentiful meaning than the present theoretical idea. This paper just tries to probe into this problem in the theory of transversal heteroclinic cycle.

I) Transversal heteroclinic cycle.

Suppose M is a smooth differentiable manifold and f is a c^r diffeomorphism. The chaotic state of f indicates a kind of dynamical behavior of f^n as $n \to \infty$. Therefore studying this kind of property, we can find that the fixed point and the k-periodic points of f have the equal effect.

Firstly, we take into consideration the hyperbolic fixed points of f. Suppose that $p_1, \cdots, p_n$ are the hyperbolic fixed points of f. If $w^u(p_i)$ and $w^s(p_{i+1})(i = 1, \cdots, n, p_{n+1} = p_1)$ intersect transversally, then we say that f possesses a transversal heteroclinic n-cycle. Obviously, if f possesses a transversal homoclinic point, then it possesses a 1-cycle. In Ref. 3 it is shown that if f possesses a transversal heteroclinic n-cycle, then it has a transversal homoclinic point, i. e. f can appear in the chaotic state in the sense of Smale horseshoe.

Secondly, we take into consideration the hyperbolic periodic points of f. Suppose that $p_1, \cdots, p_n$ are the hyperbolic k-periodic points of f. If $w^u(p_i)$ and $w^s(p_{i+1})(i =$

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$1, \cdots, n, p_{n+1} = p_1)$ intersect transversally, we say that f possesses a k-periodic transversal heteroclinic n-cycle. Lastly, we take into consideration the transversal heteroclinic cycle formed by the different period hyperbolic periodic points of f. Suppose that $p_1, \cdots, p_n$ are the hyperbolic periodic points of f whose periods are different. If $w^u(p_i)$ and $w^s(p_{i+1})(i = 1, \cdots, n, p_{n+1} = p_1)$ intersect transversally, we say that f possesses a different periodic transversal heteroclinic n-cycle. Obviously, under certain parametric conditions, if f possesses some series of hyperbolic periodic points and they can form every kind of transversal heteroclinic cycle in the ways mentioned above, the total number of transversal heteroclinic cycle will increase exponentially. So far as transversal heteroclinic cycle is concerned, the following result can be proved.

Conclusion 1. Chaos can appear in any kind of the transversal heteroclinic cycle in the sense of Smale horseshoe.

Conclusion 2. The closures of the unstable manifolds of those hyperbolic periodic points (or fixed points) that form certain kind of transversal heteroclinic cycle are identical. We simply take 2-cycle as an example. If the manifolds of p_1 and p_2 form a certain kind of transversal heteroclinic cycle, it certainly follows that $\overline{w^u(p_1)} \supset w^u(p_2)$ and $\overline{w^u(p_2)} \supset w^u(p_1)$ i.e. $\overline{w^u(p_1)} = \overline{w^u(p_2)}$ (see Fig. 1).

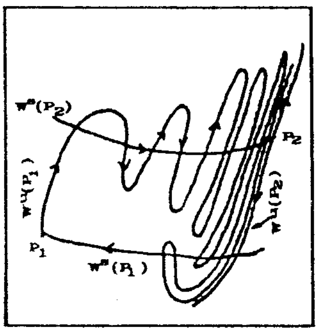

Fig. 1: Transversal heteroclinic 2-cycle.

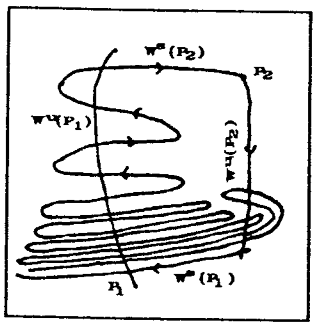

Fig. 2: 2-periodic transversal heteroclinic 2-cycle.

Conclusion 3. The closure of the stable manifolds of those hyperbolic periodic points (or fixed points) that form certain kind of transversal heteroclinic cycle are identical. Again, we simply take 2-cycle as an example. If the manifolds of p_1 and p_2 form some kind of transversal heteroclinic cycle, it certainly follows that $\overline{w^s(p_1)} \supset w^s(p_2)$ and $\overline{w^s(p_2)} \supset w^s(p_1)$ i.e. $\overline{w^s(p_1)} = \overline{w^s(p_2)}$ (see Fig. 2). As for the detailed proof, we will publish in another article.

Remark: If the manifolds of hyperbolic points only partly intersect and they do not form a cycle, conclusions 2 and 3 do not hold. Figure 3 shows that if $w^u(p_1)$ and $w^s(p_2)$ intersect transversally but $w^u(p_2)$ and $w^u(p_1)$ do not intersect, we can prove only $\overline{w^u(p_1)} \supset w^u(p_2)$ and $\overline{w^s(p_2)} \supset \overline{w^s(p_1)}$. In other words, in the sense of formed transversal heteroclinic cycle, their unstable manifolds form a holistic structure, otherwise they can not form a holistic structure.

II) Geometric structure of the strange attractor.

