Vol. 34 No. 8 SCIENCE IN CHINA (Series A) August 1991

SOME THEORETICAL ASPECTS OF THE SIMPLIFIED NAVIER-STOKES(SNS) EQUATIONS

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Received January 3, 1990.

ABSTRACT

This study deals with the formulation, mathematical property and physical meaning of the simplified Navier-Stokes (SNS) equations. The tensorial SNS equations proposed is the simplest in form and is applicable to flow fields with arbitrary body boundaries. The zones of influence and dependence of the SNS equations, which are of primary importance to numerical solutions, are expounded for the first time from the viewpoint of subcharacteristics. Besides, a detailed analysis of the diffusion process in flow fields shows that the diffusion effect has an influence zone globally windward and an upwind propagation greatly depressed by convection. The maximum upwind influential distance of the viscous effect and the relative importance of the viscous effect in the flow direction to that in the direction normal to the flow are represented by the Reynolds number, which illustrates the conversion of the complete Navier-Stokes (NS) equations to the SNS equations for flows with large Reynolds number.

Keywords: fluid mechanics, viscous flow, Navier-Stokes equation.

The simplified Navier-Stokes (SNS) equations or the so-called parabolized Navier-Stokes (PNS) equations, owing to their advantage of saving computer time and storage, have found wide applications in numerical simulations of flows with large Reynolds number.[1,2] This paper is an attempt to give the general form of the SNS equations, and to make an investigation on its mathematical and physical background so as to further develop this kind of applications.

I. FORMULATION OF THE SNS EQUATIONS

For flow fields with straight body boundary shapes, one of the coordinate axes of the Cartesian coordinate system could be set to be parallel to the boundary so as to represent the main stream direction. If the flow is of large Reynolds number, a hierarchial relation among the viscous terms in the NS equations could be obtained by an estimation of their orders of magnitude[3]. We get the Cartesian SNS equations by deleting all the minor diffusion terms in the NS equations. This procedure can be extended to the cases of curved body boundaries, if the local radius of curvature of the boundary is much larger than the thickness of the local viscous layer.

If the velocity space shares the same bases with the geometrical space that are boundary-fitted and are orthogonal at least in the regions near the boundary, the magnitude of the velocity variation in viscous zones is corresponding to the magnitude of the size of the region where the velocity variation takes place. In this case the orders of magnitude of the viscous terms contained in the NS equations can readily be estimated. So, the NS equations must be of the tensorial type. In estimating the orders of magnitude it is proper to set the equations in the following form,[4]

continuity equation

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^l}(\sqrt{g}\,\rho W^l)=0, \tag{1.1}$$

momentum equation

$$\rho\left(W^l\frac{\partial W^i}{\partial x^l}+W^lW^m\Gamma^i_{ml}\right)=-\frac{\partial P}{\partial x^l}g^{li}+f^i_v, \tag{1.2}$$

energy equation

$$\rho C_pW^l\frac{\partial T}{\partial x^l}=W^l\frac{\partial P}{\partial x^l}+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^m}\left(\sqrt{g}\,g^{ml}\lambda\frac{\partial T}{\partial x^l}\right)+\phi, \tag{1.3}$$

where the viscous force term is

$$\begin{aligned}f^i_v=&\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^l}\left\{\sqrt{g}\,\mu\left[g^{ml}\frac{\partial W^i}{\partial x^m}+g^{mi}\frac{\partial W^l}{\partial x^m}+(\Gamma^i_{nm}g^{ml}+\Gamma^l_{nm}g^{mi})W^n\right.\right.\\&\left.\left.-\frac{2}{3}g^{il}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^m}(\sqrt{g}\,W^m)\right]\right\}+\Gamma^i_{pq}\mu\left[g^{lq}\frac{\partial W^p}{\partial x^l}+g^{lp}\frac{\partial W^q}{\partial x^l}\right.\\&\left.+(\Gamma^p_{ml}g^{lq}+\Gamma^q_{ml}g^{lp})W^m-\frac{2}{3}g^{pq}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^l}(\sqrt{g}\,W^l)\right],\end{aligned} \tag{1.4}$$

and the dissipation term is

$$\begin{aligned}\phi=&\frac{1}{2}\mu\left[g_{il}\frac{\partial W^l}{\partial x^j}+g_{jl}\frac{\partial W^l}{\partial x^i}+\frac{\partial g_{ij}}{\partial x^m}W^m\right]\\&\cdot\left[g^{lj}\frac{\partial W^i}{\partial x^l}+g^{li}\frac{\partial W^j}{\partial x^l}+(\Gamma^i_{ml}g^{lj}+\Gamma^j_{ml}g^{li})W^m\right]\\&-\frac{2}{3}\mu\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^l}(\sqrt{g}\,W^l)\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^m}(\sqrt{g}\,W^m).\end{aligned} \tag{1.5}$$

