

# DISTRIBUTION, STABILITY, BIFURCATIONS AND CATASTROPHE OF STEADY ROTATION OF A SYMMETRIC HEAVY GYROSCOPE WITH VISCOUS- LIQUID-FILLED CAVITY

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**Abstract**—The number, the angles of orientation and the stability in Rumyantsev–Movchan's sense of oblique steady rotations of a symmetric heavy gyroscope with a cavity completely filled with a uniform viscous liquid, possessing a fixed point  $O$  on its symmetric axis, are given for various values of the parameters. By taking the square of the upright component of the angular momentum  $M^2$  as a control parameter, three types of bifurcation diagrams of the steady rotations, two types of jumps and two kinds of local catastrophes, one being the symmetric reduced cusp type and the other being of the symmetric reduced butterfly type, are obtained. By taking account of the  $M^2$ -damping owing to the moment of unavoidable faint friction, two different modes for the gyroscope, initially in a stable quasi-steady upright rotation with a nutation angle  $\theta$ , equal to zero, to topple over are found.

## 1. INTRODUCTION

The problem of stability of the steady rotations of a gyroscope, with a cavity filled with a viscous liquid, has attracted a great deal of attention for a long time [1–6]. In 1958, the first U.S. Satellite Explorer 1 tumbled after only a few hours of flight [7]. It was concluded that the four turnstile wire antennae were dissipating energy, thus causing a transfer of body spin axis from the axis of minimum inertia to a transverse axis of maximum inertia. Since then, the stability of gyroscopes with various kinds of energy dissipation and the modes for them to tumble or to topple over are of more interest [4, 7–13]. Most of these works discuss the stability of steady right rotations in which the angular velocity coincides with a principal axis of inertia (the steady rotations for which this condition is not satisfied are called *oblique rotations*) [9, 11, 12] and the torque-free tumbling modes [4, 8–10], and some discuss oblique rotations of a heavy gyroscope with a dissipative force depending on one generalized velocity [13].

In this paper, the distribution and stability of the oblique steady rotations of a symmetric heavy gyroscope with a cavity completely filled with a uniform viscous liquid, possessing a fixed point on its symmetric axis (called *FFVL gyroscope* for short) is discussed. Relevant bifurcations, jumps and the types of catastrophes are discussed, so that two different modes for FFVL gyroscopes, initially in a stable quasi-steady upright rotation with a nutation angle  $\theta$ , equal to zero, to topple over owing to the moment of unavoidable faint friction are found theoretically. Obviously the dissipative force depends on the distribution of the velocity of liquid, i.e. on an infinite number of generalized velocities.

Two known stability lemmas are revised in such a way that they will be appropriate for all our objectives in this paper.

## 2. BASIC EQUATIONS AND LEMMAS

Consider an FFVL gyroscope with a total mass  $m$ , three principal moments of inertia  $A$ ,  $A$  and  $C$  about a fixed point  $O$ , and with a distance  $l$  ( $l > 0$ ) between its centre of gravity  $O'$  and the point  $O$ , moving in a uniform gravitational field ( $g$ ).

Let us introduce two basic coordinates with their origins coinciding at  $O$ :

(1) A fixed coordinate system  $\{0, \xi_1, \xi_2, \xi_3\}$ . The axis  $\xi_3$  is upright and the three unit coordinate vectors are designated as  $i_1^0, i_2^0, i_3^0$ .

(2) A tied coordinate system  $\{0, x_1, x_2, x_3\}$ . This frame is tied to the gyroscope, with the symmetric axis selected as the  $x_3$ -axis and with its centre of gravity  $O'$  at  $(0, 0, l)$ . The three unit coordinate vectors are designated as  $i_1, i_2, i_3$ . The orientation of this frame relative to frame (1) is determined by Euler's angles  $(\theta, \psi, \varphi)$ , where  $\theta$  is a nutation angle,  $\psi$  a rotation angle and  $\varphi$  a precessional angle.

The dissipative non-linear evolutionary equations of this system in frame (1) are

$$\rho_1 \frac{dv}{dt} = \nabla \cdot (-p\mathbf{I} + \mu\mathbf{T}) + \rho_1 \mathbf{g} \quad (1)$$

$$\nabla \cdot \mathbf{v} = 0 \quad (2)$$

$$\frac{d}{dt}(\omega \cdot \mathbf{J}_2) = - \iint_S \mathbf{r} \times (-p\mathbf{I} + \mu\mathbf{T}) \cdot d\mathbf{S} + \mathbf{r}_O \times m_2 \mathbf{g} \quad (3)$$

with the boundary condition at the interface  $S$  of liquid and solid

$$\mathbf{v}|_S = \omega \times \mathbf{r} \quad (4)$$

where  $\rho_1, p, \nu, \mu$  and  $\mathbf{T}$  are the density of the liquid, its pressure, its velocity, the kinetic viscosity coefficient and the viscous stress tensor, respectively;  $m_2, \mathbf{J}_2, \omega$  and  $\mathbf{r}_O$  are the mass of the solid, its tensor of inertia about the point  $O$ , its angular velocity and the position vector of its centre of gravity  $O'$ , respectively.

