CHARACTER OF SIMPLIFIED NAVIER-STOKES (SNS) EQUATIONS NEAR SEPARATION POINT FOR TWO-DIMENSIONAL LAMINAR FLOW OVER A FLAT PLATE

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Received October 12, 1991.

ABSTRACT

It is proved that the simplified Navier-Stokes (SNS) equations presented by Gao Zhi[1], Davis and Golowachof-Kuzbmin-Popof (GKP)[3] are respectively regular and singular near a separation point for a two-dimensional laminar flow over a flat plate. The order of the algebraic singularity of Davis and GKP equation[2,3] near the separation point is indicated. A comparison among the classical boundary layer (CBL) equations, Davis and GKP equations, Gao Zhi equations and the complete Navier-Stokes (NS) equations near the separation point is given.

Keywords: separation flow, viscous flow, simplified Navier-Stokes equation.

I. INTRODUCTION

The simplified Navier-Stokes (SNS) equations or the so-called parabolized (PNS) equations as a kind of approximate theory suitable to the whole flow-field have found more and more applications in fluid mechanics, heat transfer and computational fluid mechanics[4]. But the characteristics of SNS equations near some key points in flow field (such as separation point, trailing edge, leading edge, stationary point) have not been studied. It is known that these key points are singular points of the classical boundary layer (CBL) equations. Glodstein's approach to the singularity near separation point of laminar flow along a flat plate and near the trailing edge of the flat plate[5] has made important fundamental contribution to the theory of classical boundary layer, giving rise to a new multi-layer boundary layer theory[6]. Furthermore, Dean[7] compared the mathematical character of NS equations with that of CBL equations near the separation point and proved that the NS equation has power series solution near the separation point, while CBL equation does not have power series solution[7]. The latter is in agreement with the conclusion that CBL equations have the Goldstein's singular near the separation point. For convenience, Dean's conclusion[7] can be briefly described as follows: NS equations are regular near the separation point and CBL equations are singular near the separation point. In this paper, the mathematical characteristics of two kinds of SNS equations presented by Gao Zhi[1], Davis[2] and Golowachof et al.[3] near the separation point for two-dimensional laminar flow over a flat plate are studied in detail. It is found that the Gao Zhi's equations have power series solution near the separation point. This conclusion is consistent with the fact that NS equations are regular near the separation point[7]. But Davis' and GKP's equations are singular near the separation point. These conclusions are consistent with those reached by making comparison among the different SNS equations[8], but there is a slight difference between the Gao Zhi's, Davis' and GKP's equations in mathematical character. SNS equations (2.3) and (2.4) (See Sec. II) are obtained according to the principles presented at first by Refs. [1—3]. Concretely speaking, Eqs. (2.3) are obtained by dropping the viscous terms smaller than the order of magnitude of $Re^{-\frac{1}{2}}$ from NS equations (2.1), while Eqs. (2.4) are obtained by dropping the viscous terms smaller than the order of magnitude of the Reynold number Re^0 from NS equations[3]. These simplification principles are obviously based on the classical boundary layer (CBL) theory. A new simplification principle is proposed recently by Gao Zhi[9,10] which is based on the viscous-inviscid flow interaction theory and completely independent of the GBL theory. Following the new simplification principle, the two most important kinds of SNS equations applicable to the whole flow field can be obtained by dropping the viscous terms smaller than the order of magnitude of $Re^{-\frac{1-3m}{2}}$ and $Re^{\frac{m}{2}}$ from NS equations, respectively, where m is the interaction parameter (a variable), $m < \frac{1}{2}$. The special cases of $m = 0$ and $1/4$ are corresponding to the classical boundary layer and boundary layer separation, respectively. For the present statement, the two most important kinds of SNS equations are just Eqs. (2.3) and (2.4). We call them SNS equations (2.3) and SNS equations (2.4), for convenience, Gao Zhi equations, Davis equations or GKP equations, respectively.

II. The Forms of SNS Equations

The complete Navier-Stokes (NS) equations for the two-dimensional steady flow of an incompressible fluid circling a flat plate are as follows:

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, & (2.1a) \\ u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + r\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right), & (2.1b) \\ u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + r\left(\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2}\right), & (2.1c) \end{cases}$$

where p is the pressure, ρ the density, $r = \mu/\rho$, r is the coefficient of viscosity, and x, y are the coordinate variables. According to the classical boundary layer theory and SNS equation theory[1-3], CBL equations and two kinds of SNS equations are, respectively,