Here $\rho, P, T, \mu, \lambda, C_p$ stand for the density, pressure, temperature, viscosity, heat conductivity and specific heat of the fluid, respectively, W is the velocity, and g, Γ the metric tensor and the Christoffel symbol of the coordinate system. Suppose that the variation of coordinate function is milder than that of the velocity field, i.e. the variation of coordinate functions can be taken as a global effect. It follows that the order of magnitude is not altered due to the occurrence of the metric tensor and the Christoffel symbol, and the tensorial SNS equations are found by estimating the orders of magnitude and deleting all the minor terms in the Cartesian system.

For 2-dimensional flows with coordinate axis x^1 boundary-fitted, we have SNS equations:

$$\rho \frac{DW^1}{Dt} = -\left(g^{11}\frac{\partial P}{\partial x^1} + g^{12}\frac{\partial P}{\partial x^2}\right) + g^{22}\frac{\partial}{\partial x^2}\left(\mu \frac{\partial W^1}{\partial x^2}\right), \tag{1.6}$$

$$\rho \frac{DW^2}{Dt} = -\left(g^{21}\frac{\partial P}{\partial x^1} + g^{22}\frac{\partial P}{\partial x^2}\right), \tag{1.7}$$

$$\rho C_p \frac{dT}{dt} = \frac{dP}{dt} + g^{22}\frac{\partial}{\partial x^2}\left(\lambda \frac{\partial T}{\partial x^2}\right) + \mu g_{11} g^{22}\frac{\partial W^1}{\partial x^2}\frac{\partial W^1}{\partial x^2}, \tag{1.8}$$

where

$$\frac{DW^i}{Dt} = W^1\frac{\partial W^i}{\partial x^1} + W^2\frac{\partial W^i}{\partial x^2} + \Gamma^i_{11}W^1W^1 + 2\Gamma^i_{12}W^1W^2 + \Gamma^i_{22}W^2W^2,$$

$$\frac{d}{dt} = W^1\frac{\partial}{\partial x^1} + W^2\frac{\partial}{\partial x^2}.$$

For 3-dimensional flows with coordinate surface x^1x^2 ($x^3 = \text{const.}$) boundary-fitted, we have SNS equations

$$\rho \frac{DW^1}{Dt} = -\left(g^{11}\frac{\partial P}{\partial x^1} + g^{12}\frac{\partial P}{\partial x^2} + g^{13}\frac{\partial P}{\partial x^3}\right) + g^{33}\frac{\partial}{\partial x^3}\left(\mu \frac{\partial W^1}{\partial x^3}\right), \tag{1.9}$$

$$\rho \frac{DW^2}{Dt} = -\left(g^{21}\frac{\partial P}{\partial x^1} + g^{22}\frac{\partial P}{\partial x^2} + g^{23}\frac{\partial P}{\partial x^3}\right) + g^{33}\frac{\partial}{\partial x^3}\left(\mu \frac{\partial W^2}{\partial x^3}\right), \tag{1.10}$$

$$\rho \frac{DW^3}{Dt} = -\left(g^{31}\frac{\partial P}{\partial x^1} + g^{32}\frac{\partial P}{\partial x^2} + g^{33}\frac{\partial P}{\partial x^3}\right), \tag{1.11}$$

$$\rho C_p \frac{dT}{dt} = \frac{dP}{dt} + g^{33}\frac{\partial}{\partial x^3}\left(\lambda \frac{\partial T}{\partial x^3}\right) + g_{11}g^{33}\mu \frac{\partial W^1}{\partial x^3}\frac{\partial W^1}{\partial x^3} + g_{22}g^{33}\mu \frac{\partial W^2}{\partial x^3}\frac{\partial W^2}{\partial x^3}. \tag{1.12}$$