From equations (1)–(4) it follows [3, 11] that an FFVL gyroscope in steady rotation must rotate as a whole with a constant nutation angle  $\theta_s$ , a constant rotation angle  $\psi_s$  and a constant angular velocity  $\omega_s$  in the direction of  $\pm i_3^0$ , with the following two equations satisfied by  $\theta_s$  and  $\omega_s$ :

$$\omega_s^2 = \frac{mgl}{(C - A) \cos \theta_s} \quad (5)$$

$$\left. \frac{dW}{d\theta} \right|_{\theta=\theta_s} = 0 \quad (6)$$

where  $W$  is

$$W = \frac{M^2}{2I} + L \quad (7)$$

and  $L, M$  and  $I$  are the potential energy of the gyroscope, the upright component of its angular momentum about the point  $O$  and its moment of inertia about the vertical line through the point  $O$ , respectively. Thus,

$$\dot{\theta}_s = \dot{\psi}_s = 0.$$

In our case of a symmetric gyroscope,  $W$  is independent of  $\psi$  and  $\varphi$ ; so in discussing the stability of steady rotations, we are concerned about the stability with respect to the metrics  $|\theta - \theta_s|$  and

$$\rho = [|\omega - \omega_s|^2 + T_r + |\theta - \theta_s|^2]^{1/2} \quad (8)$$

under the condition that the metric of the initial perturbation  $\rho_0$  is defined as [14]

$$\rho_0 = \max[|\theta - \theta_s|, |\omega - \omega_s|, |v_r(\mathbf{r})|] \quad (9)$$

where  $v_r(\mathbf{r})$  and  $T_r$  are the velocity and kinetic energy of the liquid relative to frame (2), respectively. All the quantities in equations (8) and (9) are referred to the special dimensionless quantities, i.e. the ratios of the quantities with dimension to the corresponding unit quantities.

A steady rotation of an FFVL gyroscope is said to be stable with respect to  $(\rho_0, \rho)$  [or to  $(\rho_0, |\theta - \theta_s|)$ ] if for every real number  $\varepsilon > 0$ , there exists a real number  $\delta > 0$  such that

$$\rho_0 < \delta \quad \text{for } t = 0$$

implying that

$$\rho < \varepsilon \text{ [or } |\theta - \theta_s| < \varepsilon] \quad \text{for all } t \geq 0.$$

If, in addition,  $\rho \rightarrow 0$  (or  $\theta \rightarrow \theta_s$ ) as  $t \rightarrow \infty$ , then the steady rotation is said to be asymptotically stable with respect to the metric  $(\rho_0, \rho)$  [or to  $(\rho_0, |\theta - \theta_s|)$ ].

We must use the following limit lemma [11]:

*Limit lemma.* No matter what the initial state may be, when  $t \rightarrow \infty$ , a viscous-liquid-filled gyroscope with a fixed point must tend to a steady rotation state with respect to  $\rho$ .

Using Rumyantsev–Movchan’s methods [14] and equations (1)–(4) and the limit lemma [11] we can obtain the following two lemmas.

*Lemma 1.* If for a steady rotation state  $p^{(e)}$  of an FFVL gyroscope, the generalized potential  $W$  has an isolated minimum with respect to  $\theta$  at  $\theta_s$ , then the state  $p^{(e)}$  is stable with respect to the metric  $(\rho_0, \rho)$  and is asymptotically stable with respect to the metric  $(\rho_0, |\theta - \theta_s|)$  in the case that  $\theta_s = 0$ , but is not asymptotically stable with respect to the metric  $(\rho_0, |\theta - \theta_s|)$  in the case that  $\theta_s \neq 0$ .

*Lemma 2.* If for a steady rotation  $p^{(e)}$  of an FFVL gyroscope, the value of the generalized potential  $W$  at  $\theta_s$  is only an isolated stationary value with respect to  $\theta$  but not a minimum, then the state  $p^{(e)}$  is unstable with respect to the metric  $(\rho_0, |\theta - \theta_s|)$ .

The above two lemmas are different from theorems 6 and 7 of ref. [3] in that the former are appropriate to liquid-filled symmetric gyroscopes with two cyclical coordinates, while the latter pertain to non-symmetric ones with one cyclical coordinate and in that the term instability is with respect to the metric  $(\rho_0, |\theta - \theta_s|)$  in our lemma 2, but with respect to the metric  $(\rho_0, \rho)$  in theorem 7 of ref. [3] even if two cyclical coordinates would be permitted. Obviously, our lemma 2 is stronger than theorem 7 of ref. [3].

### 3. DISTRIBUTION AND STABILITY OF OBLIQUE STEADY ROTATIONS

Introduce the following notations:

$$\begin{aligned} \Omega_0 &= \frac{M}{C}, & \bar{C} &= C\Omega_0^2/mgl, & \bar{A} &= A\Omega_0^2/mgl \\ q &= -\frac{\bar{C}}{\sqrt{|\bar{C} - \bar{A}|}}, & \gamma &= \frac{\bar{A}}{\bar{C} - \bar{A}} \\ \alpha &= \sqrt[3]{\frac{q^2}{16} + \sqrt{\frac{q^4}{256} - \frac{\gamma^3}{27}}} + \sqrt[3]{\frac{q^2}{16} - \sqrt{\frac{q^4}{256} - \frac{\gamma^3}{27}}} \end{aligned} \tag{10}$$

where it is stipulated that the principal values of both the radical expressions in equation (10) should be taken in the case of  $\gamma > 0$ , and that the real cube roots should be taken in the case of  $\gamma < 0$ .

