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Effects of contact resistance on thermal conductivity of composite media with a periodic structure

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Abstract. The thermal conductivity of periodic composite media with spherical or cylindrical inclusions embedded in a homogeneous matrix is discussed. Using Green functions, we show that the Rayleigh identity can be generalized to deal with thermal properties of these systems. A new calculating method for effective conductivity of composite media is proposed. Useful formulae for effective thermal conductivity are derived, and meanings of contact resistance in engineering problems are explained.

Nomenclature

k_m	conductivity of the matrix
k_i	conductivity of the inclusion
k	ratio of k_i to k_m
h_{im}	contact resistance or film coefficient on bound between two phases
Ω_m	domain occupied by matrix
Ω_i	domain occupied by inclusion
T_m	temperature field in matrix domain
T_i	temperature field in inclusion domain
T	temperature field: it is T_m in domain Ω_m and T_i in domain Ω_i
$\boldsymbol{q}^m$	heat flow vector in matrix domain
$\boldsymbol{q}^i$	heat flow vector in inclusion domain
$\boldsymbol{n}_{im}$	outward unit normal on surface of inclusion
T_0	applied temperature gradient
Bi	Biot number
$Y_{lm}(\theta, \varphi)$	spherical harmonic function
$U_l^m(Q)$	formal factor, defined by equation (32)
a	radius of inclusion sphere or cylinder
f_1, f_2	volume fraction of inclusion for two-dimensional and three-dimensional system
Q_{lm}, V_{lm}	definite integrals, defined in (22) and (23)
S_m, T_m	definite integrals, defined in (28) and (29)

1. Introduction

The transport properties of inhomogeneous media have been of interest since the time of Maxwell [1]. The reason for this interest is, of course, the enormous variety of physical systems, in which inhomogeneities occur: all polycrystalline and composite media, for example, are inhomogeneous systems. Recently, evaluating the property constants of composite media based on first-principle approaches has received much attention [2–5], because many important problems have been raised in the engineering field [2, 6, 7] and some controversies with profound theoretical background are related to it [8–10].

This paper deals with thermal conduction in composite media with a periodic structure. In a composite medium, heat transfer in the matrix and inclusion domain both satisfy Laplace's equation. Boundary conditions between interfaces of different phases are complex. If there is contact resistance the temperature potential suffers a transition at interfaces. The following two typical situations explain the reason for the introduction of contact resistance (or film coefficient).

(i) When two solid surfaces are pressed together they will not form a perfect thermal contact, owing to the air gaps that result from unavoidable roughnesses in the interface. Heat transfer is therefore due to conduction across the actual contact area and to conduction (or natural convection) and radiation across the air gaps [11].

(ii) Film condensation occurs when the condensate wets the solid cooling surface and forms a continuous film [12]. The heat being transferred must pass through this liquid film as it is transferred from the vapour to

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the cooling wall. Because the thermal conductivity of liquids is low, the liquid film, thin as it may be, presents a significant resistance to the flow of the heat. There may be several complicating features associated with film condensation. The character of the condensate film on the solid condensing surface can range from laminar to highly turbulent [12].

The processes in thin films are intricate. Besides conduction, convection, radiation, gravitation and other processes play an important role in heat transfer. It is therefore out of the question to describe heat transfer in the film only by the governing equation of thermal conduction. Fortunately, proper boundary conditions and the associated film coefficient are good for manifesting the effects of the presence of a thin film on thermal conduction in composite media. In fact, it is a common procedure to reflect complex processes in a thin film by proper boundary conditions [13].

The paper is arranged as follows. In section 2, governing equations and boundary conditions for thermal conduction of composite media are specified. In section 3, generalized Rayleigh identities for two-dimensional and three-dimensional composite media are derived. In section 4, a definition of effective thermal conductivities of composite media is established and a new calculating method for the effective constant of composite media is proposed. The method itself is a proof of the validity of Rayleigh technique. In section 5 we derive formulae of thermal conductivity for two dimensional and three-dimensional periodic composite media. In section 6 we apply these formulae to some practical problems.