If the boundary has abrupt turns, e. g. the boundary of flow fields in turbomachinery rotors, it is desirable to have two coordinate surfaces x^1x^2($x^3 = \text{const.}$) and x^1x^3($x^2 = \text{const.}$) boundary-fitted. Then the corresponding SNS equations are

$$\rho \frac{DW^1}{Dt} = -\left(g^{11}\frac{\partial P}{\partial x^1} + g^{12}\frac{\partial P}{\partial x^2} + g^{13}\frac{\partial P}{\partial x^3}\right) + g^{22}\frac{\partial}{\partial x^2}\left(\mu \frac{\partial W^1}{\partial x^2}\right) + g^{33}\frac{\partial}{\partial x^3}\left(\mu \frac{\partial W^1}{\partial x^3}\right), \tag{1.13}$$

$$\rho \frac{DW^2}{Dt} = -\left(g^{21}\frac{\partial P}{\partial x^1} + g^{22}\frac{\partial P}{\partial x^2} + g^{23}\frac{\partial P}{\partial x^3}\right) + g^{33}\frac{\partial}{\partial x^3}\left(\mu \frac{\partial W^2}{\partial x^3}\right), \tag{1.14}$$

$$\rho \frac{DW^3}{Dt} = -\left(g^{31}\frac{\partial P}{\partial x^1} + g^{32}\frac{\partial P}{\partial x^2} + g^{33}\frac{\partial P}{\partial x^3}\right) + g^{22}\frac{\partial}{\partial x^2}\left(\mu \frac{\partial W^3}{\partial x^2}\right), \tag{1.15}$$

$$\rho C_p \frac{dT}{dt} = \frac{dP}{dt} + g^{22}\frac{\partial}{\partial x^2}\left(\lambda \frac{\partial T}{\partial x^2}\right) + g^{33}\frac{\partial}{\partial x^3}\left(\lambda \frac{\partial T}{\partial x^3}\right) + g_{11}g^{33}\mu \frac{\partial W^1}{\partial x^3}\frac{\partial W^1}{\partial x^3} + g_{22}g^{33}\mu \frac{\partial W^2}{\partial x^3}\frac{\partial W^2}{\partial x^3} + g_{11}g^{22}\mu \frac{\partial W^1}{\partial x^2}\frac{\partial W^1}{\partial x^2} + g_{33}g^{22}\mu \frac{\partial W^3}{\partial x^2}\frac{\partial W^3}{\partial x^2}. \tag{1.16}$$

In the above 3-dimensional equations

$$\frac{DW^i}{Dt} = W^1 \frac{\partial W^i}{\partial x^1} + W^2 \frac{\partial W^i}{\partial x^2} + W^3 \frac{\partial W^i}{\partial x^3} + \Gamma^i_{11} W^1 W^1 + \Gamma^i_{22} W^2 W^2$$
$$+ \Gamma^i_{33} W^3 W^3 + 2\Gamma^i_{12} W^1 W^2 + 2\Gamma^i_{23} W^2 W^3 + 2\Gamma^i_{31} W^3 W^1,$$
$$\frac{d}{dt} = W^1 \frac{\partial}{\partial x^1} + W^2 \frac{\partial}{\partial x^2} + W^3 \frac{\partial}{\partial x^3}.$$

Such SNS equations are the simplest in form in view of the fact that the terms of viscous force, viscous dissipation and heat conduction retained in the SNS equations correspond to those in the boundary-layer equations. For 3-dimensional flows, there are several hundred viscous terms in the complete NS equations (both tensorial form or general coordinate form) and conventional SNS equations,[1] but only a few left in the tensorial SNS equations.

Adaptability of the SNS equations follows the representative form. The SNS equations consist of the terms in the Euler equations and those in the boundary-layer equations, and degrade into the Euler equations for the inviscid main flow region and into the boundary-layer equations for the viscous boundary-layer region. That is why the SNS equations are suitable for both the inviscid flow region and the viscous boundary-layer region, and can reflect with adequate accuracy the interaction between the two regions.

II. Zones of Influence and Dependence of the SNS Equations

The concept of zones of influence and dependence of partial differential equations is essential for obtaining numerical solutions and understanding intricate questions. In the research of the mathematical property of the SNS equations to date, only the characteristic roots have been considered, but how to determine the zones of influence and dependence of the SNS equations remains an open problem.