In the present case of symmetric gyroscopes, equation (7) becomes

$$W = (\cos \theta - 1)mgl + M^2/2[A + (C - A)\cos^2 \theta]. \tag{11}$$

Substituting equation (11) into equation (6), we obtain the following equations:

$$\sin \theta_s = 0 \tag{12}$$

and (if  $A \neq C$ )

$$\frac{M^2 \cos \theta_s}{mgl(C - A)} = \left( \cos^2 \theta_s + \frac{A}{C - A} \right)^2. \tag{13}$$

Equation (12) leads to  $\theta_s = 0$  and  $\pi$ , corresponding to upright and hanging-right steady rotations, respectively. Equation (13) gives oblique steady rotations with  $\theta_s$  in the open interval  $(0, \pi)$ .

The stability of the upright and hanging-right steady rotations has been clarified [11], i.e.

we know that if

$$\Omega_0^2(C - A) \begin{cases} > mgl \cos \theta_s, \\ = mgl \cos \theta_s, \end{cases} \text{ for } \theta_s = \pi, \text{ or } \theta_s = 0 \text{ and } 4A < 3C \quad (14)$$

it is stable with respect to  $(\rho_0, \rho)$ , and if

$$\Omega_0^2(C - A) \begin{cases} < mgl \cos \theta_s, \\ = mgl \cos \theta_s, \end{cases} \text{ for } 4A \geq 3C \text{ and } \theta_s = 0 \quad (15)$$

it is unstable with respect to  $(\rho_0, |\theta - \theta_s|)$ . The rest of the questions to answer are: (1) Under what conditions of the parameters  $M^2, m, l, A$  and  $C$  has equation (13) any solutions for steady oblique rotations? How many if any? (2) What is the expression for  $\theta_s$ ? (3) Are the steady oblique rotations stable? The answer to question (1) cannot be obtained by a direct analysis of the expression for  $\theta_s$ , because the roots of the quartic equation (13) of argument  $\cos \theta_s$  can only be expressed by double-radical expressions. But in the Appendix, we prove the following four theorems to answer the above three questions completely. It should be noted that any symmetric gyroscope must satisfy the inequality  $2A > C + 2ml^2$ , so that it is not necessary to list this inequality in the conditions of the following theorems.

*Theorem 1.* If an FFVL gyroscope satisfies

$$\bar{C} > \bar{A} + 1 \quad (16)$$

or

$$\bar{C} = \bar{A} + 1 \text{ and } \bar{A} < 3 \quad (17)$$

then, in the space  $(\theta, \omega, \mathbf{v}_s(\mathbf{r}))$ , it has a unique pair of steady oblique rotation states  $(\theta_s, \pm \omega_s, \mathbf{0})$ , which are unstable with respect to the metric  $(\rho_0, |\theta - \theta_s|)$ , and whose  $\theta_s$  and  $\omega_s^2$  are given by equation (5) and

$$\cos \theta_s = -\frac{q}{2\sqrt{2\alpha}} - \sqrt{-q\sqrt{\frac{\alpha}{2}} - \alpha^2} > 0. \quad (18)$$

*Theorem 2.* If an FFVL gyroscope satisfies

$$\bar{C} < \bar{A} + 1, \quad \bar{A} < 3 \quad (19)$$

and

$$\frac{256}{27} \bar{A}^3 - \bar{C}^5 + \bar{C}^4 \bar{A} \leq 0 \quad (20)$$

then, in the space  $(\theta, \omega, \mathbf{v}_s(\mathbf{r}))$ , it has two pairs of steady oblique rotation states  $(\theta_{s\pm}, \pm \omega_{s\pm}, \mathbf{0})$ , which satisfy equation (5) and

$$\cos \theta_{s\pm} = -\frac{q}{2\sqrt{2\alpha}} \pm \sqrt{-q\sqrt{\frac{\alpha}{2}} - \alpha^2} > 0 \quad (21)$$

where  $\theta_{s+}$  belongs to the pair of states stable with respect to the metric  $(\rho_0, \rho)$ , but not asymptotically stable with respect to the metric  $(\rho_0, |\theta - \theta_{s+}|)$  and  $\theta_{s-}$  belongs to the other pair of states unstable with respect to the metric  $(\rho_0, |\theta - \theta_{s-}|)$ . Condition (20) with the equality sign makes both pairs merge into a pair of steady oblique states  $(\theta_s, \pm \omega_s, \mathbf{0})$  unstable with respect to the metric  $(\rho_0, |\theta - \theta_s|)$ .

*Theorem 3.* If an FFVL gyroscope satisfies

$$\frac{256}{27} \bar{A}^3 - \bar{C}^5 + \bar{C}^4 \bar{A} > 0, \quad \bar{C} \geq \bar{A} - 1 \quad (22)$$

or

$$\frac{256}{27} \bar{A}^3 - \bar{C}^5 + \bar{C}^4 \bar{A} \leq 0, \quad \bar{C} \leq \bar{A} + 1, \text{ and } \bar{A} \geq 3 \quad (23)$$

then it has no oblique steady rotation.