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, & (2.2a) \end{cases}$$

$$\begin{cases} u\dfrac{\partial u}{\partial x}+v\dfrac{\partial v}{\partial y}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+\nu\dfrac{\partial^2 u}{\partial y}, & (2.2\text{b})\\ \dfrac{\partial p}{\partial y}=0, & (2.2\text{c})\end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0,\\ u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+\nu\dfrac{\partial^2 u}{\partial^2 y},\\ u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}+\nu\dfrac{\partial^2 v}{\partial^2 y},\end{cases} \tag{2.3}$$

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0,\\ u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+\nu\dfrac{\partial^2 u}{\partial^2 y},\\ u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}.\end{cases} \tag{2.4}$$

III. The Character of Two Kinds of Gao Zhi Equations (2.3) and Davis or GKP Equations (2.4) Near the Separation Point

1. *The Approach to Gao Zhi Equations* (2.3)

Let $x'=x/d$, $y'=Rey/d$, $u'=u/U$, $v'=Rev/U$, $p'=P/\rho U^2$, where d is the characteristic length, U the charateristic velocity, and Re the Reynolds number. For simplicity we still denote x', y', u' and v' by x, y, u, and v. With the new variables, the equations become

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0, & (3.1\text{a})\\ u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}=-\dfrac{\partial p}{\partial x}+\dfrac{\partial^2 u}{\partial^2 y}, & (3.1\text{b})\\ u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}=-R\dfrac{\partial p}{\partial y}+\dfrac{\partial^2 v}{\partial^2 y}. & (3.1\text{c})\end{cases}$$

Eq. (3.1a) is satisfied if we let $u=\dfrac{\partial\phi}{\partial y}$, $v=-\dfrac{\partial\phi}{\partial x}$. Eliminating the pressure p from (3.1b) and (3.1c) gives

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial\phi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial\phi}{\partial y}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\phi}{\partial y^2}+\frac{1}{R}\frac{\partial^2\phi}{\partial x^2}\right)=0. \tag{3.2}$$

From the boundary conditions $u|_{y=0}=v|_{y=0}=0$ at the wall and Eq. (3.1b), we know that the stream-function satisfies the following equalities:

$$\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial y}=0 \quad (y=0), \tag{3.3}$$

$$\frac{\partial^2 u}{\partial^2 x} = \frac{\partial^3 \psi}{\partial x^2 \partial y} = 0 \quad (y = 0), \tag{3.4}$$

$$\frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \psi}{\partial y^2} \quad (y = 0). \tag{3.5}$$

It is now assumed that the stream function $\psi(x,y)$ can be expanded in the form of

$$\psi(x, y) = a_2 \frac{y^2}{2!} + a_3 \frac{y^2}{3!} + b_3 \frac{y^2}{2!}\frac{x}{1!} + a_4 \frac{y^4}{4!} + b_4 \frac{y^3}{3!}\frac{x}{1!} + c_4 \frac{y^2}{2!}\frac{x^2}{2!} + \cdots. \tag{3.6}$$

Write

$$\alpha^0_{n+2} = a_{n+2},\ \alpha^1_{n+3} = b_{n+3},\ \alpha^2_{n+4} = c_{n+4}, \cdots, \ (n = 0,1,2,3,\cdots). \tag{3.7}$$

$\psi(x,y)$ can be briefly written as

$$\psi(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \alpha^m_{n+m+2} \frac{x^m}{m!} \frac{y^{n+2}}{(n+2)!}. \tag{3.8}$$

It is clear that the boundary conditions (3.3), (3.4) and (3.5) are satisfied by $\psi(x,y)$.

Theorem 1. (a) *If* $(p)|_{y=0}$ *and* $(u)|_{x=0}$ *are known functions, then SNS equations* (2.3) *have the power series solution near* (0, 0); (b) *if* $(p)|_{y=0}$ *and* $(u)|_{x=0}$ *are known functions, and* $\left(\frac{\partial^4 u}{\partial y^4}\right)_{\substack{x=0\\y=0}} = a_4 \neq 0$ *at the separation point, then SNS equation* (2.3) *has power series solution near the separation point.*

Proof. Let $\left(\frac{\partial p}{\partial x}\right)\Big|_{y=0}$ and $(u)|_{x=0}$ be the power series in the form

$$\left(\frac{\partial p}{\partial x}\right)_{y=0} = \left(\frac{\partial^3 \psi}{\partial y^3}\right)_{y=2} = a_3 + b_4 x + c_5 \frac{x^2}{2!} + \cdots, \tag{3.9}$$

$$(u)_{x=0} = \left(\frac{\partial \psi}{\partial y}\right)_{x=0} = a_2 y + a_3 \frac{y^2}{2!} + a_4 \frac{y^3}{3!} + \cdots, \tag{3.10}$$