2. Equations and boundary conditions

Consider a composite medium whose matrix, with conductivity k_m, contains inclusions of conductivity k_i, and suppose that the contact resistance (or film coefficient) on the surface of the inclusion is h_{im}, with $h_{mi} = h_{im}$. The heat flow in matrix and in inclusion have components given by

$$\boldsymbol{q}^{m} = -k_m \nabla T_m \qquad \text{in } \Omega_m \tag{1}$$

and

$$\boldsymbol{q}^{i} = -k_i \nabla T_i \qquad \text{in } \Omega_i \tag{2}$$

respectively.

In the steady state, heat flow satisfies the following equations

$$\nabla \cdot \boldsymbol{q}^{m} = 0 \qquad \text{in } \Omega_m \tag{3}$$

and

$$\nabla \cdot \boldsymbol{q}^{i} = 0 \qquad \text{in } \Omega_i. \tag{4}$$

The condition for continuity of heat flow must be applied on surfaces of inclusion:

$$\boldsymbol{n}_{im} \cdot \boldsymbol{q}^{m} = \boldsymbol{n}_{im} \cdot \boldsymbol{q}^{i} \qquad \text{on } \partial\Omega_i$$

where $\boldsymbol{n}_{im}$ is the outward unit normal vector on the surface of an inclusion. The secondary condition for composite media with contact resistance is

$$-k_i(\partial T_i/\partial n_{im}) = h_{im}(T_i - T_m) \qquad \text{on } \partial\Omega_i. \tag{5}$$

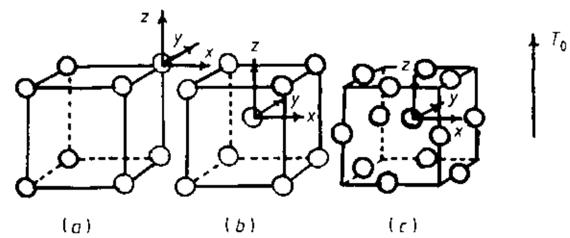

Figure 1. Layout of spherical polar coordinates and cells for different lattices: (*a*) for SC; (*b*) for BCC; and (*c*) for FCC. θ is measured from the z axis and φ is an azimuthal angle measured from the plane of xz.

3. The generalized Rayleigh identity

We first consider an array of cylinders embedded in a homogeneous medium. We apply a homogeneous temperature gradient T_0 along, say, the x axis. The radius of the cylinder is a. The fraction of inclusion is, therefore, $f_1 = \pi a^2$. The origin of cylindrical coordinates is fixed on the axis of a cylinder. It is easy to find general solutions of Laplace's equation in the inclusion and matrix regions:

$$T_i(\rho, \varphi) = C_0 + \sum_{m=1}^{\infty} \rho^m [C_m^1 \sin(m\varphi) + C_m^2 \cos(m\varphi)] \tag{6}$$

and

$$T_m(\rho,\varphi) = A_0 + \sum_{m=1}^{\infty} \{\rho^m[A_m^1 \sin(m\varphi) + A_m^2 \cos(m\varphi)] + \rho^{-m}[B_m^1 \sin(m\varphi) + B_m^2 \cos(m\varphi)]\}. \tag{7}$$

Applying boundary conditions on these solutions, we derive

$$A_0 = C_0$$

$$A_m^i = B_m^i/(H_m a^{2m}) \qquad i = 1,2$$

$$C_m^i = 2B_m^i/(a^{2m}(1 - k + mk/BI)) \qquad i = 1,2$$

where

$$H_m = (1 - k + mk/BI)/(1 + k + mk/BI)$$

$$k = k_i/k_m.$$

BI is the Biot number

$$BI = h_{im}a/k_m.$$

For a composite medium with an array of identical spheres suspended in a homogeneous matrix, we attach the origin of the spherical polar coordinates to the centre of a sphere and apply a homogeneous tem-

Table 1. Coefficients and quantities in equations (35) for SC, BCC and FCC lattices. f_c is the critical volume fraction at which the spheres touch, and a_c is the critical radius corresponding to it.

