GOVERNING EQUATION OF HRR-TYPE FIELDS FOR ORTHOTROPIC MATERIALS*

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I. CONSTITUTIVE RELATION [1]

Consider the Ramberg-Osgood materials. Their shear strain-shear stress relation takes the form

$$\gamma=\tau+\alpha\tau^{n}. \tag{1.1}$$

The nondimension stress is used throughout this note, which is nondimensionalized by shear yield stress. Meanwhile the nondimension strain is employed, which is nondimensionalized by the shear yield strain.

The axes of the orthogonal Cartesian coordinate system x_1, x_2, x_3 are chosen to be coincident with the principal axes of orthotropy symmetry of meterials. The loading function has the form[1]:

$$\Phi(\sigma_{ij})=F(\sigma_{22}-\sigma_{33})^2+G(\sigma_{33}-\sigma_{11})^2+H(\sigma_{11}-\sigma_{22})^2+2L\sigma_{23}^2+2M\sigma_{31}^2+2N\sigma_{12}^2. \tag{1.2}$$

The generalized deviatoric stresses can be written as

$$S_{ij}^{*}=\frac{\partial\Phi}{\partial\sigma_{ij}}. \tag{1.3}$$

The constitutive relation between the plastic strains and the stresses is

$$\varepsilon_{ij}^{p}=\frac{1}{2}\alpha\tau_{e}^{n-1}S_{ij}^{*}, \tag{1.4}$$

where τ_e is the generalized equivalent shear stress.

$$\tau_e=\sqrt{\Phi(\sigma_{ij})}. \tag{1.5}$$

II. PLANE STRAIN CRACK PROBLEMS

For plane strain problem, we have

$$\varepsilon_{31}=\varepsilon_{32}=\varepsilon_{33}=0, \tag{2.1}$$

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which results in

$$\sigma_{31}=\sigma_{32}=0, \tag{2.2}$$

$$(F+G)\sigma_{33}=(G\sigma_{11}+F\sigma_{22})-\frac{\varepsilon^e_{33}}{\alpha\tau_e^{n-1}} .$$

Assume the hardening exponent $n>1$. In the immediate vicinity of the crack tip, the elastic strains are negligible compared with the plastic strains. In the sense of asymptotic approximation, we have

$$\sigma_{33}=(G\sigma_{11}+F\sigma_{22})/(F+G). \tag{2.3}$$

The loading function can be written as

$$\Phi=\tau_e^2=\frac{q}{4}(\sigma_{11}-\sigma_{22})^2+\sigma_{12}^2, \tag{2.4}$$

where $\frac{q}{4}=\frac{FG}{F+G}+H$.

The plastic strain-stress relations are given by

$$\begin{cases} \varepsilon^p_{11}=-\varepsilon^p_{22}=\frac{\alpha}{2}\tau_e^{n-1}S^*_{11}, \\ \varepsilon^p_{12}=\frac{\alpha}{2}\tau_e^{n-1}\sigma_{12}, \end{cases} \tag{2.5}$$

where $S^*_{11}=\frac{q}{2}(\sigma_{11}-\sigma_{22})$.

Suppose the crack is laid on the axis x_1. The stress function takes the form

$$\Phi=Kr^{s+2}\widetilde{\Phi}(\theta), \quad s=-\frac{1}{n+1}. \tag{2.6}$$

We have[2]

$$\sigma_{ij}=Kr^s\widetilde{\sigma}_{ij}(\theta), \tag{2.7}$$

$$\begin{cases} \widetilde{\sigma}_r=\ddot{\widetilde{\Phi}}+(s+2)\widetilde{\Phi}, \\ \widetilde{\sigma}_\theta=(s+2)(s+1)\widetilde{\Phi}, \\ \widetilde{\tau}_{r\theta}=-(s+1)\dot{\widetilde{\Phi}}, \end{cases} \tag{2.8}$$

$$\begin{cases} \varepsilon^p_r=-\varepsilon^p_\theta=\alpha K^n r^{ns}\widetilde{\varepsilon}^p_r(\theta), \\ \varepsilon^p_{r\theta}=\alpha K^n r^{ns}\widetilde{\varepsilon}^p_{r\theta}(\theta), \end{cases} \tag{2.9}$$

$$\widetilde{\varepsilon}^p_r=\frac{\widetilde{\tau}_e^{n-1}}{2}\left[\frac{(\widetilde{\sigma}_r-\widetilde{\sigma}_\theta)}{2}(q\cos^2 2\theta+\sin^2 2\theta)-\frac{q-1}{2}\widetilde{\tau}_{r\theta}\sin 4\theta\right],$$

$$\widetilde{\varepsilon}^p_{r\theta}=\frac{1}{2}\widetilde{\tau}_e^{n-1}\left[\widetilde{\tau}_{r\theta}(q\sin^2 2\theta+\cos^2 2\theta)-\frac{q-1}{2}\widetilde{S}_r\sin 4\theta\right],$$

$$\tilde{\tau}_e^2 = q(\tilde{S}_r \cos 2\theta - \tilde{\tau}_{r\theta} \sin 2\theta)^2 + (\tilde{S}_r \sin 2\theta + \tilde{\tau}_{r\theta} \cos 2\theta)^2,$$

$$\begin{cases} u_r = \alpha K^n r^{ns+1} \tilde{u}_r(\theta), \\ u_\theta + \alpha K^n r^{ns+1} \tilde{u}_\theta(\theta), \end{cases} \tag{2.10}$$

$$\begin{cases} \tilde{u}_r = \dfrac{1}{(ns+1)} \tilde{\varepsilon}_r^p, \\ \tilde{u}_\theta = \dfrac{1}{ns}(2\tilde{\varepsilon}_{r\theta}^p - \tilde{u}_r). \end{cases} \tag{2.11}$$