where $a_3, b_4, \cdots, a_3, a_4, \cdots$, are determined by the known functions $(p)|_{y=0}$ and $(u)|_{x=0}$. Substituting Expression (3.8) for ψ in (3.2), collecting the terms of the same powers of x and y and writing $r = 1/R$, we have the following equations satisfied by coefficients of the stream function ψ

$$\alpha^m_{m+4} + r\alpha^{m+2}_{m+4} = 0 \tag{3.11}$$

$$\sum_{j=0}^{m} \frac{\alpha^j_{j+2}\alpha^{m-j+1}_{m-j+3}}{j!(m-j)!} = \frac{1}{m!}(\alpha^m_{m+s} + r\alpha^{m+2}_{m+5}) \tag{3.12}$$

$$\sum_{i=0}^{1}\sum_{j=0}^{m} \frac{\alpha^j_{i+j+2}\alpha^{m-j+1}_{1-i+m-j+3}}{j!(i+1)!(m-j)!(1-i)!} - \sum_{j=0}^{m} \frac{\alpha^{j+1}_{j+3}\alpha^{m-j}_{m-j+3}}{2!j!(m-j)!} = \frac{1}{2!m!}(\alpha^m_{m+6} + r\alpha^{m+2}_{m+6}) \tag{3.13}$$

$$
\begin{aligned}
&\sum_{i=0}^{n+2}\sum_{j=0}^{m}\frac{\alpha^{j}_{i+j+2}\alpha^{m-j+1}_{n+2-i+m-j+3}}{j!(i+1)!(m-j)!(n+2-i)!}\\
&+r\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\alpha^{j+1}_{i+j+2}\alpha^{m-j+3}_{n-i+m-j+s}}{j!(i+1)!(m-j)!(n-i+2)!}\\
&-\sum_{i=0}^{n+1}\sum_{m=1}^{m}\frac{\alpha^{j+1}_{i+j+3}\alpha^{m-j}_{n+1-i+m-j+3}}{j!(i+2)!(m-j)!(n+1-i)!}\\
&-r\sum_{i=0}^{n}\sum_{j=0}^{m}\frac{\alpha^{j+1}_{i+j+3}\alpha^{m-j+2}_{n-i+m-j+4}}{j!(i+2)!(m-j)!(n-i+1)!}\\
=&\frac{1}{m!(n+3)!}\left(\alpha^{m}_{n+m+7}+r\alpha^{m+2}_{n+m+7}\right)\quad(m,n=0,1,2,3,\cdots).
\end{aligned}\tag{3.14}
$$

According to the definition of α, (3.11)—(3.14) can be also written in the form

$$a_4+rc_4=0,\tag{4(1)}$$

$$a_5+rc_5-a_2b_3=0,\tag{5(1)}$$

$$b_5+rd_5=0,\tag{5(2)}$$

$$a_6+rc_6-2a_2b_4=0,\tag{6(1)}$$

$$b_6+rd_6-a_2c_4-b_3^2=0,\tag{6(2)}$$

$$c_6+re_6=0,\tag{6(3)}$$

$$a_7+rc_7+2a_4b_3-2a_3b_4-3a_2(b_5+rd_5)+3rb_3c_4=0,\tag{7(1)}$$

$$b_7+rd_7-2a_2c_5-2b_3b_4=0,\tag{7(2)}$$

$$c_7+re_7-2a_2d_5-3b_3c_4=0,\tag{7(3)}$$

$$d_7+rf_1=0,\tag{7(4)}$$

$$
\left\{
\begin{aligned}
&a_8+rc_8+5a_5b_3-a_3(5b_5+6rd_5)-4a_2(b_6+rd_5)+4rb_1c_1+6rb_3c_5=0, && 8(1)\\
&b_8+rd_8+2a_4c_4-2a_3c_5-3a_2(c_6+re_6)-2b_4^2-b_3b_5+3rc_4^2=0, && 8(2)\\
&c_8+re_8-2a_6d_6-2b_4c_4-4b_3c_5=0, && 8(3)\\
&d_8+rf_8-a_2e_6-4b_3d_5-3c_4^2=0, && 8(4)\\
&e_8+rg_8=0, && 8(5)
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
&a_9+rc_9+9a_6b_4-5a_4(b_5+2rd_5)-a_2(9b_2+10rd_6)-5a_2(b_7+rd_7) && 9(1)\\
&\quad+5rb_5c_4+10rb_4c_5+10rb_3c_6=0,\\
&b_9+rd_9+5a_5c_4-a_3(5b_6+6re_6)-4a_2(c_7+re_7)+b_3(b_6+2rd_6)\\
&\quad-b_4(5b_4+2rd_5)+10rc_4c_5=0, && 9(2)\\
&c_9+re_9+2a_4d_5-2a_3d_6-3a_2(d_7+rf_7)+b_5c_4-6b_4c_5\\
&\quad-b_3(4c_6+3re_6)+6rc_4b_5=0, && 9(3)\\
&d_9+rf_9-2a_2c_7-2b_4d_5-6b_3d_4-6c_4c_5=0, && 9(4)\\
&e_9+rg_9-a_2f_7-5b_3e_6-10c_4d_5=0, && 9(5)\\
&f_9+rh_9=0. && 9(6)
\end{aligned}
\right.
$$