	SC	BCC	FCC
a_0	11.4666	193.6554	865.634
a_1	182.5208	126.9066	2557.3744
a_2	−12.432	−8.284	−7.526
a_3	−3.4400	−58.0966	−259.6902
a_4	−26.0744	−18.1295	−365.3392
f_2	$(4/3)\pi a^3$	$(8/3)\pi a^3$	$(16/3)\pi a^3$
f_c	$\pi/6$	$(\sqrt{3}/8)\pi$	$(\sqrt{2}/6)\pi$
a_c	1/2	$\sqrt{3}/4$	$\sqrt{2}/4$

perature along the z axis. The radius of the sphere is a. The layout of coordinates for different lattices of spheres is shown in figure 1. Relations between the radius of sphere and fraction of inclusion, f_2, are given in table 1. General solutions in both regions are

$$T_i(r,\theta,\varphi) = D_0 + \sum_{l=1}^{\infty}\sum_{m=-l}^{l} D_{lm} r^l Y_{lm}(\theta,\varphi) \tag{8}$$

and

$$T_m(r,\theta,\varphi) = E_0 + \sum_{l=1}^{\infty}\sum_{m=-l}^{l} (E_{lm} r^l + F_{lm} r^{-l-1}) Y_{lm}(\theta,\varphi). \tag{9}$$

Applying boundary conditions on surface of the central sphere, we derive

$$E_{lm} = F_{lm}/(G_l a^{2l+1}) \tag{10}$$

and

$$D_{lm} = F_{lm}(2l+1)/[l(1-k+lk/Bl)a^{2l+1}] \tag{11}$$

where

$$G_l = (l-k+lk/Bl)/[k+(l+1)/l+(l+1)k/Bl].$$

According to Green function theory, discontinuities of temperature gradient and temperature at surface of inclusion are equivalent to new sources of temperature field with the intensity proportional to

$$Q(\theta,\varphi) = (\partial T_i/\partial r - \partial T_m/\partial r)_{r=a} = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} (F_{lm}/a^{l+2}) \times \frac{(2l+1) - [l(2l+1)k/Bl]}{(1-k+lk/Bl)} Y_{lm}(\theta,\varphi) \tag{12}$$

and

$$D(\theta,\varphi) = (T_i - T_m)_{r=a} = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \frac{[F_{lm}(2l+1)k/Bl]}{(1-k+lk/Bl)a^{l+1}} Y_{lm}(\theta,\varphi) \tag{13}$$

respectively.

The sources of temperature are the applied temperature gradient and the induced sources of temperature field on the surface of each sphere, so the temperature field at an arbitrary point $\boldsymbol{r}$ is given by

$$T(\boldsymbol{r}) = T_0 z + (1/4\pi)\sum_{i=0}^{\infty}\int Q_i(s)\,\mathrm{d}^2 s/|\boldsymbol{r}-\boldsymbol{s}| + (1/4\pi)\sum_{i=0}^{\infty}\int D_i(s) n_\alpha(s)\,\partial_\alpha(1/|\boldsymbol{r}-\boldsymbol{s}|)\,\mathrm{d}^2 s \tag{14}$$

where $\boldsymbol{n}_\alpha(s)$ is unit normal vector on the area element $\mathrm{d}^2\boldsymbol{s}$. Using addition theorem and orthonormality properties of spherical harmonics, we get

$$T(\boldsymbol{r}) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} (F_{lm}/a^{2l+1}) \times \frac{1-[(2l+1)k/Bl]}{1-k+lk/Bl} r^l Y_{lm}(\theta,\varphi) + \sum_{i\neq 0}^{\infty}\sum_{l=1}^{\infty}\sum_{m=-l}^{l} (F_{lm}/\rho_i^{l+1}) Y_{lm}(\theta_i,\varphi_i) + T_0 z. \tag{15}$$

We see that the terms with Bl in double summation due to $D(s)$ just make a compensation for that due to $Q(s)$, so the terms for $i \neq 0$ do not obviously depend on Bl.