Theorem 4. If an FFVL gyroscope satisfies

$$\bar{C} < \bar{A} - 1 \tag{24}$$

then, in the space  $(\theta, \omega, v, (r))$ , it has a unique pair of steady oblique rotation states  $(\theta_s, \pm \omega_s, 0)$ , which are stable with respect to the metric  $(\rho_0, \rho)$ , but not asymptotically stable with respect to the metric  $(\rho_0, |\theta - \theta_s|)$ , and in which the  $\theta_s$  and  $\omega_s^2$  are given by equation (5) and

$$\cos \theta_s = \frac{q}{2\sqrt{2\alpha}} + \sqrt{-q\sqrt{\frac{\alpha}{2}} - \alpha^2} < 0. \tag{25}$$

All the qualitative conclusions given by theorems 1-4 are shown in Fig. 1. It can be proved that the curve  $2\sqrt[5]{6} \bar{A}^3 - \bar{C}^5 + \bar{C}^4 \bar{A} = 0$  is below the straight line  $\bar{C} = \bar{A} + 1$ , being tangential to the latter at the point (3, 4) and convex at this point; and it is tangential to the  $\bar{C}$ -axis at the origin, and intersects with the straight line  $\bar{C} = \bar{A}$  at a unique point, i.e. at the origin, and tends to this line asymptotically as  $\bar{A}$  tends to  $+\infty$ . In addition, only the sector included between the  $\bar{A}$ -axis and the straight line  $\bar{C} = 2\bar{A}$  is of practical significance because of the inequality  $\bar{C} < 2\bar{A} - 2l\Omega_0^2/g$ , which must be satisfied by any symmetric gyroscope.

4. BIFURCATIONS AND CATASTROPHE

It is of significance to discuss the bifurcation set shown in Fig. 1 from the point of view of Thom's catastrophe theory (15). In doing so, the line  $\bar{C} = 2\bar{A}$  is blotted out temporarily. Near the point (3, 4), let

$$\bar{A} = 3 + \xi, \quad \bar{C} = 4 + \gamma. \tag{26}$$

By expanding  $\bar{W}$ , neglecting the terms of order higher than six, and taking the linear approximation with respect to  $\xi$  and  $\gamma$  in the coefficients we obtain

$$\bar{W} = -(x^6 + ux^4 + vx^2) + C(u, v) \tag{27}$$

with

$$u = 2^{4/3} \left( \frac{\xi}{3} - \frac{5}{24} \gamma \right), \quad v = 2^{2/3} (\xi - \gamma), \quad x = 2^{-5/6} \theta.$$

This is what we call the symmetric reduced butterfly catastrophe with all the terms of odd order in the expansion vanishing. A similar analysis shows that on the line  $\bar{C} = \bar{A} - 1$  ( $\bar{A} > 1$ ), the type of catastrophe is a symmetric reduced cusp type.

For given  $A, C, m$  and  $l$ , and  $M^2$  increases, the phase point in Fig. 1 departs more and more from the origin along a straight line through the origin,  $\bar{C} = \beta\bar{A}$ . Using this and taking account of the stability conditions (14) and (15), for upright and hanging-right steady rotations, we can obtain three kinds of bifurcations shown in Figs 2-4 for three different ranges of the values of  $\beta$ .

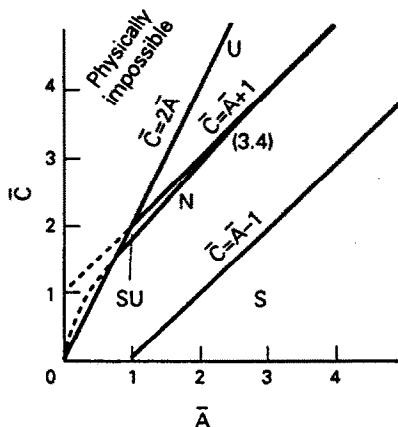


Fig. 1. Stability diagram.

Range 1:  $\beta > \frac{4}{3}$  (i.e.  $C > \frac{4}{3}A$ )

This situation is depicted in Fig. 2. Branch  $q_2$  represents the set of hanging-right steady rotations, which are all stable. Branch  $q_1$  represents the set of upright steady rotations,  $H_2$  [ $M^2 = M_2^2 = mglC^2/(C - A)$ ] is its critical point. On the left of  $H_2$ , rotations are unstable, and on the right, they are stable. A saddle-node bifurcation occurs at another critical point  $H_3$  ( $M^2 = M_3^2 = \frac{16}{\sqrt{27}} mgl \sqrt{\frac{A^3}{C - A}}$ ) with the emergence of an unstable branch of steady oblique rotations  $q_3$  and a stable branch  $q_4$ . At  $H_2$  and  $H_3$ ,  $\omega_s^2$  and  $\theta_s$  take the following values, respectively:

$$\theta_{s2} = 0, \quad \omega_{s2}^2 = mgl/(C - A) \tag{28}$$

$$\theta_{s3} = \cos^{-1} \sqrt{\frac{A}{3(C - A)}}, \quad \omega_{s3}^2 = mgl \sqrt{\frac{3}{A(C - A)}} \tag{29}$$

The  $\theta_s$  of branch  $q_3$  tends to  $\frac{\pi}{2}$ , as  $M^2$  tends to  $\infty$ . Branch  $q_4$  intersects orthogonally with branch  $q_1$  at the point  $H_2$ . Obviously, branch  $q_4$  is of practical significance: in some cases in which branch  $q_1$  has lost its stability, branch  $q_4$  can prevent the gyroscope from toppling over.