The constants included in the first four sets of equation can be arranged in the following scheme:

$$
\begin{array}{llllll}
a_2 \\
a_3 & b_3 \\
a_4 & b_4 & c_4 \\
a_5 & b_5 & c_5 & d_5 \\
a_6 & b_6 & c_6 & d_6 & e_6 \\
a_7 & b_7 & c_7 & d_7 & e_7 & f_7,
\end{array}
$$

where the constants that must take assigned values are underlined.

(i) When the point (0,0) is not the separation point ($a_2 \neq 0$), the unknown constants are uniquely determined by these equations to be functions of the constants with assigned values. In fact, Eq. 4(1) determines c_4, and, since we assume that $a \neq 0$, 5(1) determines b_3, but there is only one single equation 5(2) left for the determination of b_5 and d_5.

$$b_3 = a_2^{-1}(a_5 + rc_5), \quad c_4 = -r^{-1}a_4, \quad b_5 = -rd_5. \tag{3.15}$$

At this stage there is accordingly one undetermined constant, say d_5. But d_5 can be determined in the next stage in which the set of (7) are used. 7(1) and 7(2) can be written as

$$\begin{cases} rc_7 - 3a_2(b_5 + rd_5) = -a_7 - 2a_4b_3 + 2a_3b_4 - 3rb_3c_4, & (3.16) \\ c_7 - a_2d_5 = -re_7 + 3b_3c_4, & (3.17) \end{cases}$$

where all the terms on the right-hand sides are known, Eliminating c_7c_1 gives

$$3b_5 + 2rd_5 = a_2^{-1}(a_7 - r^2e_1 + 2a_4b_3 - 2a_3b_4 + 6rb_3c_4), \tag{3.18}$$

which, with 5(2), determines b_5 and d_5

$$d_5 = -r^{-1}a_2^{-1}(a_7 - r^2e_7 + 2a_4b_3 - 2a_3b_4 + 5rb_3c_4),$$

$$b_5 = a_2^{-1}(a_7 - r^2e_7 + 2a_4b_3 - 2a_3b_4 + 6rb_3c_4)_2.$$

c_7 can be found from either (3.16) or (3.17).

In 7(2) and 7(4) we have two equations with three unknown constants b_7, d_7 and f_7, such that there is one undetermined constant, say f_7. It is clear that the situation at the end of this stage is the same as that at the end of the stage before. It must be shown that there is no breakdown at any stage before. To avoid the use of double suffixes it is convenient to change the notation and write

$$\psi_{2n+1} = \alpha_{2n+1}\frac{y^{2n+1}}{(2n+1)!} + \alpha_{2n}\frac{y^{2n}}{(2n)!}\frac{x}{1!} + \cdots + \alpha_2\frac{y^2}{(2)!}\frac{x^{2n-1}}{(2n-1)!}, \tag{3.19}$$

$$\psi_{2n-1} = \beta_{2n-1}\frac{y^{2n-1}}{(2n-1)!} + \beta_{2n-2}\frac{y^{2n-2}}{(2n-2)!}\frac{x}{1!} + \cdots + \beta_2\frac{y^2}{(2)!}\frac{x^{2n-3}}{(2n-3)!}, \tag{3.20}$$

for the two related groups of terms. In the first group α_{2n+} and α_3 must take assigned values, and in the second group β_{2n-1} and β_3 are known. It is assumed that the equations for the coefficients β are in the same form as the sets of Egs. (7). $\beta_5, \beta_7, \cdots, \beta_{2n-3}$ can thus be determined, but for other $n-1$ coefficients there are the $n-2$ equations

$$\begin{cases} \beta_{2n-2} + r\beta_{2n-4} = N_{2n-2}, \\ \beta_{2n-4} + r\beta_{2n-6} = N_{2n-4}, \\ \cdots\cdots, \\ \beta_6 + r\beta_4 = N_6, \\ \beta_1 + r\beta_2 = N_4, \end{cases} \tag{3.21}$$

where $N_4, N_6, \cdots, N_{2n-2}$ are known. There is one undetermined coefficient, say β_2. We can write

$$\begin{cases} \beta_4 = -r\beta_2 + N_4, \\ \beta_6 = r^2\beta_2 + N_6, \\ \cdots\cdots, \\ \beta_{2n-2} = (-1)^n r^{n-2}\beta_2 + N_{2n-2}. \end{cases} \tag{3.22}$$

