# The Geometric Structure and Dynamical Properties of Lauwerier Attractor*

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## 1 Introduction

The strange chaotic attractor (ACS) is an important subject in the nonlinear field[3]. On the basis of the theory of transversal heteroclinic cycles, it is suggested that the strange attractor is the closure of the unstable manifolds of countable infinite hyperbolic periodic points[1, 2, 4, 5]. From this point of view some nonlinear phenomena are explained reasonably[6,7].

## 2 The Structure of Lauwerier Attractor

Lauwerier map is defined as

$$L: \begin{cases} x \cdots \to bx(1-2y)+y, \\ y \ldots \to 4y(1-y), \end{cases} \qquad 0<b<1/2 \tag{1}$$

where $L: Q \to Q$, $Q=[0,1]\times[0,1]$. In $y$ direction, $L$ is reduced to the logistic map with $\lambda=4$. According to the properties of periodic points of logistic map, we have

**Proposition 1.** *If* $(y_0, y_1, \cdots\cdots, y_{k-1})$ *is a periodic orbit of logistic map, there is unique point set* $(x_0, x_1, \cdots\cdots, x_{k-1})$ *such that* $((x_0, y_0), (x_1, y_1), \cdots\cdots (x_{k-1}, y_{k-1}))$ *is a saddle periodic orbit of* $L$.

**Proposition 2.** *$L$ has periodic orbits of all periods in $Q$.*

Noticing $y=\tilde{y}$ $(0 \leqslant x \leqslant 1)$ is a segment of stable manifold of periodic point $(\tilde{x}, \tilde{y})$ of $L$ in $Q$, we can easily prove

**Proposition 3.** *The unstable and stable manifolds of all periodic points of $L$ in $Q$ form various transversal heteroclinic cycles.*

Accordingly, the closure of the unstable manifolds of all periodic points is a whole

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structure, which is called Lauwerier attractor. In order to discuss its structure carefully, we perform the transformation

$$\begin{cases} x = \dfrac{1}{2} - \dfrac{1-b}{2b} u, \\ y = \dfrac{1}{2} - \dfrac{1}{2} v, \end{cases} \tag{2}$$

then $L$ becomes

$$\widetilde{L} \quad \begin{cases} u \cdots \to b(1+u)v, \\ v \cdots \to 2v^2 - 1 \quad , \end{cases} \tag{3}$$

where $L : \widetilde{Q} \to \widetilde{Q}, \widetilde{Q} = \{ |u| \leqslant \dfrac{b}{1-b}, |v| \leqslant 1 \}$. The fixed point $(0,0)$ of $L$ is replaced by the fixed point $p = (\dfrac{b}{1-b}, 1)$ of $L$. The unstable manifold $W^u(p)$ of $P$ has been derived to be[8]:

$$\begin{aligned} u(t) &= \sum_{k=1}^{\infty} b^k \Phi_k(t), \\ & \qquad\qquad t \geqslant 0 \\ v(t) &= \cos t, \end{aligned} \tag{4}$$

where $\Phi_k(t) = \sin t / 2^k \cdot \sin \dfrac{t}{2^k}$. Its closure $\overline{W^u(p)}$ can be expressed as

$$\begin{cases} u = \displaystyle\sum_{k=1}^{\infty} \left(\frac{b}{2}\right)^k \frac{\sin(0.b_{-1} b_{-2} \cdots) 2\pi}{\sin(0.b_{k-1} b_{k-2} \cdots b_v b_{-1}, \cdots) 2\pi}, \\ v = \cos(0.b_{-1} b_{-2} \cdots) 2\pi, \end{cases} \tag{5}$$

where $(0.b_{-1} b_{-2} \cdots)$ and $(0.b_{k-1} b_{k-2} \cdots b_0 b_{-1} \cdots)$ are binary expressions.

In normal cases, $\Sigma_2$ denotes the collection of bi-infinite sequences of 0 's and 1's. A map $h : \Sigma_2 \to \overline{W^u(p)}$ is constructed as follows :

$$A\, s = (\cdots a_n a_{n-1} \cdots a_0 . a_{-1} \cdots a_{-n} \cdots) \in \Sigma_2, h(s) = (u,v)$$

$$\begin{cases} u = \displaystyle\sum_{k=1}^{\infty} \left(\frac{b}{2}\right)^k \frac{\sin(0.a_{-1} a_{-2} \cdots) 2\pi}{\sin(0.a_{k-1} a_{k-2} \cdots a_0 a_{-1} \cdots) 2\pi} \\ v = \cos(0.a_{-1} a_{-2} \cdots) 2\pi. \end{cases} \tag{6}$$

Then we can show:

**Proposition 4**. *h is a continuous surjective map and satisfies the relation* $\widetilde{L} \circ h = h \circ \sigma$, *here* $\sigma$ *is the shift map on* $\Sigma_2$.