Now suppose that the initial state of the gyroscope is represented by the point  $H_1$ , and that the control parameter  $M^2$  decreases gradually; then the phase point will move continuously along a path marked with arrows through the point  $H_2$ , until  $M^2$  reaches the critical value  $M_3^2$ , i.e. the phase point arrives at  $H_3$ . At this moment a jump will occur: the phase point jumps from  $H_3$  of branch  $q_4$  to  $H_4$  of branch  $q_2$ . After this, the phase point will move continuously along branch  $q_2$  towards the left until  $M^2$  reaches zero. If the phase point is located initially on the left of  $H_4$  in branch  $q_2$  and  $M^2$  increases gradually, it will move along branch  $q_2$  towards the right continuously without any jump.

Range 2:  $\frac{4}{3} \geq \beta > 1$  (i.e.  $\frac{4}{3}A \geq C > A$ )

This situation is depicted in Fig. 3. Branch  $q_2$  represents the set of steady hanging-right rotations, which are all stable without any critical point. Branch  $q_1$  represents the set of upright steady rotations, which have a critical value  $M_6^2 = mglC^2/(C - A)$  at the point  $H_6$ . In branch  $q_1$ , the steady rotations with  $M^2$  larger than  $M_6^2$  are stable and those with  $M^2$  smaller than  $M_6^2$  are unstable. At the critical point, branch  $q_1$  bifurcates with the emergence of a branch  $q_3$  of unstable oblique steady rotations.  $\theta_s$  of states in branch  $q_3$  tends to  $\frac{\pi}{2}$  when  $M_2$  tends to  $\infty$ .

If  $M_0^2$  of an initial phase point in branch  $q_1$  is larger than  $M_6^2$ , then when  $M^2$  decreases, a jump will occur at  $H_6$ ; but wherever an initial phase point may be located in branch  $q_2$ , neither increase nor decrease of  $M^2$  can induce any jump.

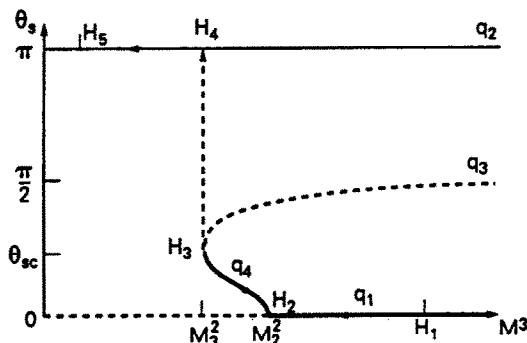


Fig. 2.  $\beta > \frac{4}{3}(C > \frac{4}{3}A)$ .

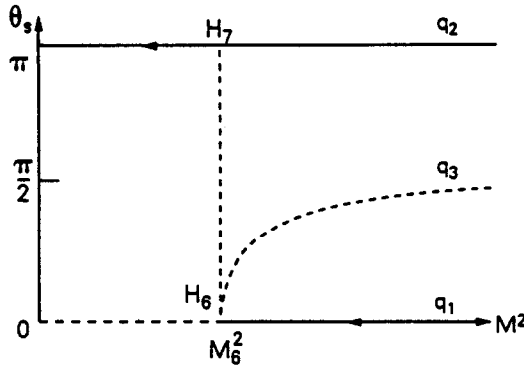


Fig. 3.  $\frac{4}{3} \geq \beta > 1 (\frac{4}{3}A \geq C > A)$ .

**Range 3:  $0 < \beta < 1$  (i.e.  $C < A$ )**

This situation is depicted in Fig. 4. Branch  $q_1$  represents the set of upright steady rotations, which are all unstable with no critical points. Branch  $q_2$  represents the set of hanging-right steady rotations, which have a critical point  $H_8$  with a critical value  $M_8^2 = mgl C^2 / (A - C)$ . The steady rotations in branch  $q_2$ , with  $M^2$  larger than  $M_8^2$  are unstable and those with  $M^2$  smaller than  $M_8^2$  are stable. At  $H_8$ , branch  $q_2$  bifurcates with the emergence of a branch  $q_3$  of stable oblique steady rotations.  $\theta_s$  of states in branch  $q_3$ , tends to  $\frac{\pi}{2}$  as  $M^2$  tends to  $\infty$ .

Since the term “steady” used above in spite of the  $M^2$ -damping induced by the unavoidable faint frictional moment, refers really to “quasi-steady” rotations on account of the existence of that moment, from the above analysis it follows that an FFVL gyroscope with  $C > A$ , initially in a stable quasi-steady upright rotation, will topple over after all owing to this frictional moment in one of the following two possible fashions. If  $C > \frac{4}{3}A$  and  $M_0^2 > M_2^2$ , according to Fig. 2,  $\theta_s$  will continue to be zero, but  $M^2$  and  $\omega_s^2$  will decrease gradually until they reach  $M_2^2$  and  $\omega_{s2}^2$  at the same time, respectively. After this,  $\theta_s$  and  $\omega_s^2$  will increase gradually until  $M^2$ ,  $\theta_s$  and  $\omega_s^2$  reach  $M_3^2$ ,  $\theta_{s3}$  and  $\omega_{s3}^2$  at the same time, respectively. At this moment, the gyroscope will start to topple over. That will be an unsteady motion. After a time it will enter into a stable quasi-steady hanging-right rotation, in which  $\theta_s$  will continue to be  $\pi$ , but  $\omega_s^2$  and  $M^2$  will decrease gradually. Finally,  $\omega_s^2$  and  $M^2$  will become zero at the same time. If  $\frac{4}{3}A \geq C > A$  and  $M_0^2 > M_6^2$ , according to Fig. 3,  $\theta_s$  will continue to be zero until the gyroscope goes on to topple over at the point  $H_6$ . The subsequent motion of the gyroscope is similar to that for the case when  $C > \frac{4}{3}A$ .