$\beta_2, \beta_4 \cdots$ appear in the equations for the coefficients with odd suffixes and they have the following forms

$$a_2(\beta_{2n-2k} + r\beta_{2n-2(k+1)}),$$

where $k = 1, 2, \cdots, n-1$. However, the forms $a_2(\beta_{2n-2k} + r\beta_{2n-2(k+1)})$ appear only in

$$-\frac{\partial\phi}{\partial y}\frac{\partial}{\partial x}\left[\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial x^2}\right] = -a_2 y\frac{\partial}{\partial x}\left[\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial x^2}\right] + \cdots. \tag{3.23}$$

Through calculation it can be known that the equations for the coefficients with odd suffixes are in the form

$$\begin{cases} \alpha_{2n+1} + \gamma\alpha_{2n+1} - (2n-3)a_2(\beta_{2n-2} + \gamma\beta_{2n-4}) = N_{2n+1}, \\ \alpha_{2n-1} + \gamma\alpha_{2n-3} - (2n-5)a_2(\beta_{2n-4} + \gamma\beta_{2n-6}) = N_{2n-1}, \\ \cdots\cdots, \\ \alpha_5 + \gamma\alpha_3 - a_2\beta_2 = N_5, \end{cases} \tag{3.24}$$

where $N_5, N_7, \cdots, N_{2n+1}$ are known. Expressing $\beta_4, \beta_6, \cdots, \beta_{2n-2}$ in terms of β_2, we have the $n-1$ equations

$$\begin{cases} \gamma\alpha_{2n+1} = N_{2n+1}, \\ \alpha_{2n+1} + \gamma\alpha_{2n-3} = N_{2n-1}, \\ \cdots\cdots, \\ \alpha_7 + \gamma\alpha_5 = N_7, \\ \alpha_5 - a_2\beta_2 = N_5, \end{cases} \tag{3.25}$$

where $N_5, N_7, \cdots, N_{2n+1}$ are known. For the $n-1$ unknowns, $\beta_2, \alpha_5, \alpha_7 \cdots, \alpha_{2n-1}, \alpha_{2n}$ and α_3 are assigned values, and the terms depending on them are therefore included in $N_5, N_7, \cdots$. Eliminating the α's, we have

$$(-1)^{n-1}\gamma^{n-2}a_2\beta_2 = N_{2n+1} - \gamma N_{2n-1} + \cdots + (-1)^{n-2}\gamma^{n-2}N_5. \tag{3.26}$$

Clearly (3.26) determines β_2 uniquely. The coefficients $\alpha_5, \cdots, \alpha_{2n-1}$ can now be found. It has been shown that the coefficients left undetermined at any stage can be found subsequently. And, since the early stages have been worked out in detail, there is no breakdown at any stage. Hence if $a_2 \neq 0$, given the pressure for all x on the boundary $y = 0$ and x component of the velocity for all y on the line $x=0$

the fluid motion is known.

(ii) $a=0, a\neq 0$ (i.e. origin (0,0) of the coordinates is the separation point) Eq. 5(1) shows that there must be the following relation between the assigned constants

$$a_5+2\gamma c_5=0. \tag{3.27}$$

In this case the pressure and the velocity could not be assigned independently because the coefficient a is common to the two series, the new relation (3.27) imposes a further restriction. The relation can be found by differentiating (3.2) with regard to y,

$$\frac{\partial^5\psi}{\partial y^5}+\gamma\frac{\partial^5\psi}{\partial y^3\partial x^2}+\left(\frac{\partial\psi}{\partial x}\frac{\partial^2}{\partial y^2}+\frac{\partial^2\psi}{\partial x\partial y}\frac{\partial}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial^2}{\partial x\partial y}-\frac{\partial^2\psi}{\partial y^2}\frac{\partial}{\partial x}\right)\cdot\left(\frac{\partial^2\psi}{\partial y^2}+\gamma\frac{\partial^2\psi}{\partial x^2}\right)=0. \tag{3.28}$$

From boundary condition and separation condition

$$\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial y}=\frac{\partial^2\psi}{\partial x\partial y}=0,\quad (y=0), \tag{3.29}$$

$$\frac{\partial u}{\partial y}=\frac{\partial^2\psi}{\partial y^2}=0,\quad (x=y=0), \tag{3.30}$$

and (3.28) we have

$$\frac{\partial^5\psi}{\partial y^5}+\gamma\frac{\partial^5\psi}{\partial y^3\partial x^2}=0,\quad (x=y=0). \tag{3.31}$$