In (15) the first sum comes from the central sphere and the second sum is over all other spheres. If we take $|\boldsymbol{r}| < a$, then

$$T(\boldsymbol{r}) = T_i(r) = \sum_{i=1}^{\infty}\sum_{m=-l}^{l} \times \left[E_{lm} + F_{lm}\left(1 - \frac{(2l+1)k/Bl}{(1-k+lk/Bl)a^{2l+1}}\right)\right] \times r^l Y_{lm}(\theta,\varphi). \tag{16}$$

Comparing (15) and (16) we arrive at the desired identity

$$\sum_{l=1}^{\infty}\sum_{m=-l}^{l} E_{lm} r^l Y_{lm}(\theta,\varphi) = \sum_{i\neq 0}^{\infty}\sum_{l=1}^{\infty}\sum_{m=-l}^{l} (F_{lm}/\rho_i^{l+1}) Y_{lm}(\theta_i,\varphi_i) + T_0 z. \tag{17}$$

A similar proof establishes the identity in the region exterior of the central sphere. This identity has the same form as the original one, but the restriction on continuity of temperature field on bound has been removed [14, 15].

With a similar procedure, we prove the generalized Rayleigh identity for a two-dimensional system. It is

$$A_0 + \sum_{m=1}^{\infty} [A_m^1 \sin(m\varphi) + A_m^2 \cos(m\varphi)]\rho^m = \sum_{i=1}^{\infty}\sum_{m=1}^{\infty} (1/\rho_i^m)[B_m^1 \sin(m\varphi_i) + B_m^2 \cos(m\varphi_i)] + T_0 x. \tag{18}$$

4. Effective thermal conductivity of composite media

In this section, we propose a new calculating method for the effective constant of composite media. To illustrate it, we discuss only composite media with cubic symmetry. For more general systems, the procedures are more complex, but have no crucial difficulties [16]. In these systems the thermal conductivity tensor reduces to a scalar. So calculating effective thermal conductivity along axis z is enough for these systems. As usual, we have [17]

$$\langle q_z\rangle = -k^*\langle \partial_z T\rangle. \tag{19}$$

The average value of the heat flow is

$$\begin{aligned}\langle q_z\rangle &= -\int_{\Omega_i} k_i\,\partial_z T_i\,\mathrm{d}\boldsymbol{x} - \int_{\Omega_m} k_m\,\partial_z T_m\,\mathrm{d}\boldsymbol{x}\\ &= -(k_i - k_m)\int_{\Omega_i}\partial_z T_i\,\mathrm{d}\boldsymbol{x} - k_m\langle\partial_z T\rangle\\ &\quad + (k_i k_m/h_{im})\int(\partial T_i/\partial r)\boldsymbol{e}_z\cdot\mathrm{d}\boldsymbol{S}\\ &= -f_2 D_{1,0}(k_i - k_m - k_i/BI) - k_m\langle\partial_z T\rangle.\end{aligned}$$

Combining these two formulae, we obtain

$$k^* = k_m + f_2 D_{1,0}(k_i - k_m - k_i/BI)/\langle\partial_z T\rangle. \tag{20}$$

Instead of applying the Green theorem, as McPhedran and McKenzie [14], or using a sample, as Suen *et al* [17], we calculate $\langle\partial_z T\rangle$ simply by integrating the expansion of T over a cell. After performing the integration, we have

$$k^*/k_m = 1 - 3f_2\left(h_0 + a^3\sum_{l=2}^{\infty}\sum_{m=-l}^{l}\times(Q_{lm}E_{lm} + F_{lm}V_{lm})/F_{1,0}\right)^{-1} \tag{21}$$

where

$$h_0 = [f_2(1 - k - 2k/BI) + 2 + k + 2k/BI]/(1 - k + k/BI)$$

$$Q_{lm} = \int_{\Omega_m}\partial_z(r^l Y_{lm}(\theta,\varphi))\,\mathrm{d}\boldsymbol{x} \tag{22}$$

and

$$V_{lm} = \int_{\Omega_m}\partial_z(r^{-l-1}Y_{lm}(\theta,\varphi))\,\mathrm{d}\boldsymbol{x}. \tag{23}$$

Owing to symmetry of the systems, we prove that the only non-zero Q_{lm} and V_{lm} are those with odd l and m divisible by 4.