From Figs 2–4, the following criterion for the stability of oblique steady rotations are obtained.

*Criterion 1.* It is stable, if  $C < A$  or  $C > \frac{4}{3}A$  and

$$1 > \cos \theta_s > \sqrt{\frac{A}{3(C - A)}}. \tag{30}$$

It is unstable, if  $\frac{4}{3}A \geq C > A$  or  $C > \frac{4}{3}A$  and

$$0 < \cos \theta_s \leq \sqrt{\frac{A}{3(C - A)}}. \tag{31}$$

In terms of the angular velocity  $\omega_s$ , the conditions (30) and (31) can be replaced by

$$\frac{mgl}{(C - A)} < \omega_s^2 < mgl \sqrt{\frac{3}{A(C - A)}} \tag{32}$$

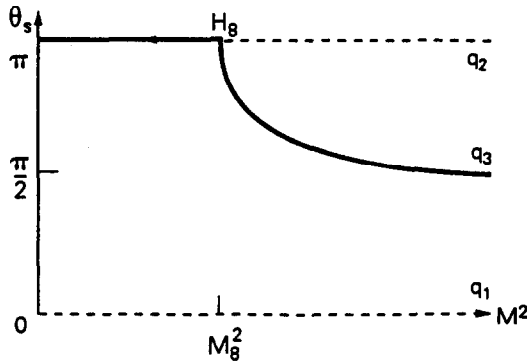


Fig. 4.  $1 > \beta > 0 (A > C)$ .

and

$$\omega_s^2 \geq mgl \sqrt{\frac{3}{A(C - A)}} \tag{33}$$

respectively.

Note that the stability and bifurcation in our case of infinite-degrees-of-freedom is similar to those in ref. [13], where no liquid-filled cavities are present and the problem of dynamics is one of finite-degrees-of-freedom.

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APPENDIX: PROOFS OF THEOREMS 1-4

Equation (13) can be rewritten in the form

$$q^2 \cos \theta_s = (\cos^2 \theta_s + \gamma)^2 \quad (\text{if } \bar{A} < \bar{C}) \tag{A.1}$$

$$-q^2 \cos \theta_s = (\cos^2 \theta_s + \gamma)^2 \quad (\text{if } \bar{A} > \bar{C}). \tag{A.2}$$

Solving equations (A.1) and (A.2) is equivalent to finding the points of intersection of the curves

$$y = (\cos^2 \theta_s + \gamma)^2 \tag{A.3}$$

and the straight line

$$y = q^2 \cos \theta_s, \quad (\text{if } \bar{A} < \bar{C}) \tag{A.4}$$



or those of equation (A.3) and

$$y = -q^2 \cos \theta_s \quad (\text{if } \bar{A} > \bar{C}) \tag{A.5}$$

in the  $y$ - $\cos \theta_s$  plane under the constraint

$$|\cos \theta_s| < 1. \tag{A.6}$$

The curves of equation (A.3) for three different ranges of  $A$  and  $C$  are shown in Figs 5-7.

In Fig. 5 ( $\bar{C} > \frac{4}{3}\bar{A}$ ) and Fig. 6 ( $\bar{A} < \bar{C} < \frac{4}{3}\bar{A}$ ), the slope of the tangent OE is

$$k_{OE} = 16 \left[ \frac{\bar{A}}{3(\bar{C} - \bar{A})} \right]^{3/2}$$

with the abscissa of the tangential point E equal to  $\sqrt{\frac{\bar{A}}{3(\bar{C} - \bar{A})}}$ . The abscissa of the point D on the curve (A.3) is equal to unity and the slope of the straight line OD is

$$k_{OD} = \left( \frac{\bar{C}}{\bar{C} - \bar{A}} \right)^2.$$

So we have  $k_{OD} \geq k_{OE}$ , where the equality sign corresponds to the case when  $\bar{C} = \frac{4}{3}\bar{A}$ .

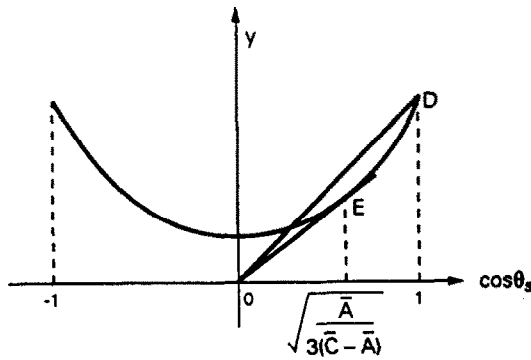


Fig. 5.  $\bar{C} > \frac{4}{3}\bar{A}$ .

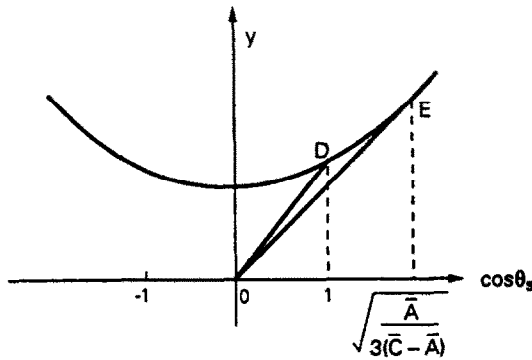


Fig. 6.  $\bar{A} < \bar{C} < \frac{4}{3}\bar{A}$ .