In the classical theory of mathematical physics,[5] the zones of influence and dependence of second-order partial differential equations with two independent variables are well known. Apart from some special cases, the extension of this concept to equations of higher order and with more than two independent variables has been limited to two limiting types of problems, the totally hyperbolic and totally elliptic. As for more general intermediate cases, no method has ever been found in pure mathematics to completely determine the zones of influence and dependence. Thus we resort to the viewpoint of subcharacteristics, as with the case of 3-dimensional boundary-layer equations[6].

For simplicity, we take as an example the Cartesian SNS equations for 3-dimensional steady flows of perfect gas. Supposing that the body boundary is composed of straight parts on which either coordinate y or z is constant, and the flow field has the x-axis fitted with its main stream direction, we have the following SNS equations:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0, \tag{2.1}$$

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu\frac{u}{\partial z}\right), \tag{2.2}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial P}{\partial y}, \tag{2.3}$$

$$\rho\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial P}{\partial z}, \tag{2.4}$$

$$\rho C_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=\left(u\frac{\partial P}{\partial x}+v\frac{\partial P}{\partial y}+w\frac{\partial P}{\partial z}\right)$$
$$+\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)+\mu\left[\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2\right]. \tag{2.5}$$

1. *Characteristics*

In the study of the characteristics, only the highest derivative terms in each question of the system are of concernment. Let Ω denote the characteristic surface, then the corresponding characteristic equation can be given as

$$\left[\left(\frac{\partial \Omega}{\partial y}\right)^2+\left(\frac{\partial \Omega}{\partial z}\right)^2\right]^2\left(u\frac{\partial \Omega}{\partial x}+v\frac{\partial \Omega}{\partial y}+w\frac{\partial \Omega}{\partial z}\right)$$
$$\cdot\left\{\left(u\frac{\partial \Omega}{\partial x}+v\frac{\partial \Omega}{\partial y}+w\frac{\partial \Omega}{\partial z}\right)^2-a^2\left[\left(\frac{\partial \Omega}{\partial y}\right)^2+\left(\frac{\partial \Omega}{\partial z}\right)^2\right]\right\}=0, \tag{2.6}$$

where a is the sound velocity.

The first factor of the above equation implies that all surfaces normal to the body boundary are characteristic surfaces. It also indicates that the speed of disturbances is infinite on the yz surfaces ($x=$ const.). The second power of the square bracket results from the viscous diffusion effect on the transfer of the momentum component in the main stream direction and the thermal diffusion effect on the energy transfer, which are all on the surfaces normal to the boundary according to Eqs. (2.2) and (2.5).

The second factor originates from the influence of the convection over the continuity, which implies that the streamlines as characteristics carry the information of mass disturbance.

The third factor, which corresponds to the propagation of pressure disturbance, represents the quadratic sheet of the characteristic normal cone. It can be well illustrated by a comparison with the characteristic equation for 2-dimensional unsteady flows of perfect gas.[5] The direction of the rays given by the ratio $d\omega:dy:dz$ ($\omega = x/u$) represents the "propagation velocities" or the ray velocities for the disturbance. The rays with parameter ω satisfy the Monge equation

$$\left(\frac{dy}{d\omega}-v\right)^2+\left(\frac{dz}{d\omega}-w\right)^2=a^2. \tag{2.7}$$

As shown in Fig. 1, for given v and w, the local ray cone or the Monge cone of the characteristic differential equation in the (x,y,z) space, whose vertex we assume to

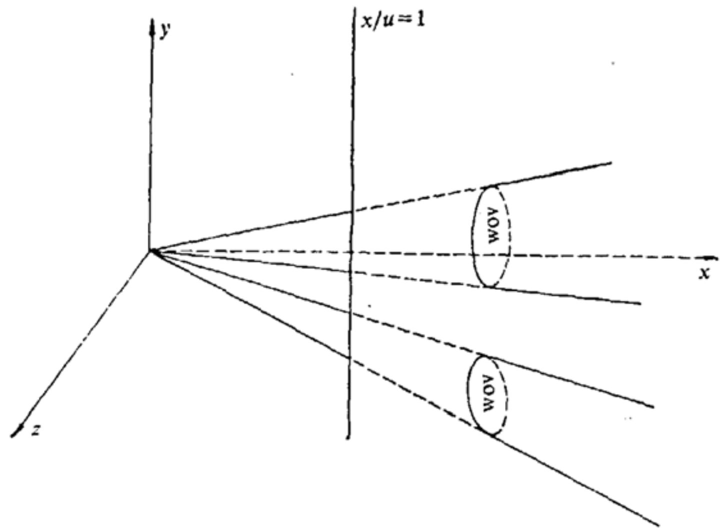

Fig. 1. Windward zone of influence of pressure disturbance, according to characteristics of SNS equations.