III. New Governing Equations and Function Equations

As shown in Fig. 1, we evaluate the J integral along the contour of $ABCDEA$. We have

$$\oint_{ABCDE} \left\{ Wn_1 - p_i \frac{\partial u}{\partial x_1} \right\} ds = 0. \tag{3.1}$$

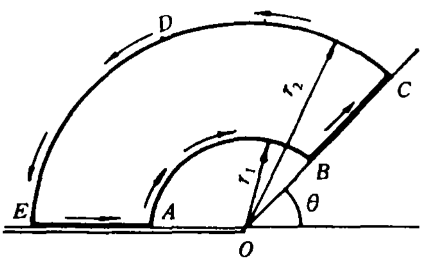

Fig. 1

In Eq. (3.1), the integral on EA is equal to zero. The integrals on AB and on CDE are cancelled, with only higher order small quantities left.

Therefore, as r_1 and r_2 tend to zero, we have

$$\int_{BC} \left(Wn_1 - p_i \frac{\partial u_i}{\partial x_1} \right) du \to 0.$$

On the other hand, we have

$$\int_{BC} \left(Wn_1 - p \frac{\partial u_i}{{}^i\partial x_1} \right) ds = \Pi(\theta) \ln(r_2/r_1).$$

Let r_1 and r_2 tend to zero simultaneously with $r_{2/\cdot 1}$ remaining constant. It follows that

$$\Pi(\theta) = \Pi_1(\theta)\sin\theta + \Pi_2 \cos\theta + (1+ns)\tilde{u}_\theta \Pi_3(\theta) = 0, \tag{3.2}$$

where

$$\tilde{\sigma}_e^2 = 3\tilde{\tau}_e^2,$$

$$\begin{cases} \Pi_1 = \dfrac{n}{1+n} \tilde{\sigma}_e^{1+n} - (\tilde{\sigma}_\theta \tilde{\varepsilon}_\theta^p + \tilde{\tau}_{r\theta} 2 \tilde{\varepsilon}_{r\theta}^p), \\ \Pi_2 = (1+ns) \tilde{u}_r \tilde{\tau}_{r\theta}, \\ \Pi_3 = \tilde{\sigma}_\theta \cos\theta + \tilde{\tau}_{r\theta} \sin\theta. \end{cases}$$

Function equation (3.2) is consistent with the function equation for isotropic hardening materials given by Wang Ke-ren and Wang Tzu-chiang[3].

From Eq. (3.2), it follows that

$$\tilde{\varepsilon}_r' = ns(\Pi_1 \sin\theta + \Pi_2 \cos\theta)/\Pi_3 + 2(1+ns)\tilde{\varepsilon}_{r\theta}. \tag{3.3}$$

Eq.(3.3) is a new governing equation. It is a nonlinear ordinary differential equation of $\tilde{\Phi}$ of order 3. The original governing equation is of order 4.

The accurate numerical calculation shows that Eq.(3.2) is truly equivalent to the original governing equation when $\Pi_3 \neq 0$.

Similarly, using the J_2 integral, another new function equation can be derived as follows.

$$\Pi^*(\theta) = \Pi_1 \cos\theta - \Pi_2 \sin\theta + (1+ns)\tilde{u}_\theta \Pi_3^* = \Pi^*(\pi), \tag{3.4}$$

where

$$\Pi_3^*(\theta) = -\tilde{\sigma}_\theta \sin\theta + \tilde{\tau}_{r\theta} \cos\theta. \tag{3.5}$$

The function equations (3.2) and (3.4) can be represented as

$$\begin{cases} \Gamma_1 = \Pi_3^*(\pi)\cos\theta, \\ \Gamma_2 = -\Pi_3^*(\pi)\sin\theta, \end{cases} \tag{3.6}$$

where

$$\begin{cases} \Gamma_1 = \dfrac{n}{1+n}\tilde{\sigma}_e^{1+n} + [\tilde{\tau}_{r\theta}(\tilde{u}_\theta - \tilde{u}_r') - \tilde{\sigma}_\theta \tilde{\varepsilon}_\theta], \\ \Gamma_2 = (1+ns)(\tilde{\tau}_{r\theta}\tilde{u}_r + \tilde{\sigma}_\theta \tilde{u}_\theta). \end{cases} \tag{3.7}$$

REFERENCES

[1] Hill, R., *The Mathematical Theory of Plasticity*, Oxford University Press, London, 1950, p. 317.
[2] Pan, J. & Shih, C. F., *Mech. Materials*, 5(1986), 299.
[3] Wang K. R. & Wang T. C., *Applied Mathematics and Mechanics*, 8(1987), 839.