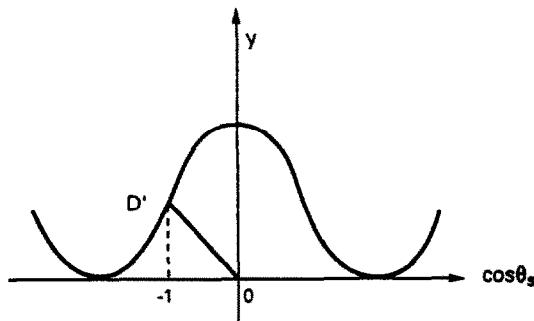


Fig. 7.  $\bar{C} < \bar{A}$ .

When

$$\bar{C} > \frac{4}{3}\bar{A} \tag{A.7}$$

the abscissa of the point E is smaller than unity, and when

$$\bar{A} < \bar{C} < \frac{4}{3}\bar{A} \tag{A.8}$$

it is greater than unity.

In Fig. 7 ( $\bar{C} < \bar{A}$ ) the abscissa of the point D' on the curve (A.3) is  $-1$ , and the slope of the straight line OD' is

$$k_{OD'} = -\left(\frac{\bar{C}}{\bar{A} - \bar{C}}\right)^2.$$

*Proof of theorem 1.* Under the condition (16), Fig. 5 or 6 are applied and the slope of the straight line (A.4) is  $k = q^2 > k_{OD}$ . So, of the two points of intersection of the straight line (A.4) and the curve (A.3), the one with the smaller abscissa satisfies the condition (A.6) and corresponds to the unique pair of steady oblique rotation states  $(\theta_s, \pm \omega_s, 0)$ .

Condition (17) implies condition (A.7), so Fig. 5 is applied and  $k = k_{OD}$ . Of the two points of intersection of curves (A.4) and (A.3), the one with the smaller abscissa satisfies condition (A.6) and corresponds to a pair of steady oblique rotation states  $(\theta_s, \pm \omega_s, 0)$ , and the other just corresponds to a pair of rotations belonging to the set of upright states because their  $\cos \theta_s$  values are equal to unity.

In order to prove equation (18), let

$$x = \sqrt{\cos \theta_s} \tag{A.9}$$

and then equation (A.1) becomes

$$x^4 + qx + \gamma = 0. \tag{A.10}$$

Of the four roots of equation (A.10), two are not real numbers, and remaining ones

$$x = \sqrt{\frac{\alpha}{2}} \pm \sqrt{\frac{-q}{2\sqrt{2\alpha}} - \frac{\alpha}{2}} \tag{A.11}$$

may be real numbers. It is shown above that equation (A.10) must have real roots and that the larger one must be ignored, and thus the negative sign must be taken. Substituting equation (A.11) into equation (A.9) we arrive at equation (18).

Besides, condition (16) gives (see Fig. 8)

$$\left. \frac{dW}{d \cos \theta} \right|_{\cos \theta = 1} < 0, \quad \left. \frac{dW}{d \cos \theta} \right|_{\cos \theta = -1} > 0$$

and condition (17) gives

$$\left. \frac{dW}{d \cos \theta} \right|_{\cos \theta = 1} = 0, \quad \left. \frac{d^2 W}{d \cos^2 \theta} \right|_{\cos \theta = 1} > 0$$

$$\left. \frac{dW}{d \cos \theta} \right|_{\cos \theta = -1} > 0.$$

Therefore, it follows from lemma 2 that the states  $(\theta_s, \pm \omega_s, 0)$  are unstable with respect to  $(\rho_0, |\theta - \theta_s|)$ .

*Proof of theorem 2.* Under the conditions (19) and (20), Fig. 5 is applied, and the slope of the straight line (A.4),  $k = q^2$ , satisfies  $k_{OD} > k \geq k_{OE}$ . Thus the straight line (A.4) and the curve (A.3) intersect at two points representing two pairs of steady oblique rotations  $(\theta_{s\pm}, \pm \omega_{s\pm}, 0)$ . In the case when the equality sign is taken in condition (20), the two points merge into a tangent point, that means the two pairs of  $(\theta_{s\pm}, \pm \omega_{s\pm}, 0)$  will merge into a two-fold pair. Substituting equation (A.11) into equation (A.9), we obtain equation (21).

It is easy to prove under condition (19) that (see Fig. 9)

$$\left. \frac{dW}{d \cos \theta} \right|_{\cos \theta = 1} > 0, \quad \left. \frac{dW}{d \cos \theta} \right|_{\cos \theta = -1} > 0.$$

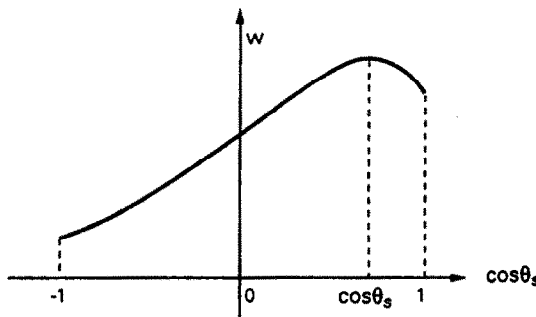


Fig. 8.  $W$ - $\cos \theta_s$  curve for condition (16).