be at the origin $x = y = z = 0$, is represented by

$$\left(\frac{y}{x} - \frac{v}{u}\right)^2 + \left(\frac{z}{x} - \frac{w}{u}\right)^2 = \left(\frac{a}{u}\right)^2. \tag{2.8}$$

The local ray cone is therefore obtained by projection of the circle

$$(y - v)^2 + (z - w)^2 = a^2 \tag{2.9}$$

in the plane $\omega = 1$ from the origin. Whether or not this circle encloses the origin $y = z = 0$, i.e. whether or not $v^2 + w^2 < a^2$, the circle cone in question either contains the x-axis or slants so much that the x-axis goes outside of it. This kind of special conic zone of influence is a result of the unsymmetrical form of the SNS equations. The momentum transfers in the y and z directions are inviscid, which cause a Mach propagation with the convective propagation along the streamlines, a global windward zone of influence can always be generated, no matter whether the flow is supersonic or subsonic. This case is quite different from those of the infinite influence zone of the NS equations or the Mach influence zone of the Euler equations.

2. *Subcharacteristics*

The subcharacteristics of the SNS equations are obtained by neglecting all the viscous terms in Eqs. (2.2) and (2.5). The subcharacteristics so obtained are just the characteristics of the corresponding inviscid equations, namely, the Euler equations. For the Euler equations, characteristics and corresponding zones of influence and dependence are well known.[9]

3. *Zones of Influence and Dependence*

The streamlines as subcharacteristics carry disturbances of mass, momentum and energy with the flow, that is to say, the disturbances are merely convected with finite local velocity along the streamlines. In the case of the SNS equations, the disturbances of energy and the momentum component in the main stream direction are also transferred instantly in the y and z directions. This is because the signal velocity of both the viscous diffusion and the thermal diffusion are infinite. Due to the vast difference in speed, a disturbance transferred through diffusion always overtakes that moved by convection. As a consequence, for the transfers of energy and momentum component in the main stream direction, a disturbance at any point influences the entire surface where the point is located normal to the body boundary, the entire flow region facing downward stands for the zone of influence, and the region upwind for the zone of dependence. For the transfers of momentum components in other directions, the streamlines are the zones of influence and dependence, which are determined by the subcharacteristics. Both the characteristics and the subcharacteristics of the pressure disturbance can be effective. Because the influence zone by the characteristics is always windward, whether the overall influence zone is windward depends on the subcharacteristics. For supersonic flow, the overall influence zone of pressure disturbance is windward, whereas for subsonic flows, the overall zone covers the entire flow field.

In contrast, the subcharacteristics do not play any role in determining the main flow structure of the NS equations,[6] the entire flow region acts as the zone of influence and dependence for all dependent variables.

III. Analysis of Diffusion Effects in Flow Fields and the Conversion of the NS Equations to the SNS Equations

The differences between the SNS and NS equations stem from different descriptions of the diffusion effects in flow fields. The conversion of the NS equation to the SNS equation can be illustrated naturally by means of a detailed study of the diffusion process occurring in flow fields. For simplicity, we start our study from the heat transfer phenomenon.

1. *Finite Diffusion Velocity——Heat Diffusion in a Steady Medium*

The Fourier heat conduction equation in parabolic form requires an infinite velocity of propagation, which implies that the effect of a thermal disturbance is instantaneously felt at any distance infinitely far from the disturbance. Even if the effect at infinity is small, the effect may be important in some cases, e.g. the small elapsed time during transient behavior. Therefore, the classical parabolic formulation has been replaced by the following damped wave equations.[7]

$$\frac{1}{C^2}\frac{\partial^2 T}{\partial t^2}+\frac{1}{\alpha}\frac{\partial T}{\partial t}=\nabla^2 T, \tag{3.1}$$

$$\frac{\alpha}{C^2}\frac{\partial q}{\partial t}+q=-\lambda\nabla T. \tag{3.2}$$

where λ is the heat conductivity, α the thermal diffusivity, q the heat flux density, $C=(\alpha/t_0)^{\frac{1}{2}}$ is called the velocity of propagation of heat wave or the second sound velocity, and t_0 the relaxation time.