It is well-known that f has a countable number of hyperbolic periodic points when f behaves chaotically. If the unstable manifolds of these points are separated from each other, they can independently form infinite inveriant sets of f. But we always find one stable strange attractor by numerical computation, and so we believe that the strange attractor possesses a holistic structure. By conclusion 2, the closure relation of the unstable manifolds is set up provided that transversal heteroclinic cycle exists. Therefore the strange attractor is a holistic closure structure containing all unstable manifolds of a countable number of hyperbolic periodic and fixed points by way of transversal heteroclinic cycle.

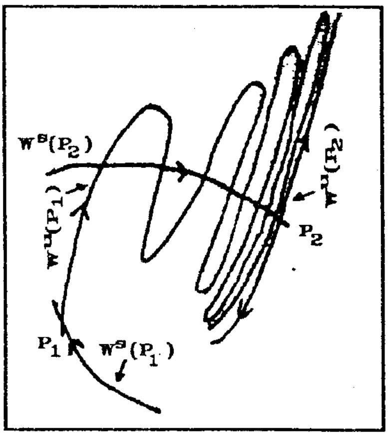

Fig. 3: Invariant manifolds of p_1 and p_2 only partly intersect, and they do not form a cycle.

Based on the proof of conclusion 1, we find that the unstable manifold and stable manifold of every hyperbolic points which consist of transversal heteroclinic cycle are transversally intersected, namely, the neighborhood of every hyperbolic point has a set in which $f^N (N \in \mathbf{Z})$ is chaotic in the sense of Smale horseshoe. Therefore the holistic geometric structure mentioned above at least have infinite Smale horseshoes. In fact, it consists of all transversal homoclinic points and heteroclinic points. This structure is more complicated than a single Smale horseshoe.

III) The strange attractor in the Hénon mapping.

Hénon mapping is

$$x(n+1) = 1 + a^* y(n) - b^* x(n)^* x(n) \ ,$$

$$y(n+1) = x(n) \ .$$

When $a = 0.3$ and $b = 1.4$, there exists a strange attractor. In this case, there exist a countable number of hyperbolic periodic points. We pick out its two hyperbolic fixed points and a series of hyperbolic 2-periodic points and hyperbolic 4-periodic points. As for every hyperbolic periodic point mentioned above, we calculate numerically the first stage of its stable and unstable manifolds. Figures 7, 8 and 9 in Ref. 8 show the state of two fixed points, periodic 2 points and periodic 4 points, respectively. From these figures, we can see that they all possess equal periodic transversal heteroclinic 1-cycle. If we overlap these figures, we can find that there exist some other types of cycles. Here we can see that, in the case of strange attractor, there exist various types of transversal heteroclinic cycles, therefore the closure of unstable manifolds of all these hyperbolic periodic points is identical, this closure forms a holistic geometric structure.

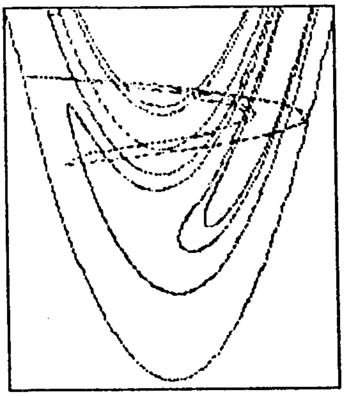

Fig. 4: Union of the first stage of the invariant manifolds of fixed points, p-2 points and p-4 points at a=0.3, b=1.4 in Hénon mapping.

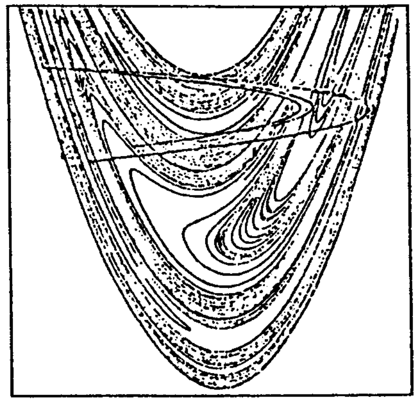

Fig. 5: Strange attractor and its attracting set of Hénon mapping at a=0.3, b=1.4.

A Comparison between Fig. 4, the combination of Figs. 1-3 of Ref. 8, and Fig. 5, the outcome of Hénon mapping by long-term iteration, shows that the two bear such a resemblance with each other. From these two figures, we can see that the strange attractor of Hénon mapping is a holistic structure formed by the unstable manifolds of all types of hyperbolic periodic points which form various transversal heteroclinic cycles. Because of the existence of all kinds of transversal heteroclinic cycles, the unstable manifolds of hyperbolic periodic points have closure relation mutually. Therefore the holistic structure is the closure of unstable manifold of anyone of the hyperbolic periodic points contained in the structure. Numerical results can be seen because of the existence of all kinds of the transversal heteroclinic cycle. Meanwhile the stable manifolds of hyperbolic periodic points have similiar results, and so holistic structure which is formed by the stable manifolds is just the attracting domain of the strange attractor. Therefore, we may conjecture that the strange attractor of the Hénon mapping is a holistic structure formed by many arbitrary Smale horseshoes.

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