From (3.31) one can obtain (3.27). From Eq. 4(1), 6(1), 6(2), 6(3), 7(1) and 7(3), we have

$$c_4=-\frac{a_4}{\gamma},\quad c_6=-\frac{a_6}{\gamma},\quad e_6=\frac{a_6}{\gamma^2},\quad b_3=\frac{1}{4a_4}(a_7-\gamma^2a_7-2a_3b_4),$$

$$b_6=-\gamma d_6+\frac{1}{16a_4^2}(a_1-\gamma^2e_7-2a_3b_4)^2,\quad c_7=-\left(\gamma e_7+\frac{3b_3a_4}{\gamma}\right).$$

There are only 5(2), 7(2) and 7(4) left of 4(1)—7(4)

$$\begin{cases} b_5+\gamma d_5=0, & 5(2)\\ b_7+\gamma d_7-2b_3b_4=0, & 7(2)\\ d_7+\gamma d_7=0. & 7(4)\end{cases}$$

The above equations contain five unknown constants, which can be determined using Eq. 9(1), 9(3), 9(5), 11(1), 11(3), 11(5) and 11(7). Eq. 9(1), 9(3), 9(5) can be written in the form

$$\begin{cases}\gamma c_9-5(a_4-\gamma c_4)b_5-10\gamma a_4d_5=-9a_6b_3-5a_5b_4,\\ \quad +a_3(9b_6+10\gamma d_6)-10\gamma b_4c_5-10\gamma b_3c_6=N_1,\\ c_9+\gamma c_9+c_4b_5+(2a_4+6\gamma c_4)d_5=2a_3d_6+6b_4c_5,\\ \quad +b_3(4c_6+3\gamma e_6)=N_2,\\ c_9-10c_4d_5=-\gamma g_9+5b_3c_6=N_3,\end{cases} \tag{3.32}$$

where N_1, N_2, N_3 are known constants. The elimination of c_9 and e_9 from (3.32) gives

$$a_4(9b_5 + 8\gamma d_5) = N_1 + \gamma N_2 - \gamma^2 N_3. \tag{3.33}$$

From Eqs. (3.32), (3.33) and 5(2), we can find b_5, d_5, c_9, e_9

$$\begin{cases} b_5 = -\dfrac{1}{a_4}(N_1 - \gamma N_2 + \gamma^2 N_3), \\ d_5 = \dfrac{1}{\gamma a_4}(N_1 - \gamma N_2 + \gamma^2 N_3), \\ c_9 = \dfrac{N_1}{\gamma}, \\ e_9 = N_3 - \dfrac{10}{\gamma^2}(N_1 - \gamma N_2 + \gamma^2 N_3). \end{cases}$$

Eqs. (8) can be used to determine all the five unknown coefficients with suffix 8. In 7(2) and 7(4) we have two equations with unknown constants b, d and f, so that there is one undetermined constant, say f. Like d, f is determined by means of Eqs. 11(1), 11(3), 11(5) and 11(7) (the detailed process of solving the equations is omitted for brevity). For general case, one can refer to 1(a). Hence SNS equations (2.3) do have power series solution at the separation point.

If

$$u' = \frac{\partial \psi'}{\partial y'} = u_1 y' + u_2 y'^2 + u_3 y'^2 + \cdots, \quad (x' = 0), \tag{3.34}$$

and

$$-\frac{\partial p'}{\partial x'} = p_0 + p_1 x' + p_2 x'^2 + \cdots, \quad (y' = 0), \tag{3.35}$$

where u, $u, \cdots$, p, $p, \cdots$ are constants with assigned values, then

$$u = Uu' = U\frac{\partial \psi'}{\partial y'} = U\sum_{n=1}^{\infty} u_n \left(\frac{R^{\frac{1}{2}}}{d}\right)^n, \quad (x = 0). \tag{3.36}$$

$$-\frac{\partial p}{\partial x} = -\rho\frac{U^2}{d}\frac{\partial p'}{\partial x'} = \rho\frac{U^2}{d}\sum_{n=0}^{\infty} p_n \left(\frac{x}{d}\right)^n, \quad (y = 0). \tag{3.37}$$

Therefore, the x component of the velocity is given as a function of y on the line $x = 0$ and the variation of the pressure is given as a function of x on the boundary $y = 0$. The relation between the new and old coefficients is

$$s!u_s = a_{s+1}(s = 1, 2, \cdots); \; p_0 = -a_3, \; p_1 = -b_4, \; 2!p_2 = -c_5, \cdots.$$