We give values of Q_{lm} and V_{lm} for $l < 7$ and $m = 0$:

$$Q_{1,0} = 1 - f_2,\qquad Q_{3,0} = 0,\qquad Q_{5,0} = -7/96,\qquad Q_{7,0} = 1/96 \tag{24}$$

$$V_{1,0} = 0,\qquad V_{3,0} = 4.105\,54,\qquad V_{5,0} = 0.341\,675,\qquad V_{7,0} = -2.954\,19. \tag{25}$$

At the lowest approximation, that is, omitting the last term in the denominator of (21), we reproduce the Lorentz–Lorenz equation or Maxwell–Garnett equation. We see that it is not the coefficients, D_{lm}, E_{lm}, F_{lm}, but the ratios of them to $F_{1,0}$ entering into the formula for effective thermal conductivity, the problem about the convergence of $U_2^0(Q)$ is, therefore, solved naturally [14, 17].

For two-dimensional periodic composite media with square symmetry, we have following formula

$$k^*/k_m = 1 - 2f_1\left(j_0 + a^2\sum_{m=2}^{\infty}(A_m^2 S_m + B_m^2 T_m)\right)^{-1} \tag{26}$$

where

$$j_0 = [(1 - k - V_1)f_1 + 1 + k + k/BI]/(1 - k + k/BI).$$

S_m and T_m are definite integrals. They can be performed analytically

$$S_1 = \int_{\Omega_m}\partial_x(\rho\cos\varphi)\,\mathrm{d}\boldsymbol{x} = 1 - f_1 \tag{27}$$

$$T_1 = \int_{\Omega_m}\partial_x(\rho^{-1}\cos\varphi)\,\mathrm{d}\boldsymbol{x} = 0 \tag{28}$$

$$\begin{aligned}S_m &= \int_{\Omega_m}\partial_x(\rho^m\cos(m\varphi))\,\mathrm{d}\boldsymbol{x}\\ &= 2^{(3-m)/2}\sin[(1+m)\pi/4]\\ &\quad - 2^{-m}\{\sin[(m-1)\pi/2]/(m-1)\\ &\quad + \sin[(m+1)\pi/2]/(m+1)\}\end{aligned} \tag{29}$$

$$\begin{aligned}T_m &= \int_{\Omega_m}\partial_x(\rho^{-m}\cos(m\varphi))\,\mathrm{d}\boldsymbol{x}\\ &= 2^{(3+m)/2}\sin[(1-m)\pi/4]\\ &\quad - 2^{m}\{\sin[(m-1)\pi/2]/(m-1)\\ &\quad + \sin[(m+1)\pi/2]/(m+1)\}.\end{aligned} \tag{30}$$

5. Formulae for effective thermal conductivities

We consider three-dimensional composite media. In order to determine unknown coefficients $F_{2l-1,m}$, we apply the Rayleigh identity at two points within the unit cell, namely $Q = (r_0, \theta_0, \varphi_0)$ and $Q = \mathbf{O} = (0, 0, 0)$, and equate odd-order derivatives with respect to z of both sides of (17). The procedure yields a

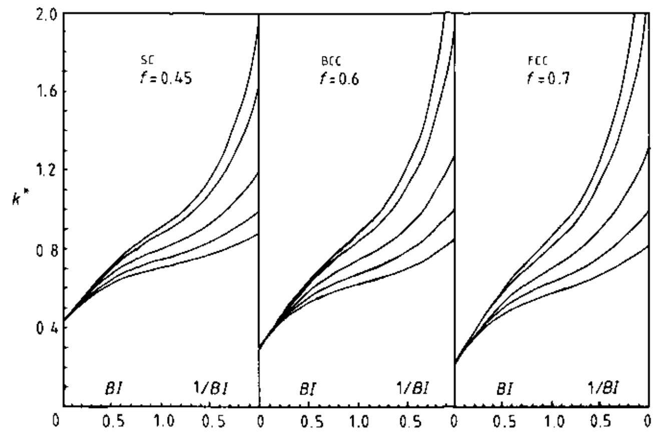

Figure 2. Dependence of effective thermal conductivity of composite media on Biot number. The curves along the positive orientation of ordinate correspond, successively, to $k = 0.75$, 1.0, 1.5, 3.0 and 4.5.