Now we consider a simple process of heat diffusion in a steady medium. Suppose that a point heat source with invariable temperature T_c is suddenly released at the origin of a coordinate system fixed in an adequately large medium with uniform temperature T_0. The change in temperature governed by the wave equation (3.1) is found to be

$$T(r,t)=T_0+\frac{1}{4}(T_c-T_0)S(Ct-r)\cdot\left\{\exp\left(-\frac{Cr}{2\alpha}\right)\right.$$
$$\left.+\int_{r/c}^{t}\frac{Cr}{2\alpha}\exp\left(-\frac{C^2\tau}{2\alpha}\right)\cdot\frac{I_1\left[\frac{C^2}{2\alpha}\sqrt{\tau^2-\left(\frac{r}{C}\right)^2}\right]}{\sqrt{\tau^2-\left(\frac{r}{C}\right)^2}}d\tau\right\}, \tag{3.3}$$

where I_1 is the Bessel function of first-order, and $S(Ct-r)$ the unit step function, which equals 1 if $Ct>r$ or 0 if $Ct\leqslant r$. Eq. (3.3) states that the heat flux propagates with a constant velocity C rather than an infinite velocity. The corresponding heat release rate at the point heat source is

$$q(0,t)=\frac{1}{4}(T_c-T_0)\frac{C\lambda}{\alpha}\left\{\exp\left(\frac{C^2t}{2\alpha}\right)\left[I_0\left(\frac{C^2t}{2\alpha}\right)-1\right]\right\}, \tag{3.4}$$

where I_0 is the Bessel function of zeroth-order. When the second sound velocity is infinitely large, we have

$$\lim_{C\to\infty}q(0,t)=\lambda(T_c-T_0)/4\sqrt{\pi\alpha t}. \tag{3.5}$$

Under the same conditions, the classical Fourier equation

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}=\nabla^2T \tag{3.6}$$

has solutions

$$T(r,t)=T_0+\frac{1}{4}(T_c-T_0)\cdot\operatorname{erf}\left[\frac{r}{2\sqrt{\alpha t}}\right]$$
$$=T_0+\frac{T_c-T_0}{2\sqrt{\pi}}\int_{\frac{r}{2\sqrt{\alpha t}}}^{\infty}\exp(-\eta^2)d\eta, \tag{3.7}$$

$$q(r,t)=\frac{\lambda(T_c-T_0)}{4\sqrt{\pi\alpha t}}\exp(-r^2/4\alpha t). \tag{3.8}$$

It can be concluded from the above solutions that (i) the temperature discontinuity across the face of the thermal wave is given by

$$\frac{T(Ct,t)-T_0}{T_c-T_0}=\frac{1}{4}\exp(-C^2t/2\alpha), \tag{3.9}$$

and the maximum heat flux in response to a step change in temperature is also finite; (ii) if the propagation velocity of heat tends to be infinite, the solution of the wave equation reduces to the solution of the parabolic equation.

The finiteness of the velocity of heat diffusion has been illustrated in the above argument. In fact, this finiteness can also be obtained from the conventional parabolic equation, for the solution of the Fourier equation involves an exponential function $\exp(-r^2/4\alpha t)$, which, as a dominant term, results in a damping structure of the temperature field. Besides, the solution of the hyperbolic equation has a damping behavior in nature although it is wavelike in form, for the temperature discontinuity becomes very small as the time becomes moderately large. It is well known that the determinacy of physical events has its intrinsic limit, the corresponding mathematical description is meaningful only within this limit. For heat conduction, it is reasonable to set a certain limit, within which the variation of temperature is considered as negligible and no heat flux exists, and the region where temperature variation is beyond this limit is defined as effective region of heat diffusion. According to the wave equation, the magnitude of the region the propagation has reached is estimated as

$$r_p \sim Ct. \tag{3.10}$$

In contrast, the magnitude of the effective region of heat diffusion according to the parabolic equation is

$$r_e \sim \sqrt{\alpha t}. \tag{3.11}$$

Since C^2 is usually much greater than α, the region of heat propagation is uaually wider than the effective region of heat diffusion. In view of this fact, the parabolic Fourier equation can be considered as more valid in expressing the finiteness of the velocity of heat diffusion than the hyperbolic equations.