According to the discussion given above, Theorem 1 can also be stated as: assuming that the stream-function can be expanded as a double series in x, y, the fluid motion is uniquely determined by the data, provided that

$$u_1 \neq 0, \; 2u_2 + p_0 = 0.$$

The fluid motion is uniquely determined if

$$u_1 = 0,\ u_3 \neq 0,\ 2u_2 + p_0 = 0,\ 4!u_4 - 4\gamma p_2 = 0.$$

2. *The Approach to GKP or Davis Equations* (2.4)

Theorem 2. *If $(p)|_{y=0}$ and $(u)|_{y=0}$ are known functions, then SNS equations* (2.4) *have power series solution near the common point located in the upstream of the separation point.*

Proof. Substituting Expression (3.8) for ψ in Eqs. (2.4), and collecting powers in terms of the same powers of x and y, we have the following conditions:

$$\alpha^m_{m+4} = 0$$

$$\sum_{j=0}^{m} \frac{\alpha^j_{i+j+2}\alpha^{m-j+1}_{m-j+3}}{j!(m-j)!(1-i)!} = \frac{\alpha^m_{m+5}}{m!}$$

$$\sum_{i=0}^{1}\sum_{j=0}^{m} \frac{\alpha^j_{i+j+2}\alpha^{m-j+1}_{n+2-i+m-j+3}}{j!(i+1)!(m-j)!(1-i)!} - \sum_{j=0}^{m}\frac{\alpha^{j+1}_{j+3}\alpha^{m-j}_{m-j+3}}{2!j!(m-j)!} = \frac{\alpha^m_{m+6}}{2!m!}$$

$$\sum_{i=0}^{n+2}\sum_{j=0}^{m} \frac{\alpha^j_{i+j+2}\alpha^{m-j+1}_{1-i+m-j+3}}{j!(i+1)!(m-j)!(n+2-i)!}$$

$$+ r\sum_{i=0}^{n}\sum_{j=0}^{m} \frac{\alpha^{j+1}_{i+j+2}\alpha^{m-j+2}_{n-i+m-j+s}}{j!(i+1)!(m-j)!(n-i+2)!}$$

$$- \sum_{i=0}^{n+1}\sum_{j=0}^{m} \frac{\alpha^{j+1}_{i+j+2}\alpha^{m-j}_{n+1-i+m-j+3}}{j!(i+2)!(m-j)!(n+1-i)!}$$

$$- r\sum_{i=0}^{n}\sum_{j=0}^{m} \frac{\alpha^{j+1}_{i+j+3}\alpha^{m-j+2}_{n-i+m-j+4}}{j!(i+2)!(m-j)!(n-i+1)!}$$

$$= \frac{\alpha^m_{n+m+7}}{m!(n+3)!},\quad (m,n = 0,1,2,\cdots)$$

From the definition of α, the above equations can also be written as

$$a_4 = 0, \qquad 4'(1)$$

$$\begin{cases} a_5 - a_2b_3 = 0, & 5'(1) \\ b_5 = 0, & 5'(2) \end{cases}$$

$$\begin{cases} a_6 - 2a_2b_4 = 0, & 6'(1) \\ b_6 - a_2c_4 - b_3^2 = 0, & 6'(2) \\ c_6 = 0, & 6'(3) \end{cases}$$

$$\begin{cases} a_7 - 2a_3b_4 - 3\gamma a_2d_5 + 3\gamma b_3c_4 = 0, & 7'(1) \\ b_7 - 2a_2c_5 - 2b_3b_4 = 0, & 7'(2) \\ c_7 - a_2d_5 - 3b_3c_4 = 0, & 7'(3) \\ d_7 = 0, & 7'(4) \end{cases}$$

$$\begin{cases} a_8 + 5a_5b_2 - a_3(5b_5 + 6\gamma d_5) - 4a_2(b_6 + \gamma d_6) + 4\gamma b_4c_4 + 6\gamma b_3c_5 = 0, & 8'(1) \\ b_8 + 2a_4c_4 - 2a_3c_5 - 3a_2(c_6 + \gamma e_6) - 2b_4^2 - b_3b_5 + 3\gamma c_4^2 = 0, & 8'(2) \\ c_8 - 2a_6d_6 - 2b_1c_1 - 4b_3c_5 = 0, & 8'(3) \end{cases}$$

$$\begin{cases} d_8 - a_2e_6 - 4b_3d_5 - 3c_4^2 = 0, & 8'(4) \\ e_8 = 0. & 8'(5) \end{cases}$$

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When the origin (0,0) of the coordinates is a common point located in the upstream of the separation point, Eq. (1) determines b_3,

$$b_3 = \frac{a_5}{a_2}.$$

From 6'(2), 7'(1) and 8'(1), c_4, d_5, b_6 can be determined (the details of solving solutions is omitted for brevity.). Hence GKP or Davis equations (2.4) have power series solution near a common point upstream of the separation point.