set of linear equations for $F_{2l-1,m}$. A typical equation, obtained from the $(2n+1)$th partial derivative is

$$\sum_{l=n+1}^{\infty}\sum_{m=0}^{2l-2n-2} A_{2n+1}^{m+2l-1}E_{2l-1,m}r_0^{2l-2n-2} \times P_{2l-2n-2}^{m}(\cos\theta_0)\cos(m\varphi_0) + \sum_{l=1}^{\infty}\sum_{m=0}^{L}\sum_{i=1}^{\infty} A_{2n+1}^{2l+2n-m}F_{2l-1,m} \times \rho_i^{-2l-2n-1}P_{2l+2n}^{m}(\cos\theta_i)\cos(m\varphi_i) = T_0\delta_{n,0} \qquad (31)$$

where

$$L = 2l - 1$$

and

$$A_k^n = n/(n-k).$$

In the sums over i we run over the lattice points (u, v, w) for all positive or negative integers (u, v, w) excluding $(0, 0, 0)$. Define

$$U_l^m(Q) = \sum_{i=1}^{\infty} \rho_i^{-l-1}P_l^m(\cos\theta_i)\cos(m\varphi_i) \qquad (32)$$

where each $U_l^m(Q)$ depends on the coordinates of Q, since

$$\rho_i = [(x_0-u)^2 + (y_0-v)^2 + (z_0-w)^2]^{1/2}$$
$$\cos\theta_i = (x_0-u)/\rho_i$$

and

$$\cos\varphi_i = (\mathrm{y}_0 - \mathrm{u})/(\mathrm{z}_0 - \mathrm{w}).$$

By using (31), (32) becomes

$$\sum_{l=n+1}^{\infty}\sum_{m=0}^{2l-2n-2} A_{2n+1}^{m+2l-1}F_{2l-1,m}r_0^{2l-2n-2}P_{2l-2n-2}^{m}(\cos\theta_0) \times \cos(m\varphi_0)/(G_{2l-1}a^{4l-1}) + \sum_{l=1}^{\infty}\sum_{m=0}^{L} A_{2n+1}^{2l+2n-m}F_{2l-1,m}U_{2l+2n}^{m}(Q) = T_0\delta_{n,0}. \qquad (33)$$

For electrical conduction, when the conductivity of the inclusions is infinite and they are nearly touching, conductivity of the composite medium will develop a singularity. Because of the presence of contact resistance, this singularity will not occur in the case of thermal conduction of composite media. We expect, therefore, that the convergence of the numerical method will be much better. Numerical calculations make us believe that the convergence of the numerical method is rather rapid and azimuthal terms have little influence on thermal conductivity of composite media.

Owing to rapid convergence of the numerical method, we can derive neat formulae for effective thermal conductivity of composite media. The solution of equation (33) to order four, without azimuthal terms, yields the following formula for effective thermal conductivity of composite media

$$k^*/k_m = 1 - 3f_2/(h_0 + a^3h_1) \qquad (34)$$

Table 2. Effective thermal conductivities of an array of copper pipes immersed in steam. The conductivity of copper pipes and conductivity of steam are 342 W m^{-1} K^{-1} and 0.6845 W m^{-1} K^{-1} respectively. The film coefficient is 11 400 W m^{-2} K^{-1}. Figures in the last row are effective conductivities of the system, if we neglect the effect of the thin film.

Length of cell side (m)	Radius of cylinder					
	0.18	0.24	0.30	0.36	0.42	0.48
0.1	0.8378	0.9838	1.2196	1.6262	2.4749	5.9816
0.05	0.8367	0.9820	1.2166	1.6207	2.4617	5.8832
0.01	0.8279	0.9678	1.1934	1.5787	2.3621	5.2189
$h_{im} = \infty$	0.8390	0.9856	1.2226	1.6317	2.4882	6.0841