2. *Finite Diffusion Region——the Diffusion Process in Flow Fields*

Let us consider a uniform flow field at uniform initial temperature. At zero time a point heat source with invariable temperature and moving along the flow is imposed upon the origin of the coordinate system in the flow field. From the standpoint of an observer standing at and moving with the heat source, the expression for the observed temperature variation process is just the same as that given in the previous analysis for steady media. If a steady coordinate system is employed, the temperature expression does not alter in form except that the distance parameter r is transformed into

$$r = [(x - ut)^2 + y^2 + z^2]^{\frac{1}{2}},$$

where r represents the distance between the point source and the point (x,y,z), u is the flow velocity, and the x-axis is oriented to the flow. For any flow field, it can be recognized from the analysis of the hyperbolic equation of heat transfer that the propagation of heat wave has behavior similar to that of the sound wave. As shown in Fig. 2, the heat conduction would influence the entire flow region in the cases where the flow velocity u is slower than the second sound velocity C, or influence a conic zone facing windward. As C is called the second sound velocity, this cone is defined as the second Mach cone accordingly.

The windward orientation of the influence zone of heat transfer can also be predicted by the parabolic equation of heat conduction. The effective velocity of heat diffusion, according to Eq. (3.11), can be obtained

$$C_e \sim \sqrt{\alpha/t} \tag{3.12}$$

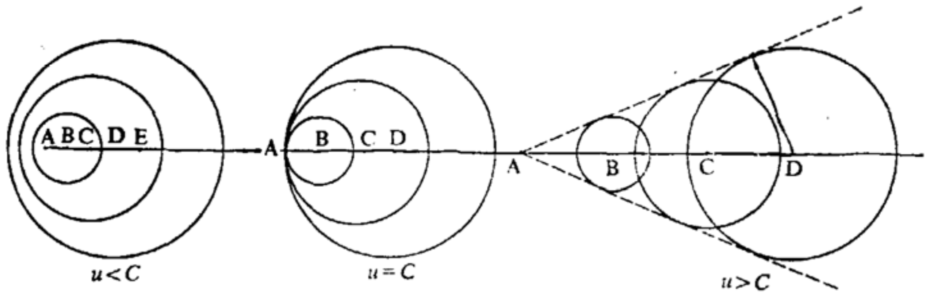

Fig. 2. The second Mach cone in a fluid flow field.

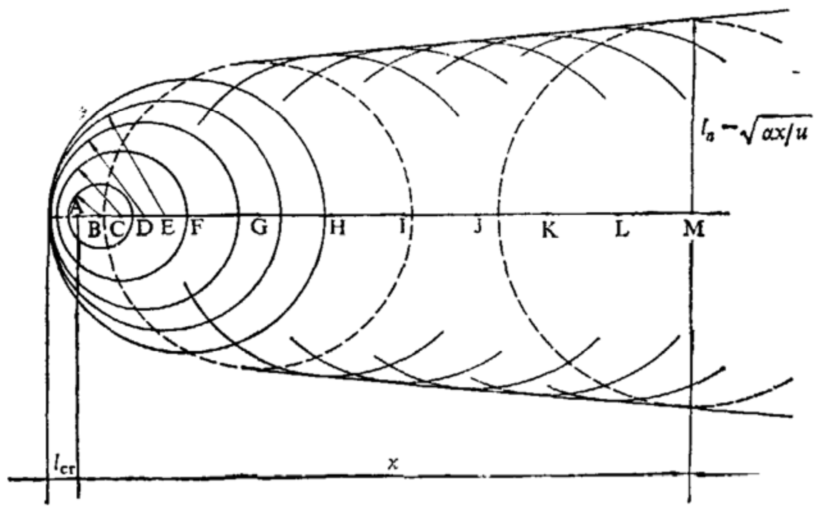

Fig. 3. Formation of the zone of influence of heat diffusion in a fluid flow field.