**Proposition 5**. *Hyperbolic periodic points, transversal homoclinic points and transversal heteroclinic points are dense in* $\overline{W^u(p)}$.

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From the above, we can draw conclusions of the structure of Lauwerier attractor:

(1) It is the closure of unstable manifolds of countable infinite hyperbolic periodic points.

(2) The stable and unstable manifolds of all periodic points form various transversal heteroclinic cycles.

(3) Periodic points, homoclinic points and heteroclinic points are dense in it.

## 3 Attraction of Lauwerier Attractor

Attractive behavior of Lauwerier attractor is given as follows :

**Proposition 6.** *$\forall \varepsilon > 0$, $\exists N$, when $n > N$, $\forall (u_0, v_0) \in \tilde{Q}$, the distance $\tilde{L}^n(u_0, v_0)$ and $W^u(\tilde{p})$ is closer than $\varepsilon$.*

*Proof.* $v_0$ can be parametrised as $v_0 = \cos z$, then $(u_n, v_n)$ is derived as

$$\begin{cases} u_n = \sum_{k=1}^{\infty} \left(\frac{b}{2}\right)^k \frac{\sin 2^n z}{\sin(2^{n-k} z)} + \left(\frac{b}{2}\right) \frac{\sin 2^n z}{\sin z} u_0, \\ v_n = \cos(2^n z). \end{cases} \tag{7}$$

Obviously, $\forall \varepsilon > 0$, $\exists N_1$, when $n > N_1$ such that

$$\left| \left(\frac{b}{2}\right)^n \frac{\sin 2^n z}{\sin z} u_0 \right| < \frac{\varepsilon}{2},$$

at the same time, $\forall \varepsilon > 0$, $\exists N_2$, when $n > N_2$ such that

$$\left| \sum_{k=h}^{\infty} \left(\frac{b}{2}\right)^k \frac{\sin 2^n z}{\sin 2^{h-k} z} \right| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$, when $n > N$, for $(u(t), v(t)) \in W^u(p)$ (where $t = 2^n z$), the distance between $(u_n, v_n)$ and $(u(t), v(t))$ is smallar than $\varepsilon$.

## 4 Chaotic Behavior on Lauwerier Attractor

Based on the fact that $h$ is a continuous surjective map and $\sigma$ has a dense orbit on $\Sigma_2$, we can prove that $\tilde{L}$ is topological transitivity on $\overline{W^u(\tilde{p})}$.

**Proposition 7.** *$\tilde{L}$ has a dense orbit on $\overline{W^u(p)}$.*

Next, we show that $\tilde{L}$ has sensitive dependence on initial conditions on $\overline{W^u(p)}$.

**Proposition 8.** *There exists $\varepsilon > 0$, for any $x \in \overline{W^u(p)}$ and any neighborhood $U$ of $x$, $\exists \bar{x} \in U$ and $n > 0$ such that $d(\tilde{L}^n(\bar{x}) - \tilde{L}^n(x)) \geq \varepsilon$.*

*Proof.* Let $\varepsilon = \min\{ |\cos x - \cos(x + \pi/2)| \mid x \in [0, \pi/2]\} > 0$, $\forall x \in \overline{W^u(p)}$,

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$\exists s=(\cdots b_n \cdots b_0 . b_{-1} \cdots) \in \Sigma_2$ such that $h(s)=x$.

$\forall S>0$, there exists sufficiently large $n$ such that $h(\overline{s})$ is contained in $\delta$-neighborhood of $x$, where $\overline{s}=(\cdots b_n \cdots b_0 . b_{-1} \cdots b_{-n+1} \overline{b}_{-n}' b_{-n-1} \cdots)$, $\overline{b}_{-n}=1$ if $b_{-n}=0$ or $\overline{b}_{-n}=0$ if $b_{-n}=1$. Based on (6) and (7), we get

$$|\widetilde{L}^{n-2}(\overline{x})-\widetilde{L}^{n-2}(x)| \geqslant \varepsilon .$$

Propositions 7 and 8 tell us that the dynamical behavior on Lauwerier attractor is chaotic.

According to the geometric structure, attraction and dynamical behavior of Lauwerier attractor, we assert that it is a strange chaotic attractor.

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