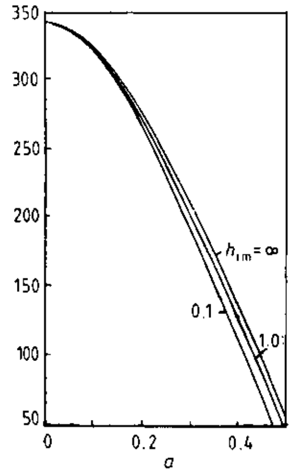

Figure 3. Effective thermal conductivities of cylindrical lead castings in a copper mould. Conductivity of the copper mould and conductivity of the lead castings are 342 W m^{-1} K^{-1} and 33.4 W m^{-1} K^{-1}, respectively. Film coefficient is 150 W m^{-2} K^{-1}.The top curve is effective conductivity of a similar system, but having an ideal thermal contact. Figures near other curves are lengths of the cell side.

where

$$
\begin{aligned}
h_1 &= V_{3,0}h_3/h_2 + (h_4/h_2)[Q_{5,0}/(G_5a^{11}) + V_{5,0}] \\
&\quad + h_5[Q_{7,0}/(G_7a^{15}) + V_{7,0}] \\
h_2 &= (G_3G_5a^{18})^{-1} + a_0/(G_5a^{11}) - a_1^2 \\
h_3 &= a_3/(G_5a^{11}) - a_1a_3 \\
h_4 &= a_3/(G_3a^7) + a_0a_3 - a_1a_2 \\
h_5 &= a_4G_7a^{15}.
\end{aligned} \tag{35}
$$

The coefficients a_0, a_1, a_2, a_3 and a_4, for different lattices are listed in table 1.

For two-dimensional periodic composite media, we derive the following formula

$$k^*/k_m = 1 - 2f/(j_0 + a^2j_1) \tag{36}$$

where

$$
\begin{aligned}
j_1 &= B_3^2T_3 + A_5^2S_5 + B_7^2T_7 \\
B_1^2 &= T_0/(j_2 - j_3j_4/j_5) \\
B_3^2 &= -H_3a^6(a_5B_1^2 + 35a_6B_5^2) \\
B_5^2 &= B_1^2j_4/j_5 \\
B_7^2 &= -H_7a^{14}(a_6B_1^2 + 330a_7B_5^2) \\
j_2 &= (H_1a^2)^{-1} - 3a_5^2H_3a^6 - 7a_6^2H_7a^{14} \\
j_3 &= 105a_5a_6H_3a^6 + 2310a_6a_7H_7a^{14} \\
j_4 &= 21a_5a_6H_3a^6 + 462a_6a_7H_7a^{14} \\
j_5 &= (H_5a^{10})^{-1} - 735a_6^2H_3a^6 - 152460a_7^2H_7a^{14} \\
a_5 &= 3.15085, \quad a_6 = 4.25577, \quad a_7 = 3.93885.
\end{aligned}
$$

7. Conclusion

Using formulae (34) and (36), we study influences of contact resistance on thermal conduction of composite media. The dependences of effective thermal conductivities of three-dimensional composite media on the Biot number are depicted in figure 2.

We use formula (34) to discuss two engineering problems.

(i) Cylindrical castings condensing in a mould: when a crust is formed at the periphery of a casting, either the casting touches the mould on an extremely rough interface, or an air gap lies between the casting and the mould. In both cases, there is a contact resistance on the bound of the castings, and it retards the condensation of castings. Figure 3 shows effective thermal conductivities of lead castings in a copper mould.

(ii) Metal pipes immersed in steam: the influence of film coefficient on thermal conduction of the system is much more subtle, and we present the effective conductivities of it in table 2.

In [5] we showed that the existence of contact resistance on surfaces between different phases of composite media changes the thermal properties dramatically. The above examples show that for some thermal conduction systems we need to take into account the effects of contact resistance. However, boundary conditions for thermal conduction complicate the problem and are an obstacle to theoretical treatment. It is, therefore, reasonable to consider first the thermal conductivity of composite media with spherical or cylindrical inclusions. Besides, the new calculating method proposed in section 4 has no problems regarding the convergence of $U_2^0(Q)$, and it is valid for complex composite media, for example, the frequency-dependent conductivity of colloid systems [13].

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