and it decreases as the time increases. For any given thermal diffusivity and flow velocity ($u \neq 0$), the convective movement will overtake inevitably the effective diffusion of heat transfer. The critical time for the gradual overtaking process can be estimated as

$$t_{cr} \sim \alpha/u^2, \tag{3.13}$$

and the corresponding critical distance of upwind propagation, which is the maximum distance the effective diffusion can influence, is then

$$l_{cr} \sim \alpha/u. \tag{3.14}$$

For media frequently utilized in engineering, thermal diffusivity of liquids, such as water, is of the order of magnitude 10^{-7} m^2/s, and for gases, such as air and steam, 10^{-5}—10^{-4} m^2/s, consequently, even for creeping flows with very small velocity, say, $u = 1$ cm/s, the maximum upwind distance is very short in the orders of magnitude 10^{-5} m and 10^{-3}—10^{-2} m for liquids and gases, respectively.

The above statement is illustrated in Fig. 3. Suppose that the initial location of the point source is at the origin of the coordinate system, and the time increases in alphabetical sequence of A, B, C, $\cdots$, M, then it can be seen that initially the effective diffusion is faster than the convective movement, and heat flux propagates gradually upwind, but at a decreasing velocity; at the critical time represented by E, the maximum upwind distance of effective diffusion is reached, and the effective velocity of heat diffusion balances the flow velocity. As the time goes on, the effective diffusion becomes slower and slower compared with the fluid motion, and the effective diffusion becomes downward to the initial location of the heat source. In this way, the overall effective region of heat diffusion forms a revolutionary body with x-axis as its symmetrical axis. The magnitude of size of the body in the direction normal to the flow should be

$$l_n \sim \sqrt{\alpha t}. \tag{3.15}$$

As the longitudinal coordinate is

$$x = ut, \tag{3.16}$$

we have

$$l_n \sim \sqrt{\alpha x/u}, \tag{3.17}$$

which means that the symmetrical body of revolution is of the parabolic type.

3. *Conversion of the NS Equations to the SNS Equations*

The relative importance of the upwind diffusion of heat can be represented by the maximum upwind distance of the effective heat transfer. Supposing that the concerned characteristic length of the flow field is L, which is a measure of the overall length of the influential revolutionary body of heat transfer, the ratio of the maximum effective distance to the overall length is estimated as

$$l_{cr}/L \sim \alpha/uL = 1/P_e, \tag{3.18}$$

where P_e is the Peclets criterion. It is well known that P_e represents the ratio of the convective effect to diffusion effect of heat transfer in flow fields[7]. Eq. (3.18) shows that P_e can also stand for a symbol of the relative importance of upwind propagation of heat diffusion. For the cases of large P_e, the upwind diffusion can be neglected, and the heat diffusion would be considered as downward.

In the general cases, heat disturbances are continuously distributed in a flow field, and at any point along a streamline there exists a revolutionary body of influence of heat transfer just similar to that discussed previously for an isolated heat source. Among these bodies, the bodies corresponding to disturbances located in the downward side of the streamline would be included in the bodies corresponding to upwind disturbances. As a result, the overall body of influence of a streamline shares the same geometrical orders of magnitude with the body of a point source. In the light of the above facts and in accordance with the energy equation of fluid flow, the ratio of the diffusion effect of heat transfer in the flow direction to that in the direction normal to the flow is obtained

$$q_l/q_n \sim 1/P_e, \tag{3.19}$$

where q_l and q_n are the diffusion terms with respect to the flow direction and to the direction normal to the flow, respectively

$$q_l \sim T/L^2, \tag{3.20}$$

$$q_n \sim T/\alpha t = Tu/\alpha L. \tag{3.21}$$

Hence, the Peclets criterion can also present the relative importance of the streamwise diffusion in heat transfer. For the cases of large P_e, the streamwise diffusion is negligible, and the energy equation converses to the parabolic form in the SNS equations.

The above conclusions are applicable, except that the criterion is altered, to the momentum transfer and the mass transfer. The Reynolds number Re acts as the criterion for momentum transfer. Apart from its classical meaning in representing the ratio of the inertia force to the viscous force of floid motions, the Reynolds number also represents the ratio of the viscous diffusion effect in the direction normal to the flow to that in the flow direction. For the flows with large Reynolds number, the streamwise viscous effect is negligible, and the complete NS equations converses to the SNS equations consequently.

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