Theorem 3. *If* $(p)|_{y=0}$ *and* $(u)|_{x=0}$ *are known functions, then GKP or Davis equations* (2.4) *do not have power series solution near the separation point in the form of* (3.8).

Proof. When the origin (0,0) is the separation point (i.e. $a_2 = 0$), Eq. (4.1) determines $a_4 = 0$. The unknown coefficients of $\phi(x,y)$ cannot be found in terms of the above equations. Accordingly, GKP or Davis equations (2.4) do not have power series solution near the separation point.

Theorem 4. *If* $\left.\frac{\partial^4 u}{\partial y^4}\right|_{(x_s,0)} \neq 0$, *where* $(x_s,0)$ *is the separation point, then GKP or Davis equations* (2.4) *have mathematical singularity near the separation point.*

Proof. Write

$$\bar{u} = \frac{\partial u}{\partial y}, \quad \bar{v} = \frac{\partial v}{\partial y}, \quad \bar{\bar{u}} = \frac{\partial^2 u}{\partial y^2}, \quad \bar{\bar{v}} = \frac{\partial^2 v}{\partial y^2}.$$

By differentiating (2.4a) once with respect to y, and (2.4b) twice to y, using boundary conditions $u|_{y=0} = v|_{y=0} = 0$, and the condition $\bar{v}|_{y=0} = 0$ determined by (2.24a), we have

$$\bar{u}\frac{\partial \bar{u}}{\partial x} = v\frac{\partial^4 u}{\partial y^4}. \tag{3.38}$$

Write

$$\left.\frac{\partial \bar{u}^2}{\partial x}\right|_{(x_s,0)} = a,$$

where $a = 2v\left(\frac{\partial u^4}{\partial y^4}\right)_{(x_s,0)}$. According to $\left(\frac{\partial \bar{u}^2}{\partial x}\right) = 2\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial^2 u}{\partial x \partial y}\right)$ and the separation condition $\left.\left(\frac{\partial u}{\partial y}\right)\right|_{(x_s,0)} = 0$, when $a \neq 0$, $\frac{\partial^2 u}{\partial x \partial y}$ has the singularity at the separation point $(x_s,0)$, that is, $\lim_{x\to x_s}\frac{\partial^2 u}{\partial x \partial y} = \infty$. Now we further examine the character of u and v near the separation point. By the separation condition and (3.38), $\bar{u}^2$ can be expanded as the power series in the form

$$\bar{u}^2 = -a(x_s - x) + \cdots (x < x_s). \tag{3.39}$$

From (3.39), we know $a < 0$. Writing $k = (-a)^{\frac{1}{2}}$, we have the following power series

$$\bar{u} = Ka(x_s - x)^{\frac{1}{2}} + \cdots \quad (x < x_s), \tag{3.40}$$

$$\frac{\partial \bar{u}}{\partial x} = -\frac{K}{2}(x_s - x)^{-\frac{1}{2}} + \cdots \quad (x < x_s). \tag{3.41}$$

By differentiating (2.4a) twice with respect to y and differentiating (2.4b) thrice with respect to y, in terms of the boundary condition $u|_{y=0} = v|_{y=0} = 0$, we have the following equation:

$$2\bar{u}\frac{\partial \bar{\bar{u}}}{\partial x} = v\frac{\partial^5 u}{\partial y^5}. \tag{3.42}$$

From (3.40), it can be known that Eq. (3.42) can be written as

$$\frac{\partial \bar{\bar{u}}}{\partial x} = \frac{\phi_1(x)}{\sqrt{x_s - x}}, \tag{3.43}$$

where $\phi_1(x) = \dfrac{v\left(\dfrac{\partial^5 u}{\partial y^5}\right)_{y=0}}{2k(1 + O(\sqrt{x_s - x}))}$.

According to Equality (3.41) and Eq. (3.43), $\dfrac{\partial u}{\partial x}$ can be expanded as power series in the form

$$\frac{\partial u}{\partial x} = \left(\frac{\partial \bar{u}}{\partial x}\right)_{y=0} y + \left(\frac{\partial \bar{\bar{u}}}{\partial x}\right)_{y=0} \frac{y^2}{2} + \cdots. \tag{3.44}$$

Finally, from the equation of continuity and the equality (3.44). u, v can be approximately written in the form of the following equalities near the separation point

$$u(x, y) = u_0(y) + \alpha(y)\sqrt{x_3 - x},$$

$$v(x,y) = \frac{\beta(y)}{\sqrt{x_3 - x}},$$

where $\alpha(y) = \dfrac{1}{2}(-ky + \phi_1(x_s)y^2)$, $\alpha(y) = 2\beta'(y)$. Hence the GKP or Davis equations (2.4) have the singularity near the separation point.

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