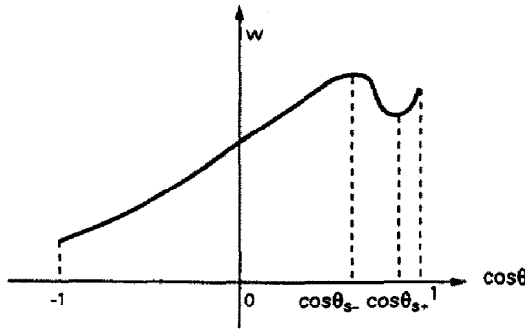


Fig. 9.  $W$ - $\cos \theta$  curve for condition (19).

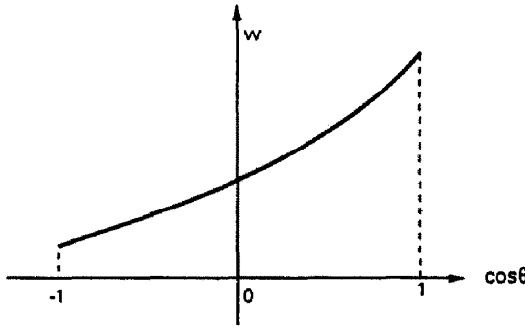


Fig. 10.  $W$ - $\cos \theta$  curve for conditions (22) and (23).

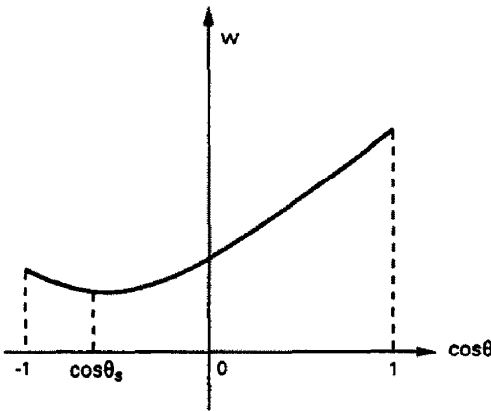


Fig. 11.  $W$ - $\cos \theta$  curve for condition (24).

Thus, it follows from lemmas 1 and 2 that the states  $(\theta_{s+}, \pm \omega_{s+}, 0)$  are stable with respect to  $(\rho_0, \rho)$ , but not asymptotically stable with respect to  $(\rho_0, |\theta - \theta_{s+}|)$ , and the states  $(\theta_{s-}, \pm \omega_{s-}, 0)$  are unstable with respect to  $(\rho_0, |\theta - \theta_{s-}|)$ . Condition (20) with the equality sign gives  $\theta_{s-} = \theta_{s+}$  corresponding to an inflection point in the curve  $W$ - $\cos \theta$ , which implies instability with respect to  $(\rho_0, |\theta - \theta_{s+}|)$ .

*Proof of theorem 3.* The whole region in the  $\bar{C}$ - $\bar{A}$  plane shown by equations (22) and (23) can be divided anew into the following three parts:

$$(1) \bar{C} \leq \bar{A} + 1, \quad \bar{A} < \bar{C} \leq \frac{2}{3}\bar{A} \tag{A.12}$$

$$(2) \bar{C} \geq \frac{2}{3}\bar{A}, \quad \frac{22}{27}\bar{A}^3 - \bar{C}^3 + \bar{C}^4\bar{A} > 0 \tag{A.13}$$

$$(3) \bar{A} \geq \bar{C} \geq \bar{A} - 1. \tag{A.14}$$

Under condition (A.12), Fig. 6 is applied and we have  $k \leq k_{0D}$ . So the straight line (A.4) and the curve (A.3) either do not intersect or intersect at some points where  $\cos \theta_s > 1$ , resulting in no steady oblique rotations.

Under condition (A.13), Fig. 5 is applied and we have  $k < k_{0E}$ . So the straight line (A.4) and the curve (A.3) do not intersect resulting in no steady oblique rotations.

Under condition (A.14), if  $A = C$ , equation (5) cannot be satisfied resulting in no steady oblique rotations. If  $\bar{A} \neq \bar{C}$ , Fig. 7 is applied and the slope of the straight line (A.5)  $k'$ , satisfies the inequality  $0 > k' \geq k_{OD}$ . Thus, both the points of intersection of the straight line (A.5) and the curve (A.3) satisfy  $\cos \theta, \leq -1$  resulting in no steady oblique rotations. The curve  $W-\cos \theta$  is shown in Fig. 10.

*Proof of theorem 4.* Under condition (24), Fig. 7 is applied and we have

$$k' < k_{OD} < 0.$$

The straight line (A.5) and the curve (A.3) intersect at two points, one of which should be ignored because the value of  $\cos \theta$ , is smaller than  $-1$ . Thus, we have one pair of steady oblique rotations  $(\theta_s, \pm \omega_s, 0)$ .

Let

$$x = \sqrt{-\cos \theta_s} \quad (\text{A.15})$$

and then we can obtain equations (A.10) and (A.11) from equation (A.2). Only the negative sign in equation (A.11) should be taken. Substituting equation (A.11) into equation (A.15) we obtain equation (25).

It follows from (see Fig. 11)

$$\left. \frac{dW}{d \cos \theta} \right|_{\cos \theta = 1} > 0, \quad \left. \frac{dW}{d \cos \theta} \right|_{\cos \theta = -1} < 0$$

and lemma 1 that the states  $(\theta_s, \pm \omega_s, 0)$  are stable with respect to  $(\rho_0, \rho)$  but not asymptotically stable with respect to  $(\rho_0, |\theta - \theta_